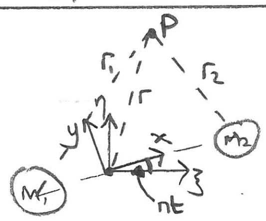
3 Three body problem

Circular → m_1 and m_2 orbit c.o.m. on circular orbit, semi major axis a

Restricted → third object of low enough mass that it does not affect orbit of m_1 and m_2 (test particle)

Circular Restricted Three Body Problem (CRTBP) [NB Also an elliptical RTBP]

(EXAM 2011 Q2)



Equation of motion for P:

As acceler^n due to ith planet is $-\mu_i \underline{r}_i / |r_i|^3$ where $\mu_i = Gm_i$

$$\ddot{\xi} = \mu_1(\xi_1 - \xi)/r_1^3 + \mu_2(\xi_2 - \xi)/r_2^3$$
$$\ddot{\eta} = \mu_1(\eta_1 - \eta)/r_1^3 + \mu_2(\eta_2 - \eta)/r_2^3$$
$$\ddot{\zeta} = \mu_1(\zeta_1 - \zeta)/r_1^3 + \mu_2(\zeta_2 - \zeta)/r_2^3$$

Let $H = \xi + i\eta$ (ie. capital η)

$$\therefore \ddot{H} = \left(\frac{\mu_1}{r_1^3}\right)(H_1 - H) + \left(\frac{\mu_2}{r_2^3}\right)(H_2 - H)$$

Consider in rotating frame x, y, z

- Let $Y = x + iy$

$$\therefore H = e^{int} Y$$
$$\therefore \dot{H} = e^{int}\dot{Y} + ine^{int} Y$$
$$\therefore \ddot{H} = e^{int}\ddot{Y} + ine^{int}\dot{Y} + ine^{int}\dot{Y} - n^2 e^{int} Y = e^{int}[\ddot{Y} + 2in\dot{Y} - n^2 Y]$$

So
$$\ddot{Y} + 2in\dot{Y} = n^2 Y + e^{-int}\ddot{H}$$
$$= n^2 Y + (\mu_1/r_1^3)(Y_1 - Y) + (\mu_2/r_2^3)(Y_2 - Y)$$

- Simplify by choosing units of mass s.t. $G(m_1 + m_2) = 1$ and so $\mu_1 + \mu_2 = 1$
  distance s.t. $a = 1$

  This means that $n = 1$ and units of time are $t_{per}/2\pi$

  and that $Y_1 = -\frac{am_2}{(m_1+m_2)} = -\mu_2$ and $Y_2 = \mu_1$

- Real: $\underline{\ddot{x} - 2\dot{y} = x - (\mu_1/r_1^3)(x + \mu_2) - (\mu_2/r_2^3)(x - \mu_1)}$

  Imaginary: $\underline{\ddot{y} + 2\dot{x} = y - (\mu_1/r_1^3)y - (\mu_2/r_2^3)y}$

- As $z = \zeta$ and $\zeta_1 = \zeta_2 = 0$ then $\underline{\ddot{z} = -(\mu_1/r_1^3)z - (\mu_2/r_2^3)z}$ $\boxed{3.1}$

- Rewrite as

$$\begin{pmatrix} \ddot{x} \overbrace{- 2\dot{y}}^{\text{Coriolis acceleration}} \\ \ddot{y} + 2\dot{x} \\ \ddot{z} \end{pmatrix} = \begin{pmatrix} \partial U/\partial x \\ \partial U/\partial y \\ \partial U/\partial z \end{pmatrix} \qquad \boxed{3.2}$$

where
$$U = \underbrace{\tfrac{1}{2}(x^2 + y^2)}_{\text{centrifugal}} + \underbrace{\mu_1/r_1 + \mu_2/r_2}_{\text{gravitational}} = \text{pseudo-potential} \qquad \boxed{3.3}$$

and
$$\left.\begin{aligned} r_1^2 &= (x + \mu_2)^2 + y^2 + z^2 \\ r_2^2 &= (x - \mu_1)^2 + y^2 + z^2 \end{aligned}\right\} \qquad \boxed{3.4}$$

Integral of motion

$$\begin{pmatrix}\dot x\\ \dot y\\ \dot z\end{pmatrix}\cdot \boxed{3.2} \rightarrow \dot x\ddot x + \dot y\ddot y + \dot z\ddot z = \dot x\,\partial U/\partial x + \dot y\,\partial U/\partial y + \dot z\,\partial U/\partial z$$

$\therefore \frac{1}{2}dv^2/dt = dU/dt$

where $v = \sqrt{\dot x^2 + \dot y^2 + \dot z^2}$ = velocity in rotating frame

$\therefore \underline{v^2 = 2U - C_J}$ $\boxed{3.5}$

where C_J = Jacobi Integral

Jacobi Integral in inertial coordinates

$$C_J = 2\mu_1/r_1 + 2\mu_2/r_2 + (x^2+y^2) - (\dot x^2 + \dot y^2 + \dot z^2)$$

Now $\dot x^2 + \dot y^2 = \dot Y \dot Y^*$

and $\dot Y = e^{-int}\dot H - inY$

so $\dot x^2 + \dot y^2 = (e^{-int}\dot H - inY)(e^{int}\dot H^* + inY^*)$

$= \dot H \dot H^* + n^2 YY^* - in[e^{int}Y\dot H^* - e^{-int}Y^*\dot H]$

$= \dot\xi^2 + \dot\eta^2 + n^2(x^2+y^2) \underbrace{- in[H\dot H^* - H^*\dot H]}_{+2n(\eta\dot\xi - \xi\dot\eta)}$

$\therefore \underline{C_J = 2\mu_1/r_1 + 2\mu_2/r_2 - (\dot\xi^2 + \dot\eta^2 + \dot\zeta^2) - 2(\eta\dot\xi - \xi\dot\eta)}$ $\boxed{3.6}$

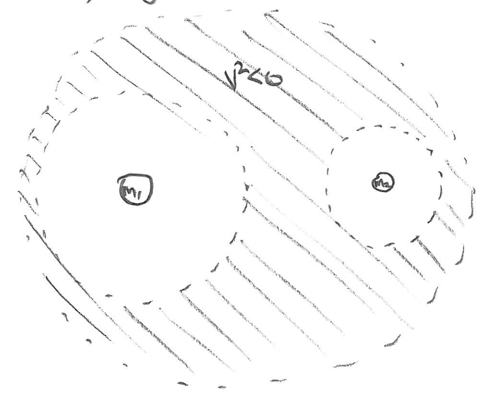
As $\underline{h} = \underline{r}\wedge\underline{\dot r} = \begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix}\wedge\begin{pmatrix}\dot\xi\\ \dot\eta\\ \dot\zeta\end{pmatrix} = \begin{pmatrix}\vdots\\ \xi\dot\eta - \eta\dot\xi\end{pmatrix}$

$C_J = -2\times$ energy/unit mass $+ 2h_z$

Zero velocity surface

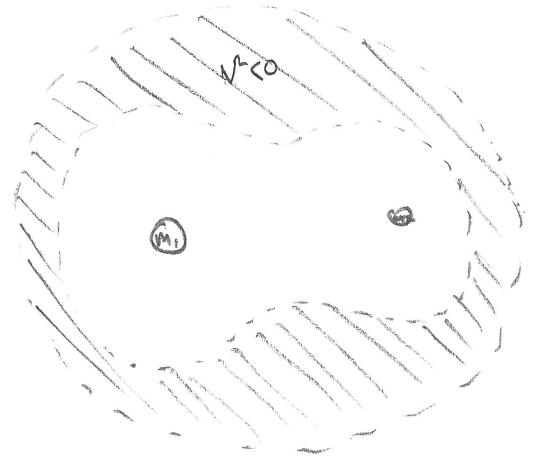
For a given C_J, regions with $v^2<0$, i.e. those with $2U < C_J$ are excluded

eg



Particle orbiting m_1 or m_2 with this C_J can never escape or switch

decreasing C_J



Can switch but never escape

→ Hill's stability

(see also Gladman 1993)

Lagrange Equilibrium Points

For equilibrium $\dot{x}=\dot{y}=\dot{z}=0$, $\ddot{x}=\ddot{y}=\ddot{z}=0$

$\boxed{3.2} \Rightarrow \partial U/\partial x_i = 0$ where $x_i = x, y, z$

$$= (\partial U/\partial r_1)(\partial r_1/\partial x_i) + (\partial U/\partial r_2)(\partial r_2/\partial x_i) + \partial U/\partial x_i$$

From $\boxed{3.4}$: $\mu_1 r_1^2 + \mu_2 r_2^2 = x^2 + y^2 + z^2 + \mu_1\mu_2$

So rewrite $\boxed{3.3}$: $U = \mu_1/r_1 + \mu_2/r_2 + \frac{1}{2}\left[\mu_1 r_1^2 + \mu_2 r_2^2 - \mu_1\mu_2 - z^2\right]$

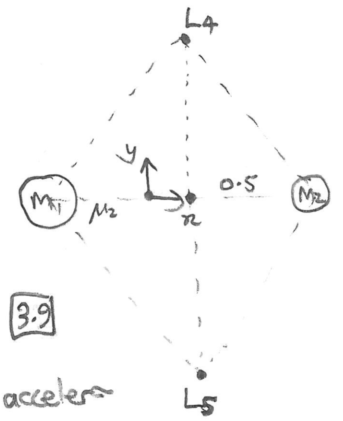
$$= \mu_1\left[\frac{1}{r_1} + \frac{1}{2}r_1^2\right] + \mu_2\left[\frac{1}{r_2} + \frac{1}{2}r_2^2\right] - \frac{1}{2}\mu_1\mu_2 - \frac{1}{2}z^2$$

Differentiating $\boxed{3.4}$: $\partial r_1/\partial x = (x+\mu_2)/r_1$, $\partial r_1/\partial y = y/r_1$, $\partial r_1/\partial z = z/r_1$ $\boxed{3.7}$

$$\left.\begin{aligned}
\therefore\ \partial U/\partial x &= \mu_1\left[-\frac{1}{r_1^2} + r_1\right]\frac{x+\mu_2}{r_1} + \mu_2\left[-\frac{1}{r_2^2} + r_2\right]\frac{x-\mu_1}{r_2} = 0\\
\partial U/\partial y &= \mu_1\left[-\frac{1}{r_1^2} + r_1\right]\frac{y}{r_1} + \mu_2\left[-\frac{1}{r_2^2} + r_2\right]\frac{y}{r_2} = 0\\
\partial U/\partial z &= \mu_1\left[-\frac{1}{r_1^2} + r_1\right]\frac{z}{r_1} + \mu_2\left[-\frac{1}{r_2^2} + r_2\right]\frac{z}{r_2} - z = 0
\end{aligned}\right\}\ \boxed{3.8}$$

There are 5 $=^{m}$ points that are sol^{n} to $\boxed{3.8}$

Triangular equilibrium points L_4, L_5



$r_1 = r_2 = 1$ and $z = 0$

From $\boxed{3.4}$: $(x+\mu_2)^2 + y^2 = 1$

$(x-\mu_1)^2 + y^2 = 1$

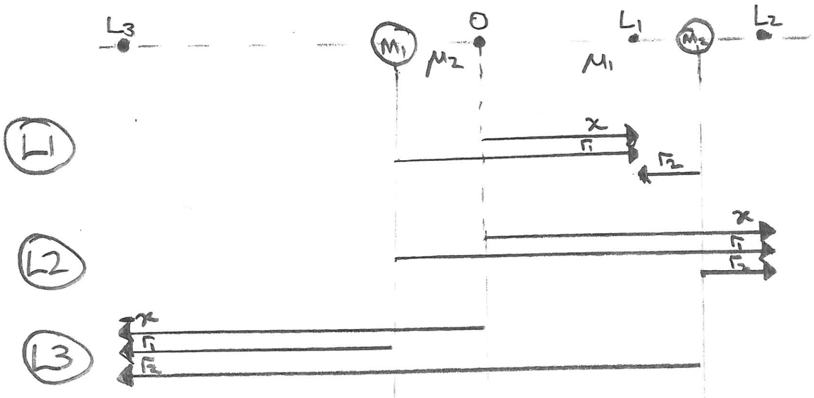
$\therefore\ 2x(\mu_2+\mu_1) + \mu_2^2 - \mu_1^2 = 0$

$$\left.\begin{aligned}
\therefore\ \underline{x} &= \tfrac{1}{2}\left[\mu_1^2 - \mu_2^2\right] = \tfrac{1}{2}\left[\mu_1 - \mu_2\right] = \underline{\tfrac{1}{2} - \mu_2}\\
\underline{y} &= \underline{\pm\sqrt{3}/2}
\end{aligned}\right\}\ \boxed{3.9}$$

These are locations where grav. potential balanced by centrifugal acceler^{n}

Colinear equilibrium points L_1, L_2, L_3

$y = z = 0$: 3 sol^{n}s to $\partial U/\partial x = 0$



$$\left.\begin{aligned}
&\underline{r_1 = 1 - r_2,\ x = \mu_1 - r_2}\\
&\underline{r_1 = 1 + r_2,\ x = \mu_1 + r_2}\\
&\underline{r_1 = r_2 - 1,\ x = \mu_1 - r_2}
\end{aligned}\right\}\ \boxed{3.10}$$

Substitute into $\boxed{3.8a}$ and solve!

L1 Get $\partial u/\partial x$ as a function of r_2 by first substituting for $x = \mu_1 - r_2$

$$\partial u/\partial x = \mu_1(r_1^3-1)r_1^{-2}(1-r_2)/r_1 + \mu_2(r_2^3-1)r_2^{-2}(-r_2/r_2) = 0$$

Then substitute for $r_1 = 1-r_2$

$$\therefore \mu_1[(1-r_2)^3-1](1-r_2)^{-2} - \mu_2(r_2^3-1)r_2^{-2} = 0$$

$$\therefore \mu_2/\mu_1 = (-3r_2+3r_2^2-r_2^3)(1-r_2)^{-2}r_2^2(r_2^3-1)^{-1}$$

$$= 3r_2^3(1-r_2+r_2^2/3)(1-r_2)^{-3}(1+r_2+r_2^2)^{-1}$$

So leading order sol^n is $r_2 = (\mu_2/3\mu_1)^{1/3} \equiv \alpha$

Rearrange and do binomial expansion

$$\alpha = r_2[1+(-r_2+r_2^2/3)]^{1/3}[1-r_2]^{-1}[1+(r_2+r_2^2)]^{-1/3}$$

$$= r_2[1+r_2/3+r_2^2/3+(53/81)r_2^3+O(r_2^4)]$$

Use Lagrange's inversion method: If $y = x + ef(y)$ where $e<1$

then $y = x + \sum_{j=1}^{\infty}\left(\frac{e^j}{j!}\right)\frac{d^{j-1}}{dx^{j-1}}[f(x)]^j$

$$\therefore r_2 = \alpha + (-\tfrac{1}{3})[r_2^2 + r_2^3 + \tfrac{53}{27}r_2^4 + \ldots]$$

$$= \alpha - \tfrac{1}{3}(\alpha^2+\alpha^3+O(\alpha^4)) + \tfrac{1}{18}(4\alpha^3+O(\alpha^4))$$

$$= \alpha - \tfrac{1}{3}\alpha^2 - \tfrac{1}{9}\alpha^3 + O(\alpha^4) \qquad \boxed{3.11}$$

L2 Same method $\rightarrow$ $r_2 = \alpha + \frac{1}{3}\alpha^2 - \frac{1}{9}\alpha^3 + O(\alpha^4)$ $\boxed{3.12}$

(ie. L_1 is slightly closer to μ_2 than L_2)

L3 Same, but substitute for r_2 and let $r_1 = 1+\beta$ and expand assuming $\beta \ll 1$

$$\rightarrow r_1 = 1 - \tfrac{7}{12}(\mu_2/\mu_1) + \tfrac{7}{12}(\mu_2/\mu_1)^2 + O((\mu_2/\mu_1)^3) \qquad \boxed{3.13}$$

Zero velocity surfaces revisited

Handout shows $-2u$ for $\mu_2 = 0.01$ in x,y plane

Remember, zero velocity surfaces for a given C_J are defined by $2u = C_J$

So an orbit to a given C_J can't go "above" the corresponding contour on plot

Although this only defines excluded regions, orbits near $=^m$ points resemble these contours

Putting locations into $C_J = x^2 + y^2 + 2\mu_1/r_1 + 2\mu_2/r_2$

to find the maximum Jacobi const for orbits to approach $=^m$ points:

$$\left.\begin{array}{ll} L_{4,5}: & C_J \doteq (\tfrac{1}{2}-\mu_2)^2 + 3/4 + 2 = 3 - \mu_2 \\ L_1: & \doteq 3 + 3^{4/3}\mu_2^{2/3} - 10\mu_2/3 \\ L_2: & \doteq 3 + 3^{4/3}\mu_2^{2/3} - 14\mu_2/3 \\ L_3 & \doteq 3 + \mu_2 \end{array}\right\} \boxed{3.14}$$

Stability (Eq 2.2)

Remember $\boxed{3.2}$ and do linear stability analysis about eq^m pt x_0, y_0, z_0

Let $x = x_0 + X$, $y = y_0 + Y$, $z = z_0 + Z$

and expand using Taylor series for $\partial U/\partial x_i$; ie., $f(\underline{x}) = f(\underline{x_0}) + \sum_i (\partial f/\partial x_i)\Delta x_i + \frac{1}{2!}\sum_i \sum_j (\partial^2 f/\partial x_i \partial x_j)\Delta x_i \Delta x_j + \ldots$

Here, consider planar motion

$$\left.\begin{aligned} \ddot{X} - 2\dot{Y} &= \cancel{(\partial U/\partial x)_0}^{0} + X(\partial^2 U/\partial x^2)_0 + Y(\partial^2 U/\partial x \partial y)_0 = XU_{xx} + YU_{xy} \\ \ddot{Y} + 2\dot{X} &= \cancel{(\partial U/\partial y)_0}^{0} + Y(\partial^2 U/\partial y^2)_0 + X(\partial^2 U/\partial x \partial y)_0 = YU_{yy} + XU_{xy} \end{aligned}\right\} \quad \boxed{3.15}$$

Rewrite $\boxed{3.15}$ as $\dot{\underline{X}} = A\underline{X}$ $\quad \boxed{3.16}$

where $\underline{X} = \begin{pmatrix} X \\ Y \\ \dot{X} \\ \dot{Y} \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ U_{xx} & U_{xy} & 0 & 2 \\ U_{xy} & U_{yy} & -2 & 0 \end{pmatrix}$

Solution to $\boxed{3.16}$ is an eigenvalue problem.

Eigenvalues

If $A\underline{w_i} = \lambda_i \underline{w_i}$ $\quad \boxed{3.17}$

then $\underline{w_i}$ is an eigenvector of A and λ_i is corresponding eigenvalue

Rewrite $\boxed{3.17}$ using the identity matrix I

$\therefore (A - \lambda_i I)\underline{w_i} = 0$

which requires $\underline{\det[A - \lambda I] = 0}$ $\quad \boxed{3.18}$

This is the characteristic eq^n used to get eigenvalues, which can be put into $\boxed{3.17}$ to get eigenvectors.

Why does this help? Use to create $W = \begin{pmatrix} w_1 & w_2 & \cdots & w_N \\ \vdots & \vdots & & \vdots \end{pmatrix}$

Rewrite $\boxed{3.17}$ as $AW = W\Lambda$ $\quad \boxed{3.19}$

where $\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & & \\ 0 & 0 & \ddots & \\ \vdots & \vdots & & \lambda_N \end{pmatrix}$

Then, let $\underline{Y} = W^{-1}\underline{X}$

$\therefore \dot{\underline{Y}} = W^{-1}\dot{\underline{X}}$

From $\boxed{3.16}$ $\dot{\underline{Y}} = W^{-1}A\underline{X} = W^{-1}AW\underline{Y}$

From $\boxed{3.19}$ $\dot{\underline{Y}} = \Lambda \underline{Y}$ $\quad \boxed{3.20}$

This can be readily solved as $\dot{y}_i = \lambda_i y_i \rightarrow y_i = y_{i0}e^{\lambda_i t}$ where y_{i0} is value of y_i @ $t=0$

$\therefore \underline{X} = W\begin{pmatrix} y_{10}e^{\lambda_1 t} \\ \vdots \end{pmatrix}$

As $\underline{Y_0} = W^{-1}\underline{X_0} \rightarrow \underline{\underline{X} = We^{\Lambda t}W^{-1}\underline{X_0}}$ where $e^{\Lambda t} = \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots \\ 0 & \ddots & \\ \vdots & & e^{\lambda_N t} \end{pmatrix}$ $\quad \boxed{3.21}$

Returning to $\boxed{3.16}$, get characteristic eq^n for A by solving $\boxed{3.18}$

$$\det\begin{bmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ U_{xx} & U_{xy} & -\lambda & 2 \\ U_{xy} & U_{yy} & -2 & -\lambda \end{bmatrix} = 0$$

$\therefore -\lambda[-\lambda(\lambda^2+4) + 1(-2U_{xy} + \lambda U_{yy})] + 1[-(-\lambda)(-\lambda U_{xx} - 2U_{xy}) + 1(U_{xx}U_{yy} - U_{xy}^2)] = 0$

$\therefore \lambda^4 + \lambda^2[4 - U_{xx} - U_{yy}] + \lambda[2U_{xy} - 2U_{xy}] + U_{xx}U_{yy} - U_{xy}^2 = 0$

$\therefore \underline{\lambda^2 = \left[U_{xx} + U_{yy} - 4 \pm \sqrt{(4 - U_{xx} - U_{yy})^2 - 4(U_{xx}U_{yy} - U_{xy}^2)}\right]/2}$ $\quad \boxed{3.22}$

To get U_{xx} etc, start from $\boxed{3.7}$ and $\boxed{3.8}$ (or $\boxed{3.3}$)

$$\therefore\ \partial^2 u/\partial x^2 = [\partial(\partial u/\partial x)/\partial r_1](x+\mu_2)/r_1 + [\partial(\partial u/\partial x)/\partial r_2](x-\mu_1)/r_2 + \partial(\partial u/\partial x)/\partial x$$

$$= \mu_1[3r_1^{-4}(x+\mu_2)^2/r_1 + (1-r_1^{-3})] + \mu_2[3r_2^{-4}(x-\mu_1)^2/r_2 + (1-r_2^{-3})]$$

$$\partial^2 u/\partial y^2 = \mu_1[3r_1^{-5}y^2 + (1-r_1^{-3})] + \mu_2[3r_2^{-5}y^2 + (1-r_2^{-3})] \qquad \boxed{3.23}$$

$$\partial^2 u/\partial x\partial y = \mu_1[3r_1^{-5}(x+\mu_2)y] + \mu_2[3r_2^{-5}(x-\mu_1)y]$$

Then substitute in for $\equiv^m$ points and put into $\boxed{3.22}$

$\underline{L_{4,5}}$ $x_0 = \frac{1}{2}-\mu_2$, $y_0 = \pm\sqrt{3}/2$, $r_{10} = r_{20} = 1$

$$\therefore\ U_{xx} = 3\mu_1(1/2)^2 + 3\mu_2(-1/2)^2 = 3/4$$

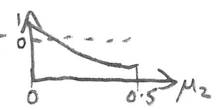
$$U_{yy} = 3\mu_1(\pm\sqrt{3}/2)^2 + 3\mu_2(\pm\sqrt{3}/2)^2 = 9/4$$

$$U_{xy} = 3\mu_1(1/2)(\pm\sqrt{3}/2) + 3\mu_2(-1/2)(\pm\sqrt{3}/2) = \pm(3\sqrt{3}/4)[1-2\mu_2]$$

$$\therefore\ \lambda^2 = \left[-1 \pm \sqrt{1-\tfrac{27}{4}(1-(1-2\mu_2)^2)}\right]/2 = \left[-1 \pm \sqrt{1-27\mu_2+27\mu_2^2}\right]/2 \qquad \boxed{3.24}$$

Linearly stable if all λ_i are imaginary

Noting that μ_2 is in range $0 \to 0.5$ and $1-27\mu_2+27\mu_2^2$



This is the case for $\mu_2 < \mu_c$ where $27\mu_c^2 - 27\mu_c + 1 = 0 \to \mu_c = 0.0385$

$\therefore\ \mu_2 < 0.0385$, λ^2 is Real+ive and eq^m linearly stable

> 0.0385 λ^2 complex " unstable (with a bit more working)

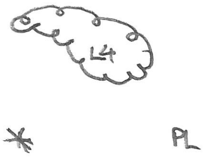
For small μ_2, expect 2 frequencies in motion around L_4 and L_5

Taking terms to μ_2 only: $\lambda_i^2 = [-1 \pm (1-\frac{27}{2}\mu_2)]/2 = -\frac{27}{4}\mu_2$ or $-(1-\frac{27}{4}\mu_2)$

As period $= 2\pi/(\lambda/i) = \dfrac{2\pi}{\sqrt{1-\frac{27}{4}\mu_2}}$ fast epicyclic motion

$\dfrac{2\pi}{\sqrt{27/4\,\mu_2}}$ longer term evolution

<u>Tadpole orbits</u>

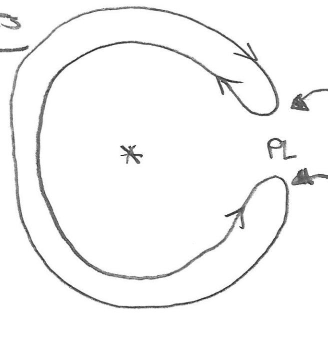


Encompass L_4 (or L_5)

eg. Jupiter's Trojans, Neptune Trojans, Tethys, Telesto+Calypso

Note similarity to zero velocity surface

<u>Horseshoe orbits</u>



Particle has close encounter w/ pl, losing angular momentum to put it onto interior orbit

Another encounter gains angular momentum to put it onto exterior orbit

Encompasses L_3, L_4, L_5

eg Janus & Epimetheus, Earth's Cruithne

$\underline{L_{1,2,3}}$ Similar analysis of characteristic eq^n shows linearly unstable (note also that these are saddle points in the U surface)

Three body problem

Zero velocity curves