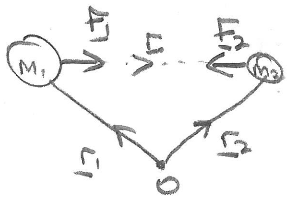
2 Body Problem



- Consider two massive objects offset from origin O in inertial space by $\underline{r}_1$ and $\underline{r}_2$

Mutual grav. attraction results in forces:

$$\underline{F}_1 = G m_1 m_2 \underline{r} / r^3 = m_1 \ddot{\underline{r}}_1$$
$$\underline{F}_2 = - G m_1 m_2 \underline{r} / r^3 = m_2 \ddot{\underline{r}}_2$$

where $G = 6.672 \times 10^{-11}\ \mathrm{N m^2 kg^{-2}}$ $(\mathrm{m^3 kg^{-1} s^{-2}})$

- Writing $\underline{r} = \underline{r}_2 - \underline{r}_1$ gives eqn of rel. motion

$$\ddot{\underline{r}} = \ddot{\underline{r}}_2 - \ddot{\underline{r}}_1 = -G(m_1 + m_2)\underline{r}/r^3$$
$$\underline{\ddot{\underline{r}} + \mu \underline{r}/r^3 = 0} \qquad \boxed{1.1}$$

where $\mu = G(m_1 + m_2)$ NOTE: take care $\bar{w}$ μ!

- To solve, take $\underline{r} \wedge \boxed{1.1}$ to get

$$\underline{r} \wedge \ddot{\underline{r}} = 0 \qquad \text{AS } \underline{r} \wedge \underline{r} = 0$$

Integrate to get

$$\underline{\underline{r} \wedge \dot{\underline{r}} = \underline{h}} \qquad \boxed{1.2}$$

where $\underline{h}$ = const vector $\perp$ to $\underline{r}$ and $\dot{\underline{r}}$

i.e. motion of m_2 rel. to m_1 is in the plane $\perp$ to $\underline{h}$

- Define polar coord. system r, θ centred on m_1 s.t. $\hat{\underline{r}}$ and $\hat{\underline{\theta}}$ are unit vectors along and $\perp$ to radius vector

Vector calculus gives

$$\left.\begin{aligned} \underline{r} &= r\hat{\underline{r}} \\ \dot{\underline{r}} &= \dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}} \\ \ddot{\underline{r}} &= (\ddot{r} - r\dot{\theta}^2)\hat{\underline{r}} + (r^{-1} d[r^2\dot{\theta}]/dt)\hat{\underline{\theta}} \end{aligned}\right\} \boxed{1.3}$$

- Substitute into $\boxed{1.2}$ to get

$$\underline{h} = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} \dot{r} \\ r\dot{\theta} \\ 0 \end{pmatrix} = r^2 \dot{\theta} \hat{\underline{z}}$$

where $\hat{\underline{z}}$ is $\perp$ to orbital plane

$$\underline{h = |\underline{h}| = r^2\dot{\theta}} \qquad \boxed{1.4}$$ NOTE: also get $r^2\dot{\theta}$ = const from $\hat{\underline{\theta}}$ component of $\boxed{1.1}$

- The $\hat{\underline{r}}$ component of $\boxed{1.1}$ becomes

$$\ddot{r} - r\dot{\theta}^2 = -\mu/r^2$$

Let $r = u^{-1}$ and use $\boxed{1.4}$ to get

$$\therefore \dot{r} = (\partial r/\partial u)(\partial u/\partial t) = -u^{-2}(du/d\theta)\dot{\theta} = -h\, du/d\theta$$

$$\therefore \ddot{r} = -h(d^2u/d\theta^2)\dot{\theta} = -h^2u^2 d^2u/d\theta^2$$

And so

$$-h^2u^2 d^2u/d\theta^2 - h^2u^3 = -\mu u^2$$

$$\therefore \underline{d^2u/d\theta^2 + u = \mu/h^2}$$

- This second order linear differential eqn can be solved

$$u = (\mu/h^2)[1 + e\cos(\theta - \varpi)]$$

where e = eccentricity, ϖ = longitude of pericentre are two constants of integration

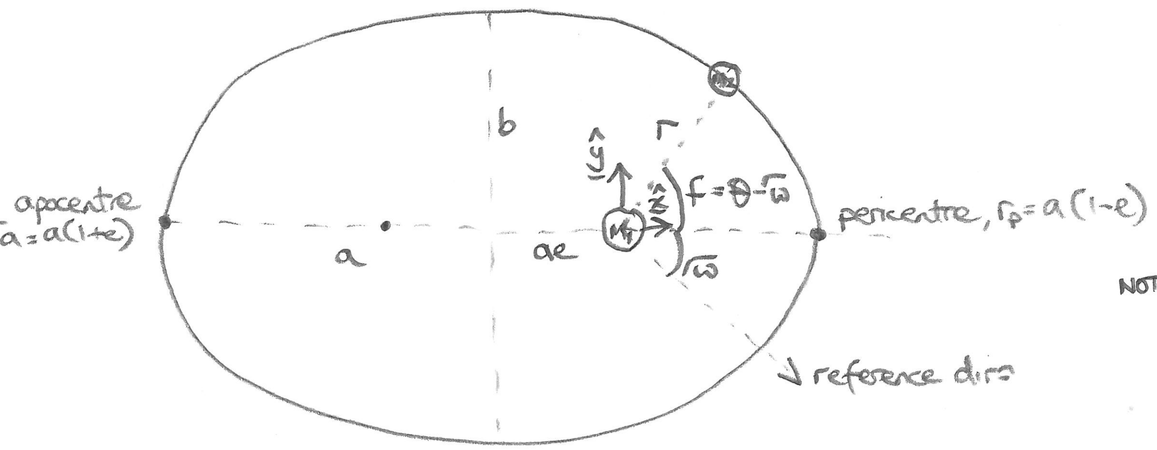
Or $\underline{r = (h^2/\mu)/[1 + e\cos(\theta - \varpi)]}$ $\boxed{1.5}$

which is the general eqn for a conic

Ellipse: $e<1$ (circle, $e=0$)

Hyperbola: $e>1$ (parabola, $e=1$)

- We'll return to hyperbolae in topic 4 and treat ellipses, the geometry of which is well defined:



NOTE: difference betw. perihelion, perigee, periastron

- m_2 follows elliptical orbit $\bar{c}$ m_1 @ focus
- a = semimajor axis

$$r_p = a(1-e) = (h^2/\mu)/(1+e)$$

$$\therefore h = \sqrt{\mu a(1-e^2)} \quad \boxed{1.6}$$

$$\therefore \underline{r = a(1-e^2)/(1+e\cos f)} \quad \boxed{1.7}$$

where f = true anomaly $= \theta - \varpi$

- b = semi minor axis

As $r_b = a(1-e^2)/(1+e(\frac{-ae}{r_b}))$

$r_b = a$

$b = a\sqrt{1-e^2}$

- Area of ellipse $= \pi ab = \pi a^2\sqrt{1-e^2}$

• <u>Orbital period</u>

$$\therefore\ dA/dt = \tfrac{1}{2} r^2 \dot{\theta}$$
$$= \tfrac{1}{2} h = \text{const (Kepler's law)} \quad (\text{from } \boxed{1.4})$$
$$= \tfrac{1}{2}\sqrt{\mu a(1-e^2)} \quad (\text{from } \boxed{1.6})$$
$$t_{per} = \pi a^2 \sqrt{1-e^2} \Big/ \tfrac{1}{2}\sqrt{\mu a(1-e^2)}$$
$$= 2\pi\sqrt{a^3/\mu} \qquad \boxed{1.8}$$

Often used: $n = 2\pi / t_{per}$ = mean motion

whence $\mu = n^2 a^3$

NOTE: This scaling also from dimensional analysis as $G = L^3 M^{-1} T^{-2}$, $\mu = L^3 T^{-2}$, $a = L$ $\therefore$ time scales $\propto \sqrt{a^3/\mu}$

• Another <u>const of motion</u>, that gives info about velocities: $\dot{\underline{r}}\ \cdot\ \boxed{1.1}$

$$\dot{\underline{r}} \cdot \ddot{\underline{r}} + \mu\, \underline{r}\cdot\dot{\underline{r}}/r^3 = 0$$

But $\underline{r}\cdot\dot{\underline{r}} = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \dot{r} \\ r\dot{\theta} \\ 0 \end{pmatrix} = r\dot{r}$

And $d(-\mu r^{-1})/dt = \mu \dot{r}/r^2$

And $d\, \dot{\underline{r}}\cdot\dot{\underline{r}}/dt = 2\dot{\underline{r}}\cdot\ddot{\underline{r}}$

$$\therefore\ 0.5\, d(\dot{\underline{r}}\cdot\dot{\underline{r}})/dt + d(-\mu r^{-1})/dt = 0$$
$$\therefore\ 0.5 v^2 - \mu/r = C \qquad \boxed{1.9}$$

where $v^2 = \dot{\underline{r}}\cdot\dot{\underline{r}} = \dot{r}^2 + r^2\dot{f}^2$ NB $\dot{f} = \dot{\theta}$

• Differentiating $\boxed{1.7}$ gives

$$\dot{r} = (dr/df)(df/dt)$$
$$= -a(1-e^2)(1+e\cos f)^{-2}(-e\sin f)\dot{f}$$
$$= r^2\dot{f}\, e\sin f / a(1-e^2)$$

But $\boxed{1.4, 1.6} \Rightarrow r^2\dot{f} = \sqrt{\mu a(1-e^2)}$

$$\therefore\ \left.\begin{aligned} \dot{r} &= Ae\sin f \\ r\dot{f} &= A(1+e\cos f) \end{aligned}\right\} \qquad \boxed{1.10}$$

where $A = \sqrt{\mu / a(1-e^2)}$

• So $v^2 = A^2[e^2\sin^2 f + (1+e\cos f)^2]$

$$= A^2[e^2 - 1 + 2(1+e\cos f)]$$
$$= \mu[2/r - 1/a] \qquad \boxed{1.11}$$

NOTE: $V_p = \sqrt{\frac{\mu}{a}}\sqrt{\frac{1+e}{1-e}}$

$V_a = \sqrt{\frac{\mu}{a}}\sqrt{\frac{1-e}{1+e}}$

Putting into $\boxed{1.9} \rightarrow C = -\mu/2a$

• <u>2D posⁿ and vel</u>

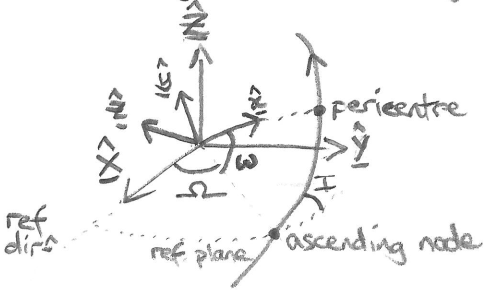
$$\left.\begin{aligned} x &= r\cos f \\ \therefore\ \dot{x} &= \dot{r}\cos f - r\dot{f}\sin f \\ &= A[e\sin f\cos f - \sin f - e\sin f\cos f] \\ &= -A\sin f \\ y &= r\sin f \\ \dot{y} &= A[e\sin^2 f + \cos f + e\cos^2 f] \\ &= A(e+\cos f) \end{aligned}\right\} \qquad \boxed{1.12}$$

- **3D Orbit** (EG1.2)

To define orbital plane wrt a ref frame $\hat{\underline{X}}, \hat{\underline{Y}}, \hat{\underline{Z}}$ req 2 angles:

I = Inclination

Ω = Longitude of Ascending Node



$I < 90°$ → prograde

$ > 90°$ → retrograde

ω = argument of pericentre

$\varpi = \omega + \Omega$ = longitude of pericentre

∴ Orbit completely defined by $a, e, \varpi, I, \Omega, f$

Transform from orbital to ref frame using matrix T defined by 3 rotations:

$$(X, Y, Z)^T = T\,(x, y, 0)^T$$
$$(\dot{X}, \dot{Y}, \dot{Z})^T = T\,(\dot{x}, \dot{y}, 0)^T \qquad \boxed{1.13}$$

where
$$T = \begin{pmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos I & -\sin I \\ 0 & \sin I & \cos I \end{pmatrix}\begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} C\Omega C\omega - S\Omega CIS\omega & -C\Omega S\omega - S\Omega CIC\omega & S\Omega SI \\ S\Omega C\omega + C\Omega CIS\omega & -S\Omega S\omega + C\Omega CIC\omega & -C\Omega SI \\ SIS\omega & SIC\omega & CI \end{pmatrix} \qquad \boxed{1.14}$$

NOTE: Abbreviations for COS and SIN

Know trig IDs: $\cos(\omega+f) = \cos\omega\cos f - \sin\omega\sin f$

$\sin(\omega+f) = \sin\omega\cos f + \cos\omega\sin f$

- Can convert $X, Y, Z, \dot{X}, \dot{Y}, \dot{Z}$ into orbital elements; get in turn:

$\boxed{a}$ New $\boxed{1.9}$ is $0.5v^2 - \mu/r = -\mu/2a$

$$\therefore\ a = (2/r - v^2/\mu)^{-1} \qquad \boxed{1.15}$$
where $r = \sqrt{X^2 + Y^2 + Z^2}$

$v = \sqrt{\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2}$

$\boxed{e}$ Rearrange $\boxed{1.6}$ to get $e = \sqrt{1 - h^2/\mu a}$ $\qquad \boxed{1.16}$

where $\underline{h}_{3D} = \underline{r} \wedge \dot{\underline{r}} = [Y\dot{Z} - Z\dot{Y}, Z\dot{X} - X\dot{Z}, X\dot{Y} - Y\dot{X}]^T$ from $\boxed{1.2}$

$\boxed{I}$ Transform also applies to $\underline{h}$ so:

$$\underline{h}_{3D} = T\begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} = h\,[\sin\Omega\sin I, -\cos\Omega\sin I, \cos I]^T \qquad \boxed{1.17}$$
$$\therefore\ I = \cos^{-1}(h_Z/h)$$

$\boxed{\Omega}$ Likewise $\quad \Omega = \tan^{-1}(-h_X/h_Y)$

$\boxed{f}$ From $\boxed{1.7}$ $\cos f = \frac{1}{e}\left[\frac{a}{r}(1-e^2) - 1\right]$ $\qquad \boxed{1.18}$

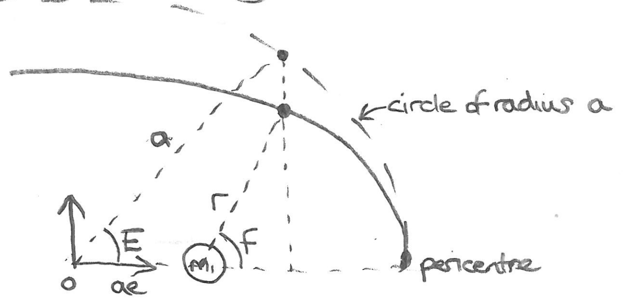
Remove ambiguity of sign noting that $\dot{r} = \dfrac{X\dot{X} + Y\dot{Y} + Z\dot{Z}}{r} \propto \sin f$

$\boxed{\omega}$ From $\boxed{1.13, 1.14}$ $Z/r = \sin I \sin(\omega + f)$

Remove ambiguity of sign c̄ X/r eqⁿ.

Mean and Eccentric Anomalies (EG 1.3)

Define eccentric anomaly, E:



$\therefore \cos E = (ae + r\cos f)/a$ from geometry

$\therefore \cos f = (\cos E - e)a/r$ rearranging

$= (\cos E - e)(1 + e\cos f)/(1-e^2)$ from $\boxed{1.7}$

$= (\cos E - e)/(1 - e\cos E)$ taking $(1-e^2)$ across and rearranging $\boxed{1.19}$

This means that:

$$2\cos^2 f/2 = 1 + \cos f = (1 - e\cos E + \cos E - e)/(1 - e\cos E)$$

$$= \left(\frac{1-e}{1-e\cos E}\right) 2\cos^2 E/2$$

$$2\sin^2 f/2 = 1 - \cos f = (1 - e\cos E - \cos E + e)/(1 - e\cos E)$$

$$= \left(\frac{1+e}{1-e\cos E}\right) 2\sin^2 E/2$$

$$\therefore \underline{\tan f/2 = \sqrt{\frac{1+e}{1-e}}\,\tan E/2} \qquad \boxed{1.20}$$

Substitute $\boxed{1.19}$ into $\boxed{1.7}$

$\therefore r = a(1-e^2)(1-e\cos E)/(1 - e\cos E + e(\cos E - e))$

$\underline{= a(1 - e\cos E)}$ $\boxed{1.21}$

Thus $\dot{r} = ae\sin E\,\dot{E} = \sqrt{\frac{\mu}{a(1-e^2)}}\,e\sin f$ from $\boxed{1.10}$

But from $\boxed{1.19} \rightarrow \sin^2 f = 1 - (\cos E - e)^2/(1 - e\cos E)^2$

$= (1 - e\cos E)^{-2}\left[1 + e^2\cos^2 E - \cancel{2e\cos E} - \cos^2 E + \cancel{2e\cos E} - e^2\right]$

$\therefore \sin f = \sqrt{1-e^2}\,\sin E/(1 - e\cos E)$

$\therefore \dot{E} = \sqrt{\frac{\mu}{a^3}}\,(1 - e\cos E)^{-1}$

$\therefore \int_0^E 1 - e\cos E\,dE = \int_\tau^t n\,dt$ where τ = time of pericentre passage

$$\therefore \underline{E - e\sin E = M = n(t - \tau)} \qquad \boxed{1.22}$$

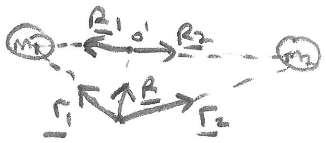
Where M = Mean anomaly

This is Kepler's eqn, which must be solved numerically

Also define $\underline{\lambda = M + \varpi}$ = mean longitude $\boxed{1.23}$

- <u>Barycentric Motion</u> (Eq 1.1)

The frame centred on M_1 is not inertial, but that on barycentre (o') is:



By definition of c.o.m: $m_1 \underline{r}_1 + m_2 \underline{r}_2 = (m_1 + m_2)\underline{R}$

But $\underline{r}_1 = \underline{R} + \underline{R}_1$

and $\underline{r}_2 = \underline{R} + \underline{R}_2$

$\therefore \; m_1 \underline{R}_1 + m_2 \underline{R}_2 = 0$

$\underline{R}_2 = -\frac{m_1}{m_2} \underline{R}_1$

Also $\underline{r} = \underline{R}_2 - \underline{R}_1 = -\underline{R}_1 (1 + m_1/m_2)$

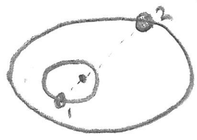
$$\left.\begin{aligned} \therefore\; \underline{R}_1 &= -\left(\frac{m_2}{m_1+m_2}\right)\underline{r} \\ \underline{R}_2 &= \left(\frac{m_1}{m_1+m_2}\right)\underline{r} \end{aligned}\right\} \quad \boxed{1.24}$$

So, both m_1 and m_2 orbit the c.o.m. on same conic as relative motion

ie, with same mean motion n, e, I, Ω

But w reduced semimajor axis and pericentres offset by 180°

eg:



- <u>Angular momentum</u>

$\dot{\theta}$ is unchanged, so the specific ang. mom. of both objects is const:

$$h_1 = R_1^2 \dot{\theta} = \left(\frac{m_2}{m_1+m_2}\right)^2 r^2 \dot{\theta} = \left(\frac{m_2}{m_1+m_2}\right)^2 h$$

$$h_2 = R_2^2 \dot{\theta} = \quad \cdots \quad = \left(\frac{m_1}{m_1+m_2}\right)^2 h$$

And total ang mom. is also const.

$$\underline{L_{tot} = m_1 h_1 + m_2 h_2 = \left(\frac{m_1 m_2}{m_1+m_2}\right) h} \qquad \boxed{1.25}$$

- <u>Energy</u> is also const:

$$E_{tot} = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 - G m_1 m_2 / r$$

$$= \tfrac{1}{2} m_1 [\dot{R}_1^2 + R_1^2 \dot{\theta}^2] + \tfrac{1}{2} m_2 [\dot{R}_2^2 + R_2^2 \dot{\theta}^2] - G m_1 m_2 / r$$

$$= \tfrac{1}{2}\left[m_1 \left(\frac{m_2}{m_1+m_2}\right)^2 + m_2 \left(\frac{m_1}{m_1+m_2}\right)^2\right](\dot{r}^2 + r^2 \dot{\theta}^2) - G m_1 m_2 / r$$

$$= \tfrac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right) v^2 - G m_1 m_2 / r$$

but $v^2 = G(m_1+m_2)[2/r - 1/a]$ from $\boxed{1.11}$

$$\therefore \underline{E_{tot} = -G m_1 m_2 / 2a} \qquad \boxed{1.26}$$

$$= \left(\frac{m_1 m_2}{m_1+m_2}\right) C$$

- Perturbed motion (EG 1.4)

Even if an additional perturbing force means that motion is not purely 2-body, the instantaneous pos^n and vel. of a particle define a set of 6 "osculating elements" which is the orbit if the pert^n were removed

Consider a small acceler^n applied to M_2

$$\underline{dF} = \bar{R}\hat{\underline{r}} + \bar{T}\hat{\underline{\theta}} + \bar{N}\hat{\underline{z}}$$

where $\hat{\underline{r}}, \hat{\underline{\theta}}, \hat{\underline{z}}$ define ref frame ctr on M_2, w $\hat{\underline{r}}$ in dir^n $r_1 \to r_2$, $\hat{\underline{z}}$ is ⊥ to orb. plane

New e.o.m $\boxed{1.1}$: $\underline{\ddot{r} + \mu \underline{r}/r^3 = \underline{dF}}$ $\boxed{1.1P}$

$\boxed{\dot{a}}$ $\int \underline{\dot{r}} \cdot \boxed{1.1P}\, dt$

From $\boxed{1.9}$: $\frac{1}{2}v^2 - \mu/r = C = \int \underline{\dot{r}} \cdot \underline{dF}\, dt$

$$\therefore \dot{C} = \begin{pmatrix} \dot{r} \\ r\dot{\theta} \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \bar{R} \\ \bar{T} \\ \bar{N} \end{pmatrix} = \bar{R}\dot{r} + \bar{T} r\dot{\theta}$$

But $C = -\mu/2a$

$$\therefore \dot{C} = \frac{\mu}{2a^2}\dot{a}$$

So from $\boxed{1.10}$:

$$\dot{a} = 2\sqrt{\frac{a^3}{\mu(1-e^2)}}\left[\bar{R}e\sin f + \bar{T}(1+e\cos f)\right] \quad \boxed{1.27}$$

$\boxed{\dot{e}}$ $d\boxed{1.2}/dt$

$$\underline{\dot{h}} = \underline{\dot{r}} \wedge \underline{\dot{r}} + \underline{r} \wedge \underline{\ddot{r}} = \underline{r} \wedge \underline{dF} = r\bar{T}\hat{\underline{z}} - r\bar{N}\hat{\underline{\theta}} \quad \boxed{1.28}$$

Only $\hat{\underline{z}}$ component changes magnitude of $\underline{h}$:

Consider $|h_{new}| = \left|\begin{pmatrix} dh_x \\ dh_y \\ h_{old}+dh_z \end{pmatrix}\right| \triangleq h_{old} + dh_z$ $\boxed{1.29}$

$$\therefore |\underline{\dot{h}}| = r\bar{T}$$ $\boxed{1.30}$

Rewrite $\boxed{1.6}$ as $e = [1 + 2Ch^2/\mu^2]^{1/2}$

$$\therefore \dot{e} = (\partial e/\partial h)\dot{h} + (\partial e/\partial C)\dot{C}$$

But $\partial e/\partial h = \frac{1}{2e}\frac{4Ch}{\mu^2} = -\frac{1}{e}\sqrt{\frac{1-e^2}{\mu a}}$

$\partial e/\partial C = \frac{1}{2e}\frac{2h^2}{\mu^2} = \frac{a(1-e^2)}{\mu e}$

$$\therefore \dot{e} = \sqrt{\frac{a(1-e^2)}{\mu}}\left[\bar{R}\sin f + \bar{T}\left[\frac{(1+e\cos f)}{e} - \frac{(1-e^2)}{e(1+e\cos f)}\right]\right]$$

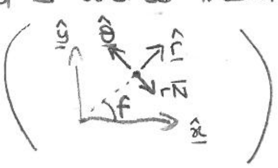
$$= \sqrt{\frac{a(1-e^2)}{\mu}}\left[\bar{R}\sin f + \bar{T}\left[\cos f + \cos E\right]\right] \quad \boxed{1.31}$$

where $\cos E = (e + \cos f)/(1 + e\cos f)$

So, only forces in orbital plane change a and e

$\boxed{\dot{I}}$ The $\hat{\underline{\theta}}$ component changes the orientation of $\underline{h}$ and so the orbital plane

$$\dot{\underline{h}}_{3D} = T\,\dot{\underline{h}}_{2D}$$



where $\dot{\underline{h}}_{2D} = \begin{pmatrix} r\bar{N}\sin f \\ -r\bar{N}\cos f \\ r\bar{T} \end{pmatrix}$ from $\boxed{1\cdot 28}$

$$\therefore \dot{\underline{h}}_{3D} = r\begin{pmatrix} \bar{T}S\Omega SI + \bar{N}[S\omega{+}f\,C\Omega + C\omega{+}f\,S\Omega CI] \\ -\bar{T}SIC\Omega + \bar{N}[S\omega{+}f\,S\Omega - C\omega{+}f\,C\Omega CI] \\ \bar{T}CI - \bar{N}[C\omega{+}f\,SI] \end{pmatrix}$$ from $\boxed{1\cdot 14}$ $\boxed{1\cdot 32}$

From $\boxed{1\cdot 17}$ $\cos I = h_z/h$

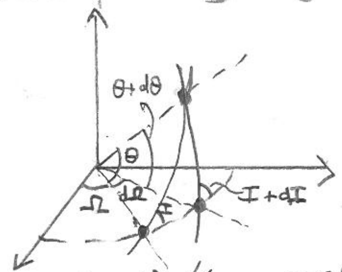
$$\therefore -\sin I\,\dot{I} = \dot{h}_z/h - h_z\dot{h}/h^2$$
$$= \frac{r}{h}[\bar{T}\cos I - \bar{N}\cos(\omega+f)\sin I - \bar{T}\cos I]$$
$$\therefore \dot{I} = r\bar{N}\cos(\omega+f)/h$$ $\boxed{1\cdot 33}$

$\boxed{\dot{\Omega}}$ from $\boxed{1\cdot 17}$ $\tan\Omega = -h_x/h_y$

$$\therefore \sec^2\Omega\,\dot{\Omega} = -\dot{h}_x/h_y + h_x\dot{h}_y/h_y^2$$
$$= \frac{1}{h}[\dot{h}_x/(\cos\Omega\sin I) + \dot{h}_y\tan\Omega/(\cos\Omega\sin I)]$$
$$\therefore \dot{\Omega} = \frac{1}{h\sin I}[\dot{h}_x\cos\Omega + \dot{h}_y\sin\Omega]$$
$$= r\bar{N}\sin(\omega+f)/(h\sin I)$$ $\boxed{1\cdot 34}$

So orbital plane only changed by forces $\perp$ to orbital plane

$\boxed{\dot{\omega}}$



NB • = node projected onto sphere

$$r = a(1-e^2)/(1+e\cos(\theta-\omega))$$
$$\therefore \cos(\theta-\omega) = \left[\left(\frac{a}{r}\right)(1-e^2)-1\right]\frac{1}{e}$$ see $\boxed{1\cdot 18}$
$$= [h^2/\mu r - 1][1+2Ch^2/\mu^2]^{-1/2}$$ eg $\boxed{1\cdot 30}$
$$\therefore -\sin(\theta-\omega)(\dot{\theta}-\dot{\omega}) = [\partial\cos(\theta-\omega)/\partial h]\dot{h} + [\partial\cos(\theta-\omega)/\partial C]\dot{C}$$ (NB r is fixed instantaneously)
$$\therefore \dot{\omega} = \dot{\theta} + e^{-1}\sqrt{\frac{a(1-e^2)}{\mu}}\left[-\bar{R}\cos f + \bar{T}\sin f\,(2+e\cos f)/(1+e\cos f)\right]$$ $\boxed{1\cdot 35}$

Now θ is posn from node, which has moved, but posn has not, doing sph. trig. on $\triangle$

$$\cos(\theta+d\theta) = \cos\theta\cos d\Omega + \sin\theta\sin d\Omega\cos I$$
$$\therefore \dot{\theta} = -\dot{\Omega}\cos I$$ $\boxed{1\cdot 36}$

Averaging around orbit

Let $\langle x\rangle = \frac{1}{t_{per}}\int_0^{t_{per}} x\,dt = \frac{1}{2\pi}\int_0^{2\pi} x\,dM$

Rewrite using $r^2 df/dt = \sqrt{\mu a(1-e^2)}$ = const $(\boxed{1\cdot 6})$

$$\therefore \langle x\rangle = \frac{1}{2\pi a^2\sqrt{1-e^2}}\int_0^{2\pi} r^2 x\,df$$ $\boxed{1\cdot 37}$