5 Collisions

- Consider a particle of mass m_1 diameter D_1 (aka target) moving through population of particles of mass m_2 (aka impactor) where n = # of m_2 particles / volume

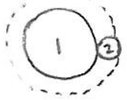
Particle-in-a-box gives rate at which m_1 encounters m_2 as

$R_{col} = n \sigma V_{rel}$ $\boxed{5.1}$

where σ = cross-section of interaction, V_{rel} = relative velocity between particles

$t_{col} = 1/R_{col}$ = collision time

$\boxed{1B}$ $\sigma_{1B} = \frac{\pi}{4}(D_1+D_2)^2$



$\boxed{2B}$ $\sigma_{2B} = \sigma_{1B}(1 + V_{esc}^2/V_{rel}^2)$

$\boxed{3B}$ If encounter is slow cf orbital motion = shear dominated regime; i.e. if $V_{rel} < V_H = \sqrt{GM_1/r_H}$ = orbital velocity at Hill radius then need to add ✱.

- Outcome depends on:

specific incident energy $Q = \frac{1}{2} m_2 V_{col}^2 / m_1 = \frac{1}{2}\left(\frac{D_2}{D_1}\right)^3 V_{col}^2$ $\boxed{5.2}$

where V_{col} = collision velocity

NB $E_{col} = \frac{1}{2} m_1 m_2 V_{col}^2/(m_1+m_2) = \frac{m_1^2}{(m_1+m_2)} Q$ $\boxed{5.3}$

as well as impact angle, internal structure, composition

Grouped by what happens to mass of m_1

- Sticking = mass gain (accretion) eg protoplanetary disk
- Bouncing = mass conservation (with energy loss) eg. planetary ring
- Fragmentation = mass loss (collisional cascade) eg. debris disk

but may also care about spin, internal structure (eg compaction), energy

Sticking

If particle accretes all the mass it encounters, it grows at a rate:

$\dot{m}_1 = m_2 R_{col} = (n m_2)\frac{\pi}{4}(D_1+D_2)^2\left[1 + \frac{4G(m_1+m_2)}{(D_1+D_2)V_{rel}^2}\right] V_{rel}$ $\boxed{5.4}$

If $D_1 \gg D_2$ and $V_{rel} \gg V_{esc}$ then $\dot{m}_1 \propto D_1^2 \propto m_1^{2/3}$ $\boxed{5.5}$

as $\dot{D}_1 = \left(\frac{2}{\pi\rho}\right) D_1^{-2}\dot{m}_1 \rightarrow D_1 = D_{10} + kt$ ie. linear growth

$V_{rel} \ll V_{esc}$ then $\dot{m}_1 \propto D_1^4 \propto m_1^{4/3}$ $\boxed{5.6}$

$\therefore D_1 = D_{10}[1-kt]^{-1}$ ie. runaway growth as $\rightarrow \infty$ in finite time

$\boxed{eg}$ Settling with coagulation (EG3.1)

For small particles in subsonic Epstein drag regime $\boxed{2.37, 2.39} \rightarrow t_s = \frac{1}{2}\rho_s D/\rho_g V_T$

If $T_s \ll 1$, $\boxed{2.50} \rightarrow dz/dt = -K_z z D$ $\boxed{5.7}$

where $K_z = \frac{1}{2}\Omega_K^2 \rho_s/\rho_g V_T$

For $V_{rel} \gg V_{esc}$, particle encounters mass at rate $f\rho_g \frac{\pi}{4} D^2 K_z z D$ where f = ratio of dust to gas

So if it accretes this mass it grows at rate $dD/dt = K_D z D$ $\boxed{5.8}$

where $K_D = \frac{1}{2} f \rho_g K_z/\rho_s$

Thus $dD/dz = -K_D/K_z = -\frac{1}{2} f\rho_g/\rho_s$

For ρ_g that is constant with z

$D = D_0 + (z_0 - z)\frac{1}{2} f\rho_g/\rho_s$ $\boxed{5.9}$

and particle grows to a final size of $\simeq \frac{1}{2} z_0 f\rho_g/\rho_s$ on reaching midplane

Thus particles grow from $\sim 0.1\mu m$ to $\sim 1cm$ as they settle to midplane on timescales of ~ 1000 years

Fragmentation

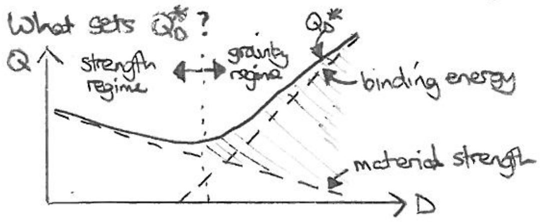
- Define dispersal threshold Q_D^*

If $Q = Q_D^*$ then largest fragment has a mass $f_{lr} m_1$ where $f_{lr} = 0.5$

$Q < Q_D^*$ then $f_{lr} \approx 1 - 0.5Q/Q_D^*$ $\rightarrow$ cratering collisions [NB $M_{cra} \propto \rho(M,Q)^{1.23}$ for small Q]

$Q > Q_D^*$ $\quad = 0.5\,(Q_D^*/Q)^{1.24}$ $\rightarrow$ catastrophic collisions

- What sets Q_D^*?



$$Q_D^* = Q_a D^{-a} + Q_b D^b \qquad \boxed{5.10}$$

eg. $Q_a = 620$ J/kg, $a = 0.5$, $Q_b = 5\times10^{-6}\rho$ J/kg, $b = 1.5$

In shaded region objects fragment but reaccumulate by self gravity
$\Rightarrow$ "rubble pile" planetesimals

Size distributions

- In general particle distributions are not mono-disperse (all same size)

$\therefore$ define $n(D)$ s.t. $n(D)dD$ is # of particles in size range $D \Rightarrow D+dD$

- Let $n(D) = KD^{-\alpha}$ from D_{min} to D_{max}

$\therefore$ distribution of cross-sectional area $\sigma(D) = K\frac{\pi}{4}D^{2-\alpha}$

distribution of mass $M(D) = K\frac{\pi\rho}{6}D^{3-\alpha}$ $\qquad \boxed{5.11}$

- Usually define K in terms of total cross-sectional area or mass in distribution

$$\sigma_{tot} = \frac{K\pi}{4(3-\alpha)}\left[D_{max}^{3-\alpha} - D_{min}^{3-\alpha}\right]$$

$$M_{tot} = \frac{K\pi\rho}{6(4-\alpha)}\left[D_{max}^{4-\alpha} - D_{min}^{4-\alpha}\right] \qquad \boxed{5.12}$$

- For $D_{min} \ll D_{max}$: $\alpha < 3$ $\quad$ σ_{tot} and M_{tot} dominated by D_{max}

$3 < \alpha < 4$ $\quad$ σ_{tot} dominated by D_{min}, M_{tot} by D_{max}

$\alpha > 4$ $\quad$ σ_{tot} and M_{tot} dominated by D_{min}

- Also defined using $n(m)$ s.t. $n(m)dm$ is # with masses $m \Rightarrow m+dm$

For $n(m) \propto m^{-q} \propto D^{-3q}$ then as $n(D) = n(m)dm/dD \propto D^{2-3q} \Rightarrow \underline{\alpha = 3q-2}$ $\qquad \boxed{5.13}$

Collision rates in size distributions

Rate of impacts onto a particle D from those in size range $D_{im} \Rightarrow D_{im}+dD_{im}$ is $n(D_{im})dD_{im}\frac{1}{V}\frac{\pi}{4}(D+D_{im})^2(1+\theta)V_{rel} =$ $R_{col}(D,D_{im})\,dD_{im}$ where

$$R_{col}(D,D_{im}) = \sigma_V(D_{im})\,(1+D/D_{im})^2(1+\theta)\,V_{rel} \qquad \boxed{5.14}$$

where $\sigma_V(D_{im})$ = cross-sectional area volume density distribution $= n(D_{im})\frac{\pi D_{im}^2}{4}\frac{1}{V} = \sigma(D_{im})/V$

V = volume filled by particles $\approx 4\pi r^3 (dr/r) I_{max}$

for a belt at $r \pm dr/2$ of scale height $h \approx I_{max} r$

$$\theta = \text{Safronov number} = (V_{esc}/V_{rel})^2 = \frac{2\pi\rho G}{3V_{rel}^2}\left(\frac{D^3+D_{im}^3}{D+D_{im}}\right)$$

Catastrophic collisions for which $Q > Q_D^*$ are all those with impactors larger than $X_c D$ where $\boxed{5.2}$ gives

$$X_c = (2Q_D^*/V_{col}^2)^{1/3} \qquad \boxed{5.15}$$

So catastrophic collision rate is

$$R_{cc}(D) = \int_{X_c D}^{D_{max}} R_{col}(D,D_{im})\,dD_{im} \qquad \boxed{5.16}$$

$\boxed{eg}$ For $\alpha < 4$ $\boxed{5.12} \rightarrow K \approx \frac{6(4-\alpha)}{\pi\rho}D_{max}^{\alpha-4}M_{tot}$ (assuming $D_{min} \ll D_{max}$) $\qquad \boxed{5.17}$

$$\therefore \sigma(D) = \frac{3}{2}\frac{(4-\alpha)}{\rho}D_{max}^{\alpha-4}M_{tot}D^{2-\alpha}$$

So, for $\theta \ll 1$ (which also means $V_{col} \approx V_{rel}$)

$$\therefore R_{cc}(D) = \int_{X_c D}^{D_{max}} A\,D_{im}^{2-\alpha}(1+D/D_{im})^2\,dD_{im} \quad \text{where } A = \frac{3}{2}\left(\frac{4-\alpha}{\rho}\right)D_{max}^{\alpha-4}M_{tot}\frac{1}{V}V_{rel}$$

$$= A\left[\frac{1}{3-\alpha}D_{im}^{3-\alpha} + \frac{2D}{2-\alpha}D_{im}^{2-\alpha} + \frac{D^2}{1-\alpha}D_{im}^{1-\alpha}\right]_{X_c D}^{D_{max}} \qquad \boxed{5.18}$$

If $\alpha > 3$ it is $X_c D$ limit that dominates, and if $X_c \ll 1$ it is last term that dominates

$$\therefore R_{cc}(D) \sim \frac{A}{\alpha-1}X_c^{1-\alpha}D^{3-\alpha}$$

$$= \left(\frac{3}{2}\frac{4-\alpha}{\alpha-1}2^{\frac{1-\alpha}{3}}\right)\left(\frac{M_{tot}}{\rho V}\right)D_{max}^{\alpha-4}\,Q_D^{*\frac{1-\alpha}{3}}\,V_{rel}^{\frac{1+2\alpha}{3}}\,D^{3-\alpha} \qquad \boxed{5.19}$$

Steady state size distribution

- Discrete form of size distribution : let m_k = mass of particles in bin k
  where bins are logarithmically spaced s.t. $D_{k+1} = D_k(1-\delta)$ [5.20]

  Integrating [5.11] for small δ gives $m_k \approx \left(\frac{\pi\rho}{6}\right) n(D_k) D_k^4 \delta$ [5.21]

  Mass conservation : $\dot{m}_k = \dot{m}_k^{+c} - \dot{m}_k^{-c}$ [5.22]
  where $+c$ and $-c$ are gain and loss from collisions

  Define a redistribution function: $f_i(k)$ is the fraction of mass leaving bin i from collisions that goes into bin k

  $\therefore \dot{m}_k^{+c} = \sum_{i=0}^{\infty} \dot{m}_i^{-c} f_i(k)$

  For steady state $\dot{m}_k = 0$ $\therefore \dot{m}_{ks}^{-c} = \sum_{i=0}^{\infty} \dot{m}_{is}^{-c} f_i(k)$

  Assuming the redistribution function is scale independent, so can be written $f_i(k) = F(k-i)$

  $\therefore \dot{m}_{ks}^{-c} = \sum_{i=0}^{\infty} \dot{m}_{is}^{-c} F(k-i)$

  As all mass goes somewhere $\sum_{i=0}^{\infty} F(k-i) = 1$, so one solution must be: $\dot{m}_{ks}^{-c} = \dot{m}_{is}^{-c} = C$ [5.23]
  ie mass loss rate in logarithmic bins is independent of size.

- If $\dot{m}_k^{-c} \approx m_k R_{cc}(D_k)$ [5.24]

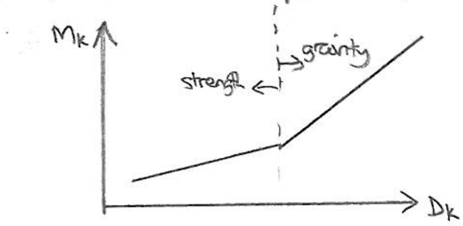
  then for a power law size distribution [5.11] we can use [5.19] to get

  $\dot{m}_k^{-c} \propto m_k Q_D^{*\frac{1-\alpha}{3}} D_k^{3-\alpha}$

  and using [5.21] and assuming $Q_D^* \propto D^b$

  $\dot{m}_k^{-c} \propto D_k^{7-2\alpha+b\left(\frac{1-\alpha}{3}\right)}$

  and for this to be independent of size $7-2\alpha+b\left(\frac{1-\alpha}{3}\right) = 0$ $\therefore \alpha = \frac{21+b}{6+b}$ [5.25]



For a dispersal threshold independent of size $\alpha = 3.5$
But for [5.10] the size distribution has different slopes in strength ($\alpha_a = 3.73$) and gravity ($\alpha_b = 3$) regimes.

Evolution of size distribution

- Let $m_k = m_{ks}(1+\epsilon_k)$

  Put into [5.22], then using [5.24] and noting $\dot{m}_k^{+c}$ is unaffected by ϵ_k:

  $m_{ks}\dot{\epsilon}_k = \dot{m}_{ks}^{+c} - m_{ks}(1+\epsilon_k) R_{cc}(D_k)$

  $\therefore \dot{\epsilon}_k / \epsilon_k = -R_{cc}(D_k)$ [5.26]

  Thus perturbations are damped on collision timescale, and the distribution achieves steady state at a given size on timescale of collisions $\rightarrow$ largest objects may retain a "primordial size distribution"

- For particles in steady state [5.22] $\rightarrow$ mass loss is balanced by gain from break-up of larger objects.
  But those at the top cannot be replenished.

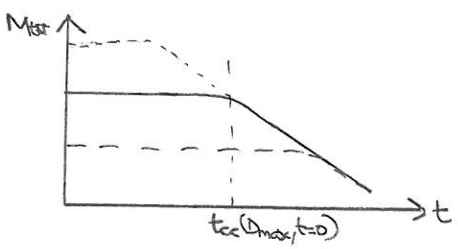
  $dM_{tot}/dt \approx -M_{tot} R_{cc}(D_{max})$ [5.27]

  From [5.19] we can write $R_{cc}(D_{max}) = B M_{tot}$

  $\therefore dM_{tot}/dt = -B M_{tot}^2$

  $\therefore \int_{M_{tot0}}^{M_{tot}} M_{tot}^{-2}\, dM_{tot} = \int_0^t -B\, dt$

  $\therefore \underline{M_{tot} = M_{tot0}\left[1 + B M_{tot0} t\right]^{-1} = M_{tot0}\left[1 + t/t_{cc}(D_{max}, t=0)\right]^{-1}}$ [5.28]



- Mass remains constant until the collision time of largest objects at which point it drops $1/t$
- As $t \rightarrow \infty$ $\underline{M_{tot} \rightarrow (Bt)^{-1}}$ [5.29]
  and so is independent of initial mass

- For a fixed shape of size distribution $\sigma_{tot} \propto M_{tot}$ (see [5.12] for a single power law distribution)
  and so dust fractional luminosity $f \approx \sigma_{tot}/4\pi r^2$ displays same behaviour [5.30]
  $\rightarrow$ debris disk luminosity falls $1/t$ at late ages at which point there is a maximum luminosity that the disk can have (for given r, Q_D^* etc)

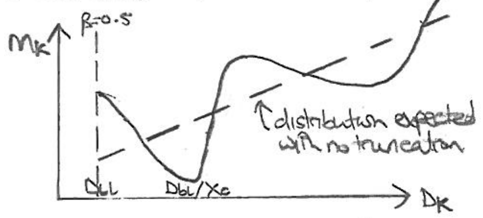
Effect of radiation forces

Radiation pressure:

To first order this truncates the size distribution at D_{bl} for which $\beta(D_{bl}) = 0.5$ $\boxed{5.31}$

$\boxed{5.23}$ and $\boxed{5.24}$ still apply, but the truncation means there are no particles of size $X_c D$ close to the truncation.

Thus $R_{cc}(D_{bl})$ is lower than "expected" and so $M_{ks}(D_{bl})$ is higher.



This causes a higher than expected collision rate for particles normally destroyed by those of size D_{bl} (ie of size D_{bl}/X_c) and so a lower $M_{ks}(D_{bl}/X_c)$, and so on.

→ wave extending to larger sizes, with amplitude and wavelength that depend on X_c (so wavelength increases with D in strength regime)

However, the high eccentricities of particles close to D_{bl} ($\boxed{2.12}$) complicates this as such particles move through a larger volume (among other effects).

P-R drag:

- Consider a planetesimal belt at r_0 creating dust all of size D (with corresponding β) that evolves due to mutual collisions (that destroy the dust) and P-R drag (that makes it spiral in)
  - If Σ = surface density of cross-sectional area and density is uniform in range $z = \pm r I_{max}$

    $\therefore \sigma_v = \Sigma 2\pi r dr / 2\pi r dr 2r I_{max} = \Sigma / 2r I_{max}$ $\boxed{5.32}$

    If $V_{rel} \simeq I_{max} V_k = I_{max} 2\pi r / t_{per}$ $\boxed{5.33}$

    then from $\boxed{5.14}$ for $\theta \simeq 0$ and $D_{im} = D$: $R_{col} = 4\sigma_v V_{rel} = 4\pi\Sigma / t_{per}$ $\boxed{5.34}$
  - For circular orbits: $\dot{r}_{pr} = -2\alpha/r$ (see $\boxed{2.18}$) where $\alpha = r_0^2/4t_{pro}$ and t_{pro} = time to go from r_0 to $r=0$ (see $\boxed{2.23}$) $\boxed{5.35}$
  - Continuity equation: $\partial n(r)/\partial t + \partial n(r)\dot{r}_{pr}/\partial r = \dot{N}^+(r) - \dot{N}^-(r)$ $\boxed{5.36}$

    where $n(r) = \#/\text{radius} = \Sigma 2\pi r/\sigma$

    $\partial n(r)/\partial t = 0$ for steady state

    $\dot{N}^+(r) = 0$ far from source

    $\dot{N}^-(r) = \#\text{lost}/\text{time}/\text{radius} = n(r) R_{col} = 4\pi\Sigma n(r)/t_{per} = 8\pi^2\Sigma^2 r/t_{per}\sigma$

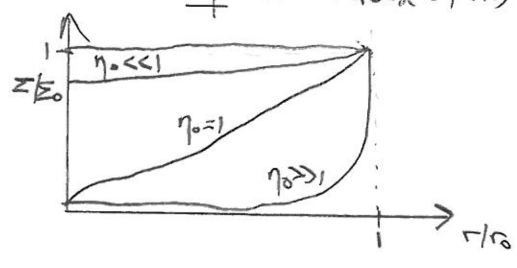
    $\therefore d[\Sigma 2\pi r(-2\alpha/r)/\sigma]/dr = -8\pi^2\Sigma^2 r/t_{per}\sigma$

    $\therefore \frac{1}{\Sigma^2} d\Sigma = \left(\frac{2\pi}{\alpha t_{per}}\right) r\,dr$

    As $t_{per} = r^{1.5} M_*^{-0.5}$

    $\therefore [-\Sigma^{-1}]_{\Sigma_0}^{\Sigma} = \frac{2\pi}{\alpha} M_*^{0.5} [2r^{0.5}]_{r_0}^{r}$

    $$\left.\begin{aligned} &\therefore \Sigma = \Sigma_0 [1 + 4\eta_0(1 - \sqrt{r/r_0})]^{-1} \\ &\text{where } \eta_0 = (4\pi\Sigma_0/t_{per0})(r_0^2/4\alpha) = t_{pro}/t_{col0} \end{aligned}\right\} \boxed{5.37}$$



- Radial distribution just depends on t_{pro}/t_{col0}
- If $\eta_0 \ll 1$ Σ = const P-R drag-dominated

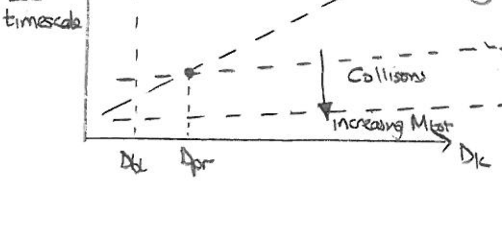
  $\eta_0 \gg 1$ Σ is depleted Collision-dominated tending to $\Sigma_0/4\eta_0$ at $r \to 0$ $\boxed{5.38}$ which is independent of Σ_0

- Effect of P-R drag on size distribution: mass conservation should include extra $-\dot{m}_k^{-pr}$ term on RHS of $\boxed{5.22}$

  where $\dot{m}_k^{-pr} = m_k R_k^{pr}$

  $R_k^{pr} \propto \beta \simeq A_{pr} D_k^{-1}$ $\boxed{5.39}$

In the steady state collisional regime, $\boxed{5.21, 5.23, 5.24}$ → $R_{cc}(D_k) \propto D_k^{\alpha-4}$ $\boxed{5.40}$



Thus at some size D_{pr} $R_{cc}(D_{pr}) = R_k^{pr}$

If this occurs at $D_{pr} > D_{bl}$ → P-R drag dominated

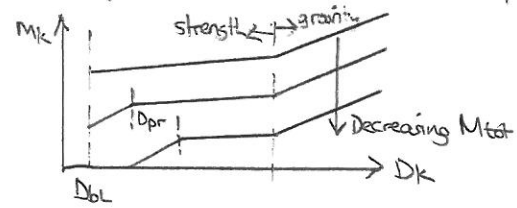
$D_{pr} < D_{bl}$ → Collision dominated

In a P-R drag dominated disk, ignore collisional loss for $D < D_{pr}$, s.t. steady state means

$$\dot{M}_{ks}^{+c} = \sum_{i=0}^{i_{pr}} \dot{M}_{is}^{-c} F(k-i) = \dot{M}_{ks}^{-pr} = M_{ks} A_{pr} D_k^{-1}$$

Substituting for $\boxed{5.23}$ $M_{ks} = A_{pr}^{-1} C D_k \sum_{i=0}^{i_{pr}} F(k-i)$ $\boxed{5.41}$

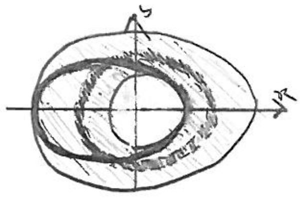
So the shape of the size distribution for $D < D_{pr}$ is determined by the redistribution function



Accurate collision rates

Collision rates for specific orbits are calculated by assuming $\lambda, \tilde{\omega}$ and Ω to be randomly distributed (due to orbital motion and precession), and needs to consider how the rates and velocity of collisions vary around orbit.
As simple case, consider particle with a, e interacting with particle also with a, e and random $\lambda, \tilde{\omega}, \Omega$ and I distributed s.t. density is uniform between $0 \to I_{max}$
Local collision rate of $D_{im} \to D_{im} + dD_{im}$ onto D is: $R_{col}(r, D, D_{im}) dD_{im} = \sigma_V(r, D_{im}) dD_{im} \left[1 + \frac{D}{D_{im}}\right]^2 [1 + \theta(r, D, D_{im})] V_{rel}(r)$ $\boxed{5.42}$

$\boxed{\sigma_V}$



Consider region $r \to r + dr$: Volume of torus is $dV = 4\pi r^2 dr I_{max}$
Define fraction of orbit spent in region as $F_r dr$ where $F_r = 2/\dot{r} t_{per}$
and $\dot{r}^2 = v^2 - (r\dot{\theta})^2 = \mu[2/r - 1/a] - \mu a(1-e^2)/r^2$ $\boxed{5.43}$
Let $u = a/r$ $\therefore F_u = F_r dr/du = -\pi^{-1} u^{-2} [u^2(e^2-1) + 2u - 1]^{-1/2}$ $\boxed{5.44}$
$\therefore \sigma_V(r, D_{im}) = (\sigma(D_{im})/dV) F_u du$ $\boxed{5.45}$

$\boxed{V_{rel}}$ Since $V_\theta = r\dot{\theta} = h/r = \sqrt{\mu a(1-e^2)}/r$, $V_{\theta 1} = V_{\theta 2}$
$\therefore V_{rel} = 2\dot{r}$
$\therefore V_{rel}^2(r) = 4(\mu/a)[u^2(e^2-1) + 2u - 1]$ $\boxed{5.46}$

So, for $\theta \approx 0$:

$R_{col}(r, D, D_{im})$: $R_{col}(r, D, D_{im}) = \sigma(D_{im})[1 + D/D_{im}]^2 \sqrt{\mu/a} \,/\, [2\pi^2 a r^2 I_{max}]$ $\boxed{5.47}$

$R_{col}(D, D_{im})$: Need to integrate around orbit weighting by fraction of time spent in each region F_u

$$\therefore R_{col}(D, D_{im}) = \int_{(1+e)^{-1}}^{(1-e)^{-1}} R_{col}(r, D, D_{im}) \pi^{-1} u^{-2} [u^2(e^2-1) + 2u - 1]^{-1/2} du$$
$$= \sigma(D_{im})[1 + D/D_{im}]^2 \sqrt{\mu/a(1-e^2)} \frac{1}{2\pi^2 a^3 I_{max}}$$ $\boxed{5.48}$

$R_{cc}(D)$:

$$R_{cc}(r, D) = \underbrace{\int_{X_c D}^{D_{max}} \sigma(D_{im})[1 + D/D_{im}]^2 dD_{im}}_{\propto V_{rel}^{\frac{2}{3}(\alpha-1)}} \frac{1}{dV} V_{rel}(r) F_u du$$

which then needs to be integrated around orbit:

$$R_{cc}(D) = \int_{(1+e)^{-1}}^{(1-e)^{-1}} R_{cc}(r, D) F_u du$$ $\boxed{5.49}$