4 Close Approaches

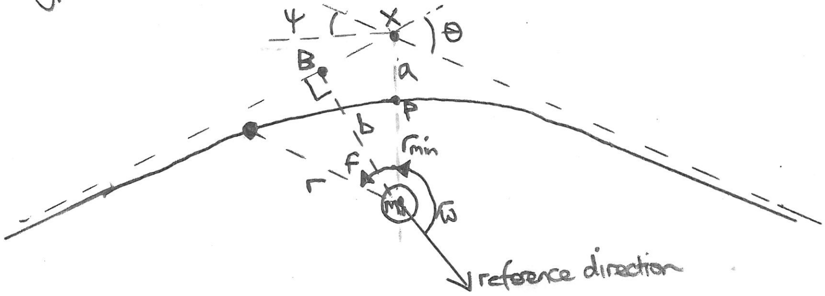
Geometry of hyperbolic orbit

Remember, general sol^n to $\ddot{\underline{r}} + \mu\underline{r}/r^3 = 0$ is $\boxed{1.5}$:

$$r = (h^2/\mu)[1 + e\cos(\theta - \varpi)]^{-1}$$

where $h = r^2\dot\theta$, $\mu = G(m_1+m_2)$, e and ϖ are constants of integration

Hyperbolic orbits have $e > 1$ and we define "a" and "f" thus:



$\therefore$ $\underline{r = a(e^2-1)/(1 + e\cos f)}$ $\boxed{4.1}$

$\underline{h = \sqrt{\mu a(e^2-1)}}$ $\boxed{4.2}$

Closest approach: is when $f = 0$ $\therefore$ $\underline{r_{min} = a(e-1)}$ $\boxed{4.3}$

Collisions: If $\underline{r_{min} < (D_1 + D_2)/2}$ $\boxed{4.4}$

where D_i are diameters of bodies

For spherical objects of uniform density ρ: $\underline{D_i = \left(\frac{6M_i}{\pi\rho}\right)^{1/3}}$ $\boxed{4.5}$

Impact parameter, b

For $r \to \infty$, $\boxed{4.1}$ gives $1 + e\cos f_\infty \to 0$

$\therefore f_\infty = \cos^{-1}(-e^{-1}) = \pi \pm \cos^{-1}(e^{-1})$

But $f_\infty = \pi/2 + \psi$

$\therefore$ $\underline{\psi = \pi/2 - \cos^{-1}(e^{-1})}$ $\boxed{4.6}$

Looking at M_1BX, and noting that $M_1X = ae$ and $\angle BXM_1 = \pi/2 - \psi$

$\therefore BX = ae\cos(\pi/2 - \psi) = a$

$\therefore$ $\underline{b = a\sqrt{e^2-1}}$ $\boxed{4.7}$

Scattering angle, θ

$\theta = 2\psi = \pi - 2\cos^{-1}(e^{-1})$

$\underline{\sin\theta/2} = BX/M_1X = e^{-1} = \underline{[1 + (b/a)^2]^{-1/2}}$ $\boxed{4.8}$

$\tan\theta/2 = a/b$ $\therefore$ $\underline{b = a\cot\theta/2}$ $\boxed{4.9}$

Velocity, V

Same eqns also apply from elliptical motion:

$\boxed{1.9}$ $C = \frac{1}{2}V^2 - \mu/r$, $\boxed{1.4}$ $h = r^2\dot f$

Differentiating $\boxed{4.1}$, also gives same expression, but differs after substituting for h from $\boxed{4.2}$

$\dot r = r\dot f\, e\sin f/(1 + e\cos f) = \underline{\sqrt{\frac{\mu}{a(e^2-1)}}\, e\sin f}$ $\boxed{4.10}$

$\underline{r\dot f = \sqrt{\frac{\mu}{a(e^2-1)}}(1 + e\cos f)}$ $\boxed{4.11}$

Thus $V^2 = \dot r^2 + r^2\dot f^2 = \frac{\mu}{a(e^2-1)}[e^2\sin^2 f + 1 + 2e\cos f + e^2\cos^2 f]$

$\underline{= \mu\left[\frac{1}{a} + \frac{2}{r}\right]}$ $\boxed{4.12}$

And so $\underline{C = \mu/2a}$ $\boxed{4.13}$

i.e. +ve whereas was −ve for elliptical motion

Velocities V_r, V_∞, V_{max}

$$\dot r^2 = V^2 - (r\dot f)^2$$
$$= \mu\left[\frac{1}{a} + \frac{2}{r}\right] - \mu a(e^2-1)/r^2$$
$$= \frac{\mu}{a}\left[1 + 2\frac{a}{r} - \left(\frac{a}{r}\right)^2(e^2-1)\right] \qquad \boxed{4.14}$$

As $r \to \infty$, $\boxed{4.12}$ gives that $V_\infty = \sqrt{\mu/a}$ $\qquad \boxed{4.15}$

More often we know V_∞ but not $a = \mu/V_\infty^2$ $\qquad \boxed{4.16}$

Which means we can rewrite $\boxed{4.9}$ as $\sin\theta/2 = [1 + b^2V_\infty^4/\mu^2]^{-1/2}$ $\qquad \boxed{4.17}$

Or if we know r_{min} and V_∞: use $\boxed{4.3}$ $\quad = e^{-1} = [1 + r_{min}/a]^{-1} = [1 + r_{min}V_\infty^2/\mu]^{-1}$ $\qquad \boxed{4.18}$

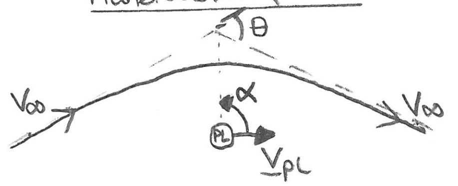
Maximum velocity occurs at pericentre at $f=0, r=r_{min}$, so $\boxed{4.12}$:

$$\therefore\ V_{max}^2 = \mu\left[\frac{1}{a} + \frac{2}{a(e-1)}\right] = \left(\frac{\mu}{a}\right)\left(\frac{1+e}{e-1}\right) \qquad \boxed{4.19}$$

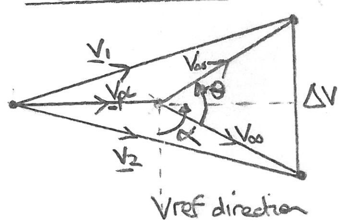
Patched Conics

For encounter in presence of third object, use patched conic approximation:
Assume orbits are described by: ellipse, hyperbola (impulsive change in $\underline{V}$), ellipse
eg Sun + planet + spacecraft

Planetocentric frame



Inertial frame



From geometry, $\Delta V = 2V_\infty \sin\theta/2$
$$= 2V_\infty[1 + b^2V_\infty^4/\mu^2]^{1/2} = 2V_\infty[1 + r_{min}V_\infty^2/\mu]^{-1} \qquad \boxed{4.20}$$

Whether this change increases ($V_2 > V_1$) or decreases ($V_2 < V_1$) velocity in inertial frame depends on α

$\alpha > \pi/2$ → encounter is behind planet, velocity increases, so semimajor axis of heliocentric orbit decreases increases
$\alpha < \pi/2$ → " in front of " " decreases "

Gravity Assist: : encounters with planets are used to get spacecraft to inner/outer Solar System in this way

∴ ΔV has its maximum value when $(\partial \Delta V/\partial V_\infty)_{r_{min}} = 0$ for a given r_{min}

$$\therefore\ 2[1 + r_{min}V_\infty^2/\mu]^{-1} - (4V_\infty^2 r_{min}/\mu)[1 + r_{min}V_\infty^2/\mu]^{-2} = 0$$
$$\therefore\ V_\infty = \sqrt{\mu/r_{min}}$$

From $\boxed{4.20}$ $\Delta V_{max} = V_\infty = \sqrt{\mu/r_{min}}$ $\qquad \boxed{4.21}$

From $\boxed{4.4}$ there is a maximum possible ΔV before the spacecraft hits the planet

$$\Delta V_{max} = \sqrt{2\mu/(D_1+D_2)} \qquad \boxed{4.22}$$

Escape velocity

From $\boxed{4.13}$, two objects are unbound if $C = \frac{1}{2}V^2 - \mu/r > 0$
As closest possible separation is r_{min} from $\boxed{4.4}$, escape velocity is defined as:

$$\frac{1}{2}V_{esc}^2 - 2\mu/(D_1+D_2) = 0$$
$$\therefore\ V_{esc} = 2\sqrt{\mu/(D_1+D_2)} \qquad \boxed{4.23}$$

For a planet $V_{esc} \approx 2\sqrt{GM_{pl}/D_{pl}} = \sqrt{\frac{2\pi G\rho}{3}}\, D_{pl}$ $\qquad \boxed{4.24}$

So V_{esc} in m/s is roughly $\frac{1}{2}D_{pl}$, where D_{pl} is in km

Also means $\Delta V_{max} = \frac{1}{\sqrt{2}} V_{esc}$ $\qquad \boxed{4.25}$

Gravitational focussing

Collisions occur if $b < b_{crit}$

To get b_{crit}, set $r_{min} = (D_1+D_2)/2 = 2\mu/V_{esc}^2$

$\therefore a(e-1) = 2\mu/V_{esc}^2$

As $a = \mu/V_\infty^2$, $e = 1 + 2V_\infty^2/V_{esc}^2$

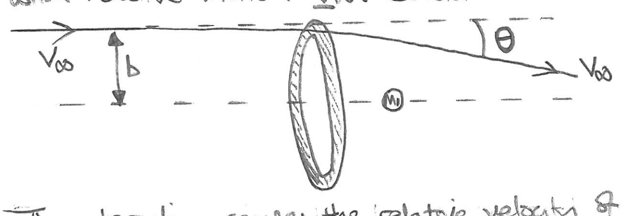
$$\therefore b_{crit}^2 = a^2(e^2-1) = \frac{\mu^2}{V_\infty^4}\left[4V_\infty^4/V_{esc}^4 + 4V_\infty^2/V_{esc}^2\right] = \frac{4\mu^2}{V_{esc}^4}\left[1 + V_{esc}^2/V_\infty^2\right]$$

$$\therefore \underline{b_{crit} = \left(\frac{D_1+D_2}{2}\right)\left[1 + V_{esc}^2/V_\infty^2\right]^{1/2}} \qquad \boxed{4.26}$$

Thus a planet's collision cross-sectional area can be significantly enhanced if $V_{esc} \gg V_\infty$.
This occurs either for large planets (with high V_{esc}) or low velocity dispersion of objects encountering the pl.

Dynamical friction (EG 2.5)

Consider a massive object m_1 (eg a planet) moving through a "sea" of objects m_2 (eg. planetesimals) with relative motion $\underline{V_\infty}$. Consider now those objects at impact parameters $b \pm db/2$ in frame on m_1



The interaction causes the relative velocity of m_2 to change by:

$$\Delta V_{\parallel} = V_\infty(\cos\theta - 1) = -2V_\infty \sin^2\theta/2 = \underline{-2V_\infty\left[1 + b^2V_\infty^4/\mu^2\right]^{-1}} \qquad (\text{from } \boxed{4.17})$$

$$\Delta V_{\perp} = -V_\infty \sin\theta = -2V_\infty \sin\theta/2\sqrt{1-\sin^2\theta/2}$$

$$= -2V_\infty\left[1+b^2V_\infty^4/\mu^2\right]^{-1/2}\left[1-\left[1+b^2V_\infty^4/\mu^2\right]^{-1}\right]^{1/2}$$

$$= -2(bV_\infty^3/\mu)\left[1+b^2V_\infty^4/\mu^2\right]^{-1}$$

To get the effect on m_1, note that $m_1\Delta\underline{V_1} + m_2\Delta\underline{V_2} = 0$ and $\Delta\underline{V} = \Delta\underline{V_2} - \Delta\underline{V_1}$

$$\therefore \Delta\underline{V_1} = -\left(\frac{m_2}{m_1+m_2}\right)\Delta\underline{V}$$

$$\left.\begin{aligned} \therefore |\Delta V_{1,\parallel}| &= \frac{2m_2}{m_1+m_2}V_\infty\left[1+b^2V_\infty^4/\mu^2\right]^{-1} \\ |\Delta V_{1,\perp}| &= \frac{2m_2 bV_\infty^3}{G(m_1+m_2)^2}\left[1+b^2V_\infty^4/\mu^2\right]^{-1} \end{aligned}\right\} \qquad \boxed{4.27}$$

Assuming m_1 moves at $\underline{V_1}$ and m_2 move at a range of $\underline{V_2}$ described by a phase space number density $f(\underline{V_2})$ where $f(\underline{V_2})d^3\underline{V_2}$ is #/unit volume in velocity space element $d^3\underline{V_2}$

For each $\underline{V_2}$ and b, $\sum \Delta V_{1,\perp} = 0$ so only change to $\underline{V_1}$ is from $V_{1,\parallel}$

Now, # of objects m_2 with velocity $\underline{V_2}$ and impact params $b \pm db/2$ per unit time is $2\pi b\,db\,V_\infty f(\underline{V_2})d^3\underline{V_2}$

Integrating over all impact params:

$$d\underline{V_1}/dt\,\Big|_{\underline{V_2}} = 4\pi\left(\frac{m_2}{m_1+m_2}\right)V_\infty^2 f(\underline{V_2})d^3\underline{V_2}\int_{b_{min}}^{b_{max}} b\left[1+b^2V_\infty^4/\mu^2\right]^{-1} db\,\frac{\underline{V_2}-\underline{V_1}}{|\underline{V_2}-\underline{V_1}|}$$

$$= -2\pi G^2 m_2(m_1+m_2) f(\underline{V_2})d^3\underline{V_2}\ln\left[1+\Lambda^2\right](\underline{V_1}-\underline{V_2})/|\underline{V_1}-\underline{V_2}|^3 \qquad \boxed{4.28}$$

where $\Lambda = b_{max}V_\infty^2/G(m_1+m_2)$

ie dynamical friction is dominated by long range interactions; b_{max} could be scale height of disk.

If $\Lambda \gg 1$, $\frac{1}{2}\ln\left[1+\Lambda^2\right] \simeq \ln\Lambda$ = Coulomb logarithm

If all particles move at same $\underline{V_2}$ then $f(\underline{V_2})d^3\underline{V_2} = n$, and if $m_2 \ll m_1$

$$\underline{d\underline{V_1}/dt = -4\pi G^2 m_1(nm_2)\ln\Lambda\,(\underline{V_1}-\underline{V_2})/|\underline{V_1}-\underline{V_2}|^3} \qquad \boxed{4.29}$$

ie dynamical friction is independent of m_2, just on nm_2 = mass volume density of m_2

Tisserand parameter

One approach to getting the change in orbital elements after encounter involves the CRTBP.
Remember the Jacobi constant is an integral of motion, and in inertial coordinates relative to barycentre:

$$\boxed{3.6} \Rightarrow C_J = 2\left(\frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}\right) + 2(\xi\dot{\eta} - \eta\dot{\xi}) - (\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2)$$

where $\mu_1 = \frac{m_1}{m_1+m_2}$, $\mu_2 = \frac{m_2}{m_1+m_2}$, $\mu = G(m_1+m_2) = 1$, $n = 1$

If $m_1 \gg m_2$, $\mu_1 \simeq 1$ and $r_1 \simeq r$
If also far from m_2, $\mu_2/r_2 \ll \mu_1/r_1$
Now $\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2 = v^2 = [2/r - 1/a]$ (from $\boxed{1.11}$)
And $\xi\dot{\eta} - \eta\dot{\xi} = h_z = h\cos I = \sqrt{a(1-e^2)}\cos I$ (from $\boxed{1.17, 1.6}$)

So: $C_J \simeq \cancel{\frac{2}{r}} + 2\sqrt{a(1-e^2)}\cos I - \cancel{\frac{2}{r}} + \frac{1}{a}$

Rewriting, noting that length unit of CRTBP was a_{pl}, we find the Tisserand parameter:

$$T_{pl} = \left(\frac{a_{pl}}{a}\right) + 2\sqrt{\frac{a}{a_{pl}}(1-e^2)}\cos I \qquad \boxed{4.30}$$

that must be constant in scattering problems. (for above assumptions)

Interpretation of T_{pl}

In rotating coords, $\boxed{3.33, 3.5} \Rightarrow C_J = 2\left(\frac{\mu_1}{r_1} + \frac{\mu_2}{r_2}\right) + (x^2+y^2) - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

At close encounter, $x \simeq 1$, $y \simeq 0$, $r_1 \simeq 1$, so for $m_2 \ll m_1$

$$C_J \simeq 3 - V_{rel}^2 \qquad \boxed{4.31}$$

$\boxed{4.32}$

In other words $V_{rel} \simeq V_{pl}\sqrt{3 - T_{pl}}$

ie. T_{pl} is a measure of the relative velocity of encounters

If $T_{pl} \simeq 3$ encounter is slow (and strong)
> 3 objects cannot cross orbit of planet

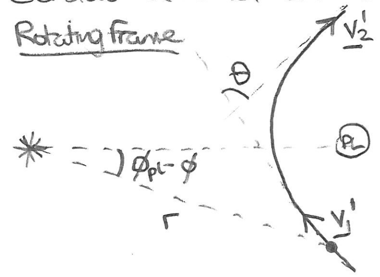
Comet taxonomy is defined by T_{Jup}

$T_{Jup} < 2$ → Oort Cloud comet
> 2 → Ecliptic comet: $2 < T_{Jup} < 3$ = Jupiter family comet
$T_{Jup} > 3$ = Encke-type ($a < a_{Jup}$), Centaurs ($a > a_{Jup}$)

Cometary dynamics

Consider a comet on orbit a_1, e_1 in orbital plane of a planet on a circular orbit at a_{pl} from M_*

Rotating frame



In inertial frame $V_{pl} = \sqrt{\mu/a_{pl}}$

and comet is initially moving at $V_1^2 = \mu[2/r - 1/a_1] = V_{pl}^2\left[2\frac{a_{pl}}{r} - \frac{a_{pl}}{a_1}\right]$

But $V_{\phi 1} = r\dot{\phi} = \sqrt{\mu a_1(1-e_1^2)}/r = V_{pl}\sqrt{(a_{pl}/r)(a_1/r)(1-e_1^2)}$

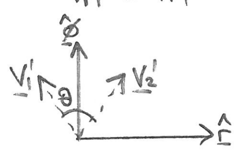
$\therefore V_{r1}^2 = V_1^2 - V_{\phi 1}^2 = V_{pl}^2[2(a_{pl}/r) - (a_{pl}/a_1) - (a_{pl}/r)(a_1/r)(1-e_1^2)]$

So, for $r \simeq a_{pl}$

$$\left.\begin{aligned} V_{\phi 1} &= V_{pl}\sqrt{\frac{a_1}{a_{pl}}(1-e_1^2)} \\ V_{r1} &= \pm V_{pl}\sqrt{2 - (a_{pl}/a_1) - (a_1/a_{pl})(1-e_1^2)} \end{aligned}\right\} \quad \boxed{4.33}$$

To get velocity in inertial frame after encounter, get v' in rotating frame, rotate then convert to inertial frame:

$$V'_{r1} = V_{r1} \quad \text{and} \quad V'_{\phi 1} = V_{\phi 1} - V_{pl} \qquad \boxed{4.34}$$



$$\therefore \begin{pmatrix} V'_{\phi 2} \\ V'_{r2} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} V'_{\phi 1} \\ V'_{r1} \end{pmatrix}$$

$$\therefore V_{r2} = V'_{r2} \quad \text{and} \quad V_{\phi 2} = V'_{\phi 2} + V_{pl}$$

$$\text{So, } V_2^2 = \left[(V_{\phi 1} - V_{pl})\sin\theta + V_{r1}\cos\theta\right]^2 + \left[V_{pl} + (V_{\phi 1} - V_{pl})\cos\theta - V_{r1}\sin\theta\right]^2$$
$$= V_1^2 + 2V_{pl}\left[(V_{pl} - V_{\phi 1})(1-\cos\theta) - V_{r1}\sin\theta\right] \qquad \boxed{4.35}$$

To get resulting change in energy, define $x = 1/a$

$$\therefore \Delta x = \frac{1}{a_2} - \frac{1}{a_1} = \frac{1}{\mu}\left[V_1^2 - V_2^2\right]$$ NB encounter is treated impulsively s.t. r is unchanged.

$$= \frac{2V_{pl}}{\mu}\left[(V_{\phi 1} - V_{pl})(1-\cos\theta) + V_{r1}\sin\theta\right] \qquad \boxed{4.36}$$

Now, assume scattering angle is small, s.t. $\cos\theta \simeq 1$ and $\sin\theta \simeq \theta = 2\mu_{pl}/bV_1'^2$

Use $\boxed{4.34, 4.33}$ $\quad V_1'^2 = V_{r1}^2 + (V_{\phi 1} - V_{pl})^2 = V_{pl}^2\left[3 - (a_{pl}/a_1) - 2\sqrt{\frac{a_1}{a_{pl}}(1-e_1^2)}\right]$

Assume $a_1 \gg a_{pl}$ and $q \simeq a_{pl} - b$ giving $(a_1/a_{pl})(1-e_1^2) \simeq 2(1 - b/a_{pl})$ (where q = pericentre distance $= a(1-e)$)

$\therefore V_{r1} \simeq \pm V_{pl}\sqrt{2b/a_{pl}}$ and $V_1'^2 \simeq V_{pl}^2(3 - 2\sqrt{2})$

Putting this into $\boxed{4.36}$ with further definition $b = k r_H$ where $r_H = a_{pl}(M_{pl}/3M_*)^{1/3}$

$$\Delta x = \pm\left[\frac{2^{5/2}3^{1/6}}{3-2\sqrt{2}}\right]\left(\frac{M_{pl}}{M_*}\right)^{5/6}\left(\frac{1}{a_{pl}}\right)k^{-1/2} \qquad \boxed{4.37}$$

Cometary diffusion

- Since Tisserand parameter $\boxed{4.30}$ is conserved, diffusion in q and I are less important than that in x as $\Delta q/q \simeq (\Delta x/x)(a_{pl}/a)$ see $\boxed{\text{EG 2.5a}}$
- Since kicks in x can be +ve or −ve, comet keeps q and I constant, but performs random walk in x defined by diffusion coefficient $D_x = \langle \Delta x^2 \rangle^{1/2}$ where simulations show $\simeq (10/a_{pl})(M_{pl}/M_*)$ $\boxed{4.38}$
- Characteristic diffusion time is $t_{diff} = t_{per} N_x$ where N_x = # of passages required to change x by $O(x) \simeq (x/D_x)^2$

$$\therefore t_{diff} = 10^{-2} t_{per,pl}(a_{pl}/a)^{1/2}(M_{pl}/M_\odot)^{-2}(M_*/M_\odot)^2 \qquad \boxed{4.39}$$

- The diffusion problem can be written (Yabushita 1980) for $n(x,\tau)dx$ = # of comets in $x \to x + dx$

$$dn/d\tau = \tfrac{1}{2}x_0^{1/2}\,\partial^2(nx^{3/2})/\partial x^2 \qquad \text{where } \tau = t/t_{diff}(x_0)$$

And solved for $n(x,0) = \delta(x - x_0)$ to find

$$n(x,\tau) = \left(\frac{4}{x\tau}\right)e^{-\frac{8}{\tau}(1+\sqrt{x/x_0})}\, I_2\left[\frac{16}{\tau}(x/x_0)^{1/4}\right] \qquad \boxed{4.40}$$

where I_2 is modified Bessel function

Giving the # of comets remaining $N(\tau) = \frac{1}{\tau}e^{-(8/\tau)}\left[\tau e^{8/\tau} - \tau - 8\right]$ and a half-life of $\tau_{1/2} \simeq 4.8$

Galactic tides

A comet on $e \simeq 1$ orbit being scattered by a planet has specific ang. mom. $h \simeq \sqrt{2\mu q}$ $\boxed{4.40}$

Nearby stars exert a torque (Heisler & Tremaine 1986): $dh/dt = 5\pi G\rho_0 a^2 e^2 \sin^2 I \sin(2\omega)$

Differentiating $\boxed{4.40}$ gives $\dot{h} = \sqrt{\frac{\mu}{2}}\, q^{-1/2}\dot{q}$

As the time to change q by Δq is $t_{\Delta q} = \Delta q/\dot{q}$, the time for tides to change q by $O(q)$ is $t_{tide} = \sqrt{\frac{\mu}{2}}\, q^{1/2}/\dot{h}$

Plugging in $e \simeq 1$, $I = 60.2°$, $\langle \sin 2\omega \rangle = 1/\sqrt{2}$ and normalising to local stellar density $\rho_0 \simeq 0.15 M_\odot/pc^3$, setting $q = a_{pl}$

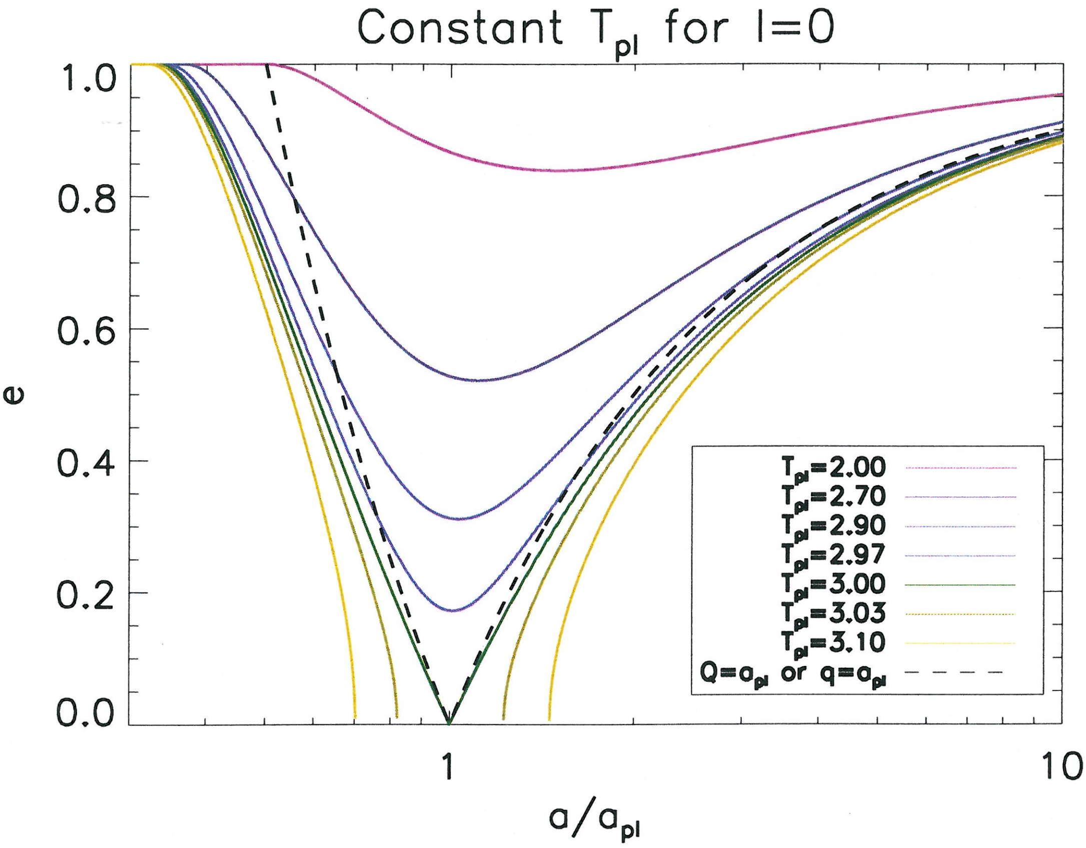
$$t_{tide} = 10^{15} t_{per,pl}(a/a_{pl})a^{-3}(M_*/M_\odot)(\rho_0/\rho_\odot)^{-1} \qquad \boxed{4.41} \qquad \text{where } a \text{ is in AU.}$$

Thus tides freeze a comet's random walk in energy at a_{tide} (where $t_{tide} = t_{diff}$) then changing q, and I to isotropic distn:

$$a_{tide} = 10^4 a_{pl}^{-1}(M_{pl}/M_\odot)^{4/3}(M_*/M_\odot)^{-2/3}(\rho_0/\rho_\odot)^{-2/3} \qquad \boxed{4.42}$$

which can be compared with the semimajor axis beyond which a single encounter removes comets ie $D_x = x$

$$\therefore a_{ej} = 0.1\, a_{pl}(M_{pl}/M_*)^{-1} \qquad \boxed{4.43}$$ showing that some planets eject comets, others put them into Oort Cloud.


Constant T_{pl} for I=0
1.0
0.8
0.6
0.4
0.2
0.0
e
1
10
a/a_{pl}
T_{pl}=2.00
T_{pl}=2.70
T_{pl}=2.90
T_{pl}=2.97
T_{pl}=3.00
T_{pl}=3.03
T_{pl}=3.10
$Q=a_{pl}$ or $q=a_{pl}$