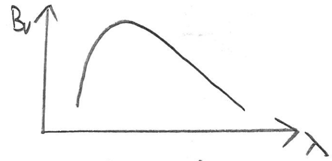
2 Small body dynamics

Stellar radiation

Stars emit approximately like a black body at a temperature T_* (in K) for which the energy emitted / time / surface area / Hz / solid angle at a frequency $\nu = c/\lambda$ Hz is

$$B_\nu = (2h\nu^3/c^2)\left[e^{h\nu/kT_*} - 1\right]^{-1} \quad \text{in Jy/sr} \quad (1\,\text{Jy} = 10^{-26}\,\text{W/m}^2/\text{Hz}) \qquad \boxed{2.1}$$

where $h = 6.626\times10^{-34}$ Js (Planck const), $k = 1.381\times10^{-23}$ J/K (Boltzmann const), $c = 2.998\times10^{8}$ m/s



Peaks at $\lambda_{max} = 5100/T_*$ μm $\boxed{2.2}$

Total energy/time/surface area $= \int B_\nu d\nu \int d\Omega = \pi\int_0^\infty B_\nu d\nu = \sigma T_*^4$ $\boxed{2.3}$

where $\sigma = 2\pi^5 k^4/15c^2h^3 = 5.67\times10^{-8}$ W/m²/K⁴ (Stefan Boltzmann const)

So, for a star of radius R_*:

Luminosity: $L_* = 4\pi R_*^2 \sigma T_*^4$ $\boxed{2.4}$

Flux density received at distance d: $F_{\nu*} = B_\nu \pi R_*^2/d^2 = 1.77(L_*/L_\odot)T_*^{-4}(d/\text{pc})^{-2} B_\nu$ Jy $\boxed{2.5}$

where $L_\odot = 3.826\times10^{26}$ W, 1 pc $= 648000/\pi$ AU (distance at which 1 AU subtends 1″)

Usually care about energy received, but if photons are important, note $E_{phot} = h\nu$

Radiation forces (see Burns, Lamy & Soter 1979, Icarus, 40, 1)

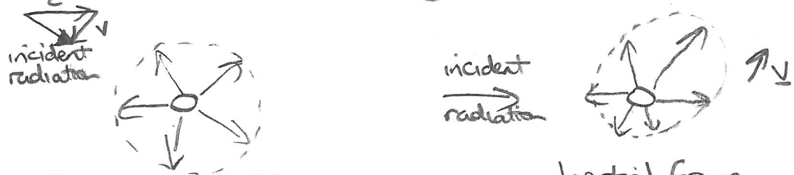
A particle w cross-sectional area $\sigma = \frac{\pi}{4}D^2$ intercepts an amount of energy $\sigma F_*(1-\dot{r}/c)$ per unit time

where $F_* =$ integrated flux density $= \int_0^\infty F_{\nu*}\,d\nu = L_*/4\pi r^2$ $\boxed{2.6}$ (and $\dot{r} = \underline{v}.\hat{\underline{r}}$)

Perfect absorption

Rate of loss of momentum from radiation beam $= \sigma F_*(1-\dot{r}/c)\frac{1}{c}\hat{\underline{r}}$ which is given to the particle

The absorbed energy is continuously reradiated, and for spherical isothermal grains, is isotropic in frame moving w particle



Frame moving w particle

Inertial frame

as all energy absorbed is reemitted in $\Rightarrow$

Thus the radiation is equivalent to a momentum loss rate $-\sigma F_*(1-\dot{r}/c)\frac{1}{c^2}\underline{v}$

Keeping terms $O(v/c)$:

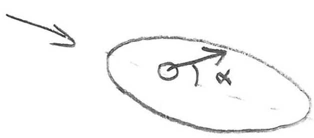
$$\therefore \underline{F_{rad}} \simeq (\sigma F_*/c)\left[(1-\dot{r}/c)\hat{\underline{r}} - \underline{v}/c\right] \qquad \boxed{2.7}$$

Realistic absorption and scattering

Define optical properties of particles that are functions of particle size, composition and wavelength.

Q_{abs} = fraction of energy absorbed (which is also that reemitted)

Q_{sca} = fraction of energy scattered, generally not isotropically, but symmetrical about incident beam.



Phase function $f(\alpha)$, where $f(\alpha)d\Omega$ is fraction of energy scattered into solid angle $d\Omega = 2\pi\sin\alpha\,d\alpha$

Rate of loss of momentum from beam is modified by $Q_{pr} = Q_{abs} + Q_{sca}(1 - \langle\cos\alpha\rangle)$ $\boxed{2.8}$

where $\langle\cos\alpha\rangle = \int f(\alpha)\cos\alpha\,d\Omega$ accounts for anisotropy (ie. some momentum is scattered forward)

Average over wavelengths of incident radiation $\langle Q_{pr}\rangle = \int Q_{pr}F_{\nu*}d\nu \Big/ \int F_{\nu*}d\nu$

eg: Perfect absorber: $\langle Q_{pr}\rangle = 1$

backscatterer: $\langle Q_{pr}\rangle = 2$

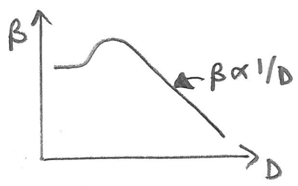
forward scatterer: $\langle Q_{pr}\rangle = 0$

Realistic radiation forces

So $\boxed{2.7}$ becomes $\underline{F}_{rad} = (\sigma F_{*}/c)\langle Q_{pr}\rangle[(1-\dot{r}/c)\hat{\underline{r}} - \underline{v}/c]$

For $\underline{v} = \dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}}$ where coords are centred on star

$$\underline{F}_{rad} = (\sigma F_{*}/c)\langle Q_{pr}\rangle[(1-2\dot{r}/c)\hat{\underline{r}} - (r\dot{\theta}/c)\hat{\underline{\theta}}]$$



Define $\beta = F_{rad}/F_{grav} = \left(\frac{\sigma L_{*}\langle Q_{pr}\rangle}{4\pi r^2 c}\right)\Big/\left(\frac{GM_{*}m}{r^2}\right)$

$$= \left(\frac{\sigma}{m}\right)\left(\frac{L_{*}}{M_{*}}\right)\left(\frac{1}{4\pi cG}\right)\langle Q_{pr}\rangle = 0.43\left(\frac{2700\,kg/m^3}{\rho}\right)\left(\frac{L_{*}}{M_{*}}\right)\langle Q_{pr}\rangle D^{-1} \qquad \boxed{2.9}$$

where L_{*} is in $L_\odot$, M_{*} in $M_\odot$, D in μm

Giving $\underline{F}_{rad} = \beta F_{grav}[\underbrace{(1-2\dot{r}/c)\hat{\underline{r}}}_{\text{radiation pressure}} - \underbrace{(r\dot{\theta}/c)\hat{\underline{\theta}}}_{\text{Poynting-Robertson drag}}] \qquad \boxed{2.10}$

Radiation pressure

New e.o.m. is $\ddot{\underline{r}} = -\mu(1-\beta)\underline{r}/r^3$

∴ equivalent to reducing stellar mass by $(1-\beta)$ in all equations, ie $\mu \to \mu(1-\beta)$ $\boxed{2.11}$

∴ dust orbits are conic sections and for $\beta > 1$ orbits are hyperbolic

Sudden change in β (EG1.5)

Consider dust released (eg in sublimation or collision) from parent planetesimal in orbit $a, e, I, \varpi, \Omega, f, \beta = 0$

New orbit ($a'e'$etc) differs, but $\underline{r}$ and $\underline{v}$ are unchanged, ie. $\underline{r}' = \underline{r}$ and $\underline{v}' = \underline{v}$

Since $\underline{r}'$ and $\underline{v}'$ define orbital plane ($\boxed{1.2}$), this is unchanged and $I' = I$, $\Omega' = \Omega$

As $v'^2 = v^2$, from $\boxed{1.11}$: $\mu(2/r - 1/a) = \mu(1-\beta)(2/r - 1/a')$

$$\therefore a'/a = (1-\beta)/(1-2\beta a/r)$$

As $r'^2\dot{f}' = r^2\dot{f}$, from $\boxed{1.6}$: $\mu a(1-e^2) = \mu(1-\beta)a'(1-e'^2)$

$$\therefore e'^2 = (e^2 + \beta^2 + 2e\beta\cos f)/(1-\beta)^2$$

As $r' = r$, from $\boxed{1.7}$: $a(1-e^2)/(1+e\cos f) = a'(1-e'^2)/(1+e'\cos f')$

$$\therefore \cos f' = (\beta + e\cos f)/\sqrt{e^2+\beta^2+2e\beta\cos f}$$

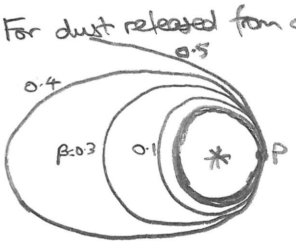
From geometry: $\varpi' + f' = \varpi + f$

$\boxed{2.12}$

As dust is unbound if $e' > 1$, this occurs when $\beta > \frac{1}{2}(1-e^2)/(1+e\cos f)$ $\boxed{2.13}$

which means $\beta > \frac{1}{2}(1-e)$ for dust released at pericentre and $\beta > 0.5$ for dust released from circular orbit

For dust released from circular orbit at point P:



For hyperbolic orbits, energy integral $\boxed{1.9} \Rightarrow \frac{1}{2}V_\infty^2 = \frac{1}{2}V_0^2 - \mu(1-\beta)/r_0$

$$\therefore V_\infty^2 = (\mu/a)[2\beta a/r_0 - 1] \qquad \boxed{2.14}$$

Continuous change in β (eg sublimation, sputtering, evolution of L_{*})

Rewriting $\boxed{1.11} \to a^{-1} = 2/r - v^2/\mu(1-\beta)$

As v is const: $\dot{a}/a = (2a/r - 1)(1-\beta)^{-1}\dot{\beta}$ $\boxed{2.15}$

Rewriting $\boxed{1.6} \to e^2 = 1 - h^2/[\mu(1-\beta)a]$

As h is const: $\dot{e} = [(a/r - 1)(1-e^2)/e(1-\beta)]\dot{\beta}$ $\boxed{2.16}$

So, as β increases: orbit expands (a increases)

eccentricity increases if close to pericentre (decreases at apocentre)

Poynting – Robertson Drag (P-R)

From $\boxed{2.10}$ $\bar{R} = -2(\beta\mu/c)\dot{r}/r^2$

$\bar{T} = -(\beta\mu/c)\, r\dot{\theta}/r^2$

where $\boxed{1.10} \rightarrow \dot{r} = Ae\sin f$, $r\dot{\theta} = A(1+e\cos f)$, $A = \sqrt{\frac{\mu(1-\beta)}{a(1-e^2)}}$

Use Gauss' perturbation equations and average around orbit.

Remember $\langle \dot{x} \rangle = \frac{1}{t_{per}} \int_0^{t_{per}} \dot{x}\, dt$

but $r^2\, df/dt = \sqrt{\mu(1-\beta)a(1-e^2)}$ ($\boxed{1.4, 1.6}$)

and $t_{per} = 2\pi\sqrt{a^3/\mu(1-\beta)}$ ($\boxed{1.9}$)

$\therefore \langle \dot{x} \rangle = \frac{1}{2\pi a^2\sqrt{1-e^2}} \int_0^{2\pi} \dot{x} r^2\, df$ $\boxed{2.17}$

From $\boxed{1.27}$ $\dot{a} = 2\sqrt{\frac{a^3}{\mu(1-\beta)(1-e^2)}}\left[\bar{R}e\sin f + (1+e\cos f)\bar{T}\right]$

$= -2\left(\frac{a}{1-e^2}\right)\left(\frac{\beta\mu}{c}\right)\left(\frac{1}{r^2}\right)\left[2e^2\sin^2 f + (1+e\cos f)^2\right]$

$\therefore \langle \dot{a} \rangle = -2\left(\frac{a}{1-e^2}\right)\left(\frac{\beta\mu}{c}\right)\frac{1}{2\pi a^2\sqrt{1-e^2}} \int_0^{2\pi} 1 + \frac{3}{2}e^2 + \underbrace{\cancel{2e\cos f}}_{0} - \underbrace{\cancel{\tfrac{1}{2}e^2\cos 2f}}_{0}\, df$

$= -\left(\frac{\beta\mu}{c}\right)\frac{1}{a}\frac{2+3e^2}{(1-e^2)^{3/2}}$ $\boxed{2.18}$

where $\beta\mu/c = 6.24\times10^{-4}(M_*/M_\odot)\beta$ AU^2/yr

From $\boxed{1.31}$ $\dot{e} = -\left(\frac{\beta\mu}{c}\right)\left(\frac{1}{r^2}\right)\left[2e\sin^2 f + (1+e\cos f)\left(\cos f + \frac{e+\cos f}{1+e\cos f}\right)\right]$

$\therefore \langle \dot{e} \rangle = -\left(\frac{\beta\mu}{c}\right)\frac{1}{2\pi a^2\sqrt{1-e^2}} \int_0^{2\pi} \frac{5}{2}e - \underbrace{\cancel{\tfrac{1}{2}e\cos 2f}}_{0} + \underbrace{\cancel{2\cos f}}_{0}\, df$

$= -\frac{5}{2}\left(\frac{\beta\mu}{c}\right)\frac{1}{a^2}\frac{e}{\sqrt{1-e^2}}$ $\boxed{2.19}$

From $\boxed{1.33, 1.34}$ as $\bar{N} = 0$, $\dot{I} = \dot{\Omega} = 0$ ie orbital plane is unchanged $\boxed{2.20}$

From $\boxed{1.35}$ $\dot{\varpi} \propto -\bar{R}\cos f + \bar{T}\sin f(2+e\cos f)/(1+e\cos f)$

$\propto \left[2e\sin f\cos f - (2+e\cos f)\sin f\right]/r^2$

$\therefore \langle \dot{\varpi} \rangle = 0$ $\boxed{2.21}$

So, orbit shrinks, and for initial orbit $a_0, e_0 = 0$, at time t : $e = 0$ and

$a^2 = a_0^2 - 2.5\times10^{-3}(M_*/M_\odot)\beta t$ $\boxed{2.22}$

and timescale to get to $a=0$ is (and quicker for initially eccentric orbit)

$t_{pr} = 400\, a_0^2 / \left(\frac{M_*}{M_\odot}\right)\beta$ yr $\boxed{2.23}$

P-R constants

$\boxed{2.18, 2.19} \rightarrow (da/de) = \frac{2}{5}a\frac{2+3e^2}{e(1-e^2)} = \frac{2}{5}a\left[\frac{2}{e} + \frac{5e}{1-e^2}\right]$

$\therefore \int \frac{5}{2}a^{-1}\, da = \int 2/e + 5e/(1-e^2)\, de$

$\therefore a(1-e^2)e^{-4/5} = C_1$ is constant along a trajectory $\boxed{2.24}$

Consider $n(x_i, t, \beta)$ = # of particles β / unit volume of phase space, x_i are coords.

Continuity: $\underbrace{\partial n/\partial t}_{0 \text{ for steady state}} + \underbrace{\partial n\dot{x}_i/\partial x_i}_{\text{slow diffusive processes}} = \underbrace{N^+ - N^-}_{\text{fast processes}}$ $\boxed{2.25}$

Consider steady state distribution of particles far from sources or sinks: $\partial n\dot{x}_i/\partial x_i = 0$

For $n(a,e,\beta)$: $\partial n\dot{a}/\partial a + \partial n\dot{e}/\partial e = n(\partial\dot{a}/\partial a + \partial\dot{e}/\partial e) + \dot{a}\,\partial n/\partial a + \dot{e}\,\partial n/\partial e = 0$

$\therefore \partial n/\partial e + (\dot{a}/\dot{e})\partial n/\partial a = -(n/\dot{e})\left[\partial\dot{a}/\partial a + \partial\dot{e}/\partial e\right]$

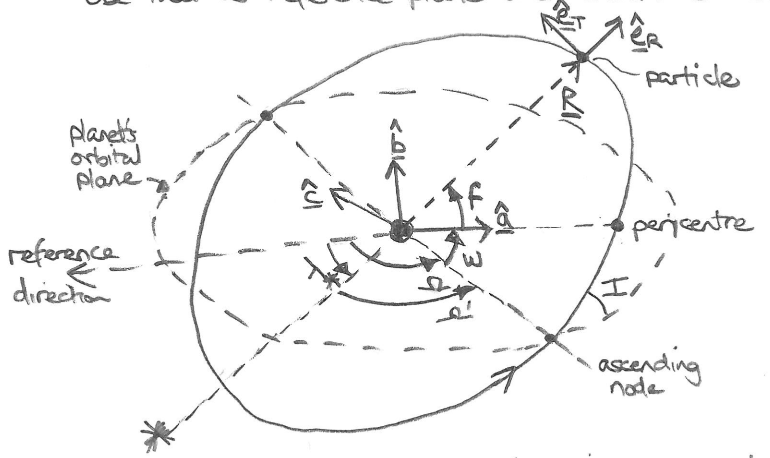
Now $dn = da\,\partial n/\partial a + de\,\partial n/\partial e = de\left[\partial n/\partial e + (\dot{a}/\dot{e})\partial n/\partial a\right]$

$\therefore dn/n = de\left[e/(1-e^2) - 1/5e\right]$

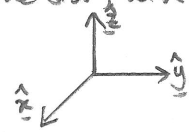
$\therefore n e^{1/5}(1-e^2)^{1/2} = C_2$ $\boxed{2.26}$

Planetocentric orbits: radiation pressure

In planetocentric frame, the star orbits the planet in the planet's orbital plane.
Use that as reference plane and consider circular planet orbit so ✱ remains at distance a_{pl} and $\lambda_* = n_{pl} t$



- Particle's orbital plane defined by normal $\hat{c}$ which is defined by I, Ω w.r.t. ref direction.
- Axis $\hat{a}$ points to particle's pericentre defined by ω
- Define a further coordinate system w $\hat{x}$ pointing to the star, with $\hat{z}$ normal to planet's orbital plane



Define Ω' and use to get transformation matrix s.t. $\underline{F}_{xyz} = T\,\underline{F}_{abc}$ (where T is $\boxed{1.14}$ with $\Omega \to \Omega'$)

Since $\underline{F}_{xyz} = [-(\mu\beta/a_{pl}^2), 0, 0]^T$ (ie, acceleration not force)

$$\underline{F}_{abc} = T^{-1}\underline{F}_{xyz} = -[\mu\beta/a_{pl}^2]\begin{pmatrix} \cos\Omega'\cos\omega - \sin\Omega'\sin\omega\cos I \\ -\cos\Omega'\sin\omega - \sin\Omega'\cos\omega\cos I \\ \sin\Omega'\sin I \end{pmatrix} \qquad \boxed{2.27}$$

Now $\bar{R} = F_a\cos f + F_b\sin f$, $\bar{T} = -F_a\sin f + F_b\cos f$, $\bar{N} = F_c$

Put into Gauss' perturbation equations to find $\boxed{1.27, 1.31, 1.33, 1.35}$ (NB $\mu \to \mu_{pl} = GM_{pl}$)

$$\left.\begin{aligned}
\dot{a} &= 2\sqrt{a^3/\mu_{pl}(1-e^2)}\,[\bar{R}e\sin f + \bar{T}(1+e\cos f)] = 2\sqrt{\frac{a^3}{\mu_{pl}(1-e^2)}}\,[-F_a\sin f + F_b(e+\cos f)] \\
\dot{e} &= \sqrt{a(1-e^2)/\mu_{pl}}\,[F_b + F_b\cos f\cos E - F_a\sin f\cos E] \\
\dot{I} &= [\mu_{pl}a(1-e^2)]^{-1/2}[rF_c\cos(\omega+f)] \\
\dot{\omega} + \dot{\Omega}\cos I &= e^{-1}\sqrt{a(1-e^2)/\mu_{pl}}\,[1+e\cos f]^{-1}[F_b\sin f\cos f - F_a(1+\sin^2 f + e\cos f)]
\end{aligned}\right\} \quad \boxed{2.28}$$

Assume particle orbits planet faster than planet orbits ✱ and average over particle's orbit:

Use $\langle\dot{x}\rangle = \frac{1}{2\pi}\int_0^{2\pi}\dot{x}\,dM$

and relations $dM = (1-e\cos E)dE$, $\sin f\,dM = (1-e^2)^{1/2}\sin E\,dE$, $\cos f\,dM = (\cos E - e)dE$ $\Big\}$ $\boxed{2.29}$

that are readily derived from $\boxed{1.22, 1.19, 1.20}$

$$\left.\begin{aligned}
\therefore \langle\dot{a}\rangle &= \frac{1}{\pi}\sqrt{\frac{a^3}{\mu_{pl}(1-e^2)}}\left[-F_a(1-e^2)^{1/2}\int\sin E\,dE + F_b\int e\,dM + F_b\int \cos E - e\,dE\right] = 0 \\
\langle\dot{e}\rangle &= \frac{3}{2}\sqrt{\frac{a(1-e^2)}{\mu_{pl}}}\,F_b \\
\langle\dot{I}\rangle &= -\frac{3}{2}\sqrt{\frac{a}{\mu_{pl}(1-e^2)}}\,e\cos\omega F_c \\
\langle\dot{\omega} + \dot{\Omega}\cos I\rangle &= -\frac{3}{2}\sqrt{\frac{a(1-e^2)}{\mu_{pl}}}\,\frac{1}{e}F_a
\end{aligned}\right\} \quad \boxed{2.30}$$

ie, radiation pressure does not change a, but e oscillates

$\boxed{\text{eg}}$ Solve for $I = 0$

Noting that $\varpi = n_{pl}t + \Omega' + \omega$, $\boxed{2.27}$ gives $\underline{F}_{abc} = -(\mu\beta/a_{pl}^2)[\cos(\overbrace{\varpi - n_{pl}t}^{\Omega'+\omega}), -\sin(\overbrace{\varpi - n_{pl}t}^{\Omega'+\omega}), 0]^T$

So, letting $k = (\mu\beta/a_{pl}^2)\frac{3}{2}\sqrt{a(1-e^2)/\mu_{pl}} \simeq \frac{3}{2}\beta n_{pl}\sqrt{\frac{M_*}{M_{pl}}}\sqrt{\frac{a}{a_{pl}}}$ $\boxed{2.31}$

$\boxed{2.30} \Rightarrow \langle\dot{e}\rangle = k\sin(\varpi - n_{pl}t)$ and $\langle\dot{\varpi}\rangle = (k/e)\cos(\varpi - n_{pl}t)$

Let $z = e\,e^{i\varpi}$ s.t. $\langle\dot{z}\rangle = (\partial z/\partial e)\dot{e} + (\partial z/\partial\varpi)\dot{\varpi}$

$\therefore \langle\dot{z}\rangle = e^{i\varpi}k\sin(\varpi - n_{pl}t) + ie\,e^{i\varpi}\frac{k}{e}\cos(\varpi - n_{pl}t) = ki\,e^{in_{pl}t}$ $\boxed{2.32}$

$\langle z\rangle = \langle z_0\rangle + (k/n_{pl})[e^{in_{pl}t} - 1]$ $\boxed{2.33}$

Thus for $e \ll 1$ at all times

$2k/n_{pl} \ll 1$

$\therefore \beta \ll \frac{1}{3}\sqrt{M_{pl}/M_*}\sqrt{a_{pl}/a}$ $\boxed{2.34}$

NB as $\langle\dot{a}\rangle = 0$, maximum distance from planet is $2a$ and particles are lost by pericentre decreasing s.t. they collide with the planet.

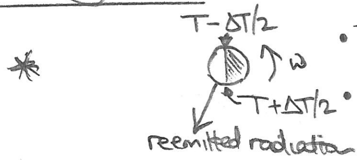
Stellar wind forces

Stream of charged particles (e^- and p^+) from upper atmosphere of $*$
Exerts a force directly analogous to radiation forces, but $c \rightarrow V_{sw}$ = velocity of wind and $\langle Q_{pr} \rangle \rightarrow \langle Q_{sw} \rangle$ = efficiency of momentum transfer.

$$\left.\begin{array}{ll} \underline{\text{Pressure force}}: & F_{sw}/F_{rad} \simeq (\dot{M}_*/L_*)(\langle Q_{sw}\rangle/\langle Q_{pr}\rangle)\, c V_{sw} \\ \underline{\text{Drag force}}: & F_{sw}/F_{rad} \simeq (\dot{M}_*/L_*)(\langle Q_{sw}\rangle/\langle Q_{pr}\rangle)\, c^2 \end{array}\right\} \quad \boxed{2.35}$$

ie. effect of drag is much stronger (compared with radiation forces) than pressure as wind is slower than light.
Solar wind drag is ~ 0.3x P-R drag at present, but would have been stronger at young ages and may dominate for low luminosity M stars.

Yarkovsky force



- The morning hemisphere is cooler than evening hemisphere and so emits less energy and momentum
- Direction of force depends on spin direction → random walk
- Can be important for ~ km-sized planetesimals and is source of Near Earth Asteroids

Gas drag – see Armitage's "Astrophysics of planet formation" book (chapters 2, 4)

A particle of diameter D, density ρ_s, moving through gas of density ρ_g with relative velocity $\underline{\Delta V}$ experiences an aerodynamic drag force

$$\underline{F_{gas} = -\tfrac{1}{8} C_D \pi D^2 \rho_g |\Delta V| \underline{\Delta V}} \qquad \boxed{2.36}$$

where C_D = drag coefficient that depends on
λ = mean free path of molecules $= m_g/\rho_g \sigma_{col}$
in which m_g = mean molecular mass $\simeq 2.34 m_H$, m_H = mass of Hydrogen atom
σ_{col} = cross-section for molecular collisions $\simeq 2\times10^{-19}\ m^2$ for molecular hydrogen

For $\underline{D < \tfrac{9}{2}\lambda}$ (ie. small particles or low densities)

$\underline{\text{Epstein drag}}$ $\quad C_D = \tfrac{8}{3} V_T/|\Delta V|$ or 2 if supersonic $|\Delta V| >> V_T$ $\quad \boxed{2.37}$

where V_T = mean thermal velocity of gas $= \sqrt{\tfrac{8}{\pi}}\, c_s$
c_s = sound speed $= \sqrt{kT/m_g}$ and k = Boltzmann const
ie. difference in momentum flux from windward and leeward faces of $\sigma \rho_g [(V_T+\Delta V)^2 - (V_T - \Delta V)^2]$

For $\underline{D > \tfrac{9}{2}\lambda}$ (ie. large particles or high densities)

$\underline{\text{Stokes drag}}$ $\quad C_D$ depends on Reynolds number, the ratio of inertial to viscous forces
$Re = D|\Delta V|/\nu$
ν = molecular viscosity $= \lambda V_T/2$
$C_D = 24/Re \ (Re < 1)$, $24/Re^{0.6} \ (1 < Re < 800)$, $0.44 \ (Re > 800)$ $\quad \boxed{2.38}$

$\underline{\text{Stopping time}}$, t_s is time for gas drag to reduce ΔV to zero

$$\therefore \underline{t_s} = m|\Delta V|/|F_{gas}| = \left(\frac{4}{3C_D}\right)\left(\frac{\rho_s D}{\rho_g|\Delta V|}\right) \qquad \boxed{2.39}$$

Also quoted as dimensionless stopping time $T_s = \Omega_K t_s$ $\quad \boxed{2.40}$
where $\Omega_K = n = 2\pi/t_{per} = \sqrt{GM_*/r^3} = V_K/r$ $\quad \boxed{2.41}$

If $T_s << 1$ dust is strongly coupled to gas (small grains at large distance)
$T_s >> 1$ dust is decoupled from gas (large grains at small distance)

Acceleration due to gas drag: $\underline{F_{gas}/m = -\underline{\Delta V}/t_s}$ $\quad \boxed{2.42}$

Motion of particle through protoplanetary disk

To get a fiducial gas disk model, consider an axisymmetric disk in cylindrical polar coords.

Conservation of momentum:

Vertical $\quad \underbrace{-(GM_*/r^3)z}_{\text{gravity}} - \underbrace{\frac{1}{\rho_g}\partial P/\partial z}_{\text{pressure}} = 0$

If isothermal in vertical direction, then as $P = \rho_g kT/m_g$ $\quad \boxed{2.43}$

$-(GM_*/r^3)z - \frac{1}{\rho_g}(\partial\rho_g/\partial z)kT/m_g = 0$

$\therefore \rho_g = \rho_{gmid}\, e^{-z^2/2h_g^2}$ $\quad \boxed{2.44}$

where $h_g = \sqrt{r^3 kT/GM_* m_g} = c_s/\Omega_K$ $\quad \boxed{2.45}$

Mass surface density $\Sigma_g = \int_{-\infty}^{\infty} \rho_g\, dz = \sqrt{2\pi}\, h_g \rho_{gmid}$ $\quad \boxed{2.46}$

Usually assume $\Sigma_g = \Sigma_{g0}(r/r_0)^{-\beta}$ $\quad \boxed{2.47}$

where $r_0 = 1\,AU$, $\beta = 1.5$, $\Sigma_{g0} = 1700\,g/cm^2$ for a "minimum mass solar nebula"

Radial $\quad \underbrace{-(GM_*/r^2)}_{\text{gravity}} - \underbrace{\frac{1}{\rho_g}\partial P/\partial r}_{\text{pressure}} + \underbrace{r\Omega_g^2}_{\text{centrifugal}} = 0$

$\therefore \Omega_g^2 = \frac{GM_*}{r^3}\left[1 + \left(\frac{r^2}{\rho_g GM_*}\right)\partial P/\partial r\right] = \Omega_K^2[1-\eta]$ $\quad \boxed{2.48}$

where $\eta = \frac{-1}{\rho_g \Omega_K^2 r}\partial P/\partial r$

and order of magnitude $\partial P/\partial r \simeq P/r$ $\quad \therefore \eta = \frac{1}{v_K^2}kT/m_g = c_s^2/v_K^2$ $\quad \boxed{2.49}$

So gas orbits at a height z above the midplane at subKeplerian velocity

(but there may be locations, eg near gaps, where $\partial P/\partial r > 0$ so gas is super Keplerian)

Settling

If $T_s \ll 1$: Particle orbits with gas at z, but vertical component of gravity accelerates it to midplane

Gas drag → reaches terminal velocity

$\therefore -GM_*z/r^3 = v_z/t_s$

Settling timescale $t_{settle} = z/v_z = (r^3/GM_*)/t_s = (\Omega_K T_s)^{-1}$ $\quad \boxed{2.50}$

$T_s \gg 1$: Particle on orbit inclined by I to midplane and I is damped:

$\boxed{1.33} \rightarrow \dot{I} = \frac{r}{h}\cos(\omega+f)\bar{N}$ $\quad$ where $\boxed{2.42} \rightarrow \bar{N} = -v_z/t_s$

For $e \simeq 0$, $I \ll 1$, $\boxed{1.13, 1.14} \rightarrow v_z \simeq \cos(\omega+f) I v_K$ and $\boxed{1.6} \rightarrow h \simeq v_K r$

$\therefore \dot{I} = -\cos^2(\omega+f) I/t_s$

$\therefore t_{settle} = I/\langle \dot{I} \rangle = 2t_s = 2T_s/\Omega_K$ $\quad \boxed{2.51}$

So, grains that settle fastest are those with $T_s \simeq 1$ that settle on orbital timescales

Radial drift

New e.o.m. $\boxed{1.1P}$ with $\boxed{1.3}$ and $\boxed{2.42}$: $(\ddot{r} - r\dot{\theta}^2 + \mu/r^2)\hat{\underline{r}} + [r^{-1}d(r^2\dot{\theta})/dt]\hat{\underline{\theta}} = -\Delta\underline{v}/t_s$ $\quad \boxed{2.52}$

$\hat{\underline{\theta}}$ component with $v_{\theta d} = r\dot{\theta}$ is

$\frac{1}{r}\frac{d}{dt}(r v_{\theta d}) = -(v_{\theta d} - v_{\theta g})/t_s$

But $d(r v_{\theta d})/dt \simeq d(r v_K)/dt = \frac{1}{2} v_{rd} v_K$ $\quad$ where $v_{rd} = \dot{r}$

As $v_{\theta g} = \Omega_g r = v_K[1-\eta]^{0.5}$ $\quad$ (from $\boxed{2.48}$)

$\therefore v_{\theta d} \simeq v_K[1 - \eta/2 - v_{rd}t_s/2r]$ $\quad \boxed{2.53}$

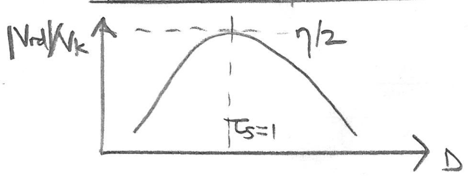
$\hat{\underline{r}}$ component

$dv_{rd}/dt = v_{\theta d}^2/r - \mu/r^2 - (v_{rd} - v_{rg})/t_s$

If $dv_{rd}/dt \simeq 0$ and $v_{rg} \simeq 0$ then using $\boxed{2.53}$

$(v_K^2/r)[1 - \eta - v_{rd}t_s/r] - v_K^2/r - v_{rd}/t_s = 0$

$\therefore v_{rd}/v_K = -\eta\,[T_s + 1/T_s]^{-1}$ $\quad \boxed{2.54}$



→ Big problem, as cm-sized grains at 1 AU accrete onto $*$ on timescales of 100 yr