UNIVERSITY OF CAMBRIDGE

MAMA/316, NST3AS/316, MAAS/316

MAT3

MATHEMATICAL TRIPOS Part III

Thursday 30 May 2024 1:30 pm to 4:30 pm

PAPER 316

PLANETARY SYSTEM DYNAMICS

Before you begin please read these instructions carefully

Candidates have THREE HOURS to complete the written examination.

Attempt no more than **THREE** questions.
There are **FOUR** questions in total.
The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury tag
Script paper
Rough paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Consider a coordinate system that is centred on a star with $\hat{\mathbf{z}}$ pointing towards the observer, so that the $\hat{\mathbf{x}} - \hat{\mathbf{y}}$ plane is that of the sky, with $\hat{\mathbf{x}}$ pointing towards North. The circular orbit of radius a of a planet around the star is defined by the inclination I of the orbit to the sky plane, and the longitude of ascending node Ω, i.e., the angle in the sky plane between North and where the planet passes through the sky plane while moving towards the observer. The location of the planet within the orbit is defined by the angle f subtended at the star between its location and the ascending node. Provide a sketch of this coordinate system, noting the orbit and location of the planet, and the angles I, Ω and f.

(ii) Derive expressions for the x, y, z location of the planet within this coordinate system.

(iii) The observer sees the planet projected onto the sky plane, with observables being the projected separation from the star $R_{\rm sky}$ and the position angle ϕ, which is the angle subtended at the star between North and the planet's projected location, as measured anticlockwise from North. Give expressions for $R_{\rm sky}$ and ϕ.

(iv) If the orbit is close to edge-on, such that $I = \pi/2 - I'$, where $I' \ll 1$, show that to second order in I': $\tan(\phi - \Omega) \approx I' \tan f$ and $R_{\rm sky} \approx [1 + 0.5(I' \tan f)^2] \cos f$. You may use without proof that $\tan(A - B) = (\tan A - \tan B)/(1 + \tan A \tan B)$.

(v) The planet is observed in a system which also hosts an axisymmetric circumstellar belt of dust at radius $a_{\rm d}$. Observations constrain the position angle of the dust belt's ascending node as seen in projection $\phi_{\rm d}$, as well as its inclination to the sky plane $I_{\rm d}$. Derive an expression for the mutual inclination $I_{\rm m}$ between the orbital planes of the dust belt and the planet in terms of $I_{\rm d}$, $\phi_{\rm d}$, I and Ω, and check your expression for consistency (e.g., by considering special cases such as $\phi_{\rm d} = \Omega$). You may use without proof the spherical trigonometric identity $\cos A = \cos a \sin B \sin C - \cos B \cos C$, where A is the internal angle opposite the arc of angular length a.

(vi) Derive an expression for the difference in position angles seen for the planet and the belt, $\tan(\phi - \phi_{\rm d})$, in terms of $\Omega - \phi_{\rm d}$, I and f, and give an approximation for $\phi - \phi_{\rm d}$ for $I' \ll 1$.

(vii) Comment on how the difference in position angle between the planet's location and the belt's ascending node can be used to assess whether their orbits are coplanar.

(viii) If $a \gg a_{\rm d}$ and the planet's orbit is misaligned with the belt (i.e., $I_{\rm m} > 0$), comment on the expected evolution of the orbits of material within the belt.

2

(i) Consider a spherical planetesimal of diameter D_{t} and density ρ. The planetesimal suffers a catastrophic collision such that half of its mass ends up in a single fragment and the remaining mass is distributed into fragments with sizes between $D_{\mathrm{min,f}}$ and $D_{\mathrm{max,f}} \gg D_{\mathrm{min,f}}$, in a size distribution such that the number of fragments with diameters in the range D_{f} to $D_{\mathrm{f}} + dD_{\mathrm{f}}$ is $n(D_{\mathrm{f}})dD_{\mathrm{f}} \propto D_{\mathrm{f}}^{-\alpha_{\mathrm{f}}}$, where α_{f} is a constant in the range 3 to 4. Derive an expression for the total cross-sectional area of the fragments $\sigma_{\mathrm{tot},1}$.

(ii) If instead the fragments created in the collision, other than the single largest fragment, are all of the same size $D_{\mathrm{f}} \ll D_{\mathrm{t}}$, then determine a new expression for their total cross-sectional area, showing that this is $\sigma_{\mathrm{tot},2} \approx (\pi/8)D_{\mathrm{t}}^3 D_{\mathrm{f}}^{-1}$. Comment on the ratio $\sigma_{\mathrm{tot},2}/\sigma_{\mathrm{tot},1}$.

(iii) Consider a belt of planetesimals of total mass M in which planetesimals have sizes in the range D_{min} to D_{max}, drawn from a power law size distribution $n(D) \propto D^{-\alpha}$, where α is a constant in the range 3 to 4. The volume of the belt is V and the relative velocity at which planetesimals collide v_{rel}. Planetesimals can be assumed to all have the same dispersal threshold Q_{D}^*, independent of size, and gravitational focussing can be ignored. Derive an expression for the rate at which a planetesimal in the belt of diameter D_{t} suffers catastrophic collisions, stating any assumptions made.

(iv) You may assume that the cross-sectional area created in all catastrophic collisions is given by $\sigma_{\mathrm{tot},2}$, and that this cross-sectional area remains in a dust clump for a fixed duration Δt. Derive an expression for $N(>\sigma_{\mathrm{tot}})$, the number of dust clumps that would be expected to be present at any given time with more cross-sectional area than a given level σ_{tot}.

(v) If the size distribution is such that $\alpha = 3.5$, give and comment on the dependence of $N(>\sigma_{\mathrm{tot}})$ on the different parameters, one of which is $N(>\sigma_{\mathrm{tot}}) \propto \sigma_{\mathrm{tot}}^{-1}$.

(vi) Discuss the dynamical processes which might cause the clumps to disperse, and how these might depend on the size of the fragments created, D_{f}, and whether a fixed duration for the clump lifetime is a reasonable assumption.

3

(i) Two bodies of mass M_1 and $M_2 \ll M_1$ are on a circular orbit about their centre of mass O. Units are chosen such that both the distance between the bodies and their mean motion are unity. A test particle P is orbiting in the binary's orbital plane, and its location in this plane is given by (x, y) in the rotating frame $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}})$ that is centred on O with $\hat{\boldsymbol{x}}$ pointing towards M_2. Sketch the location of M_1, M_2 and P in an inertial frame centred on O, and give expressions for the distances OM_1, OM_2, M_1P and M_2P in terms of x, y, μ_1 and μ_2, where $\mu_i = GM_i$.

(ii) The test particle's equation of motion can be written $\ddot{x} - 2\dot{y} = F$ and $\ddot{y} + 2\dot{x} = G$, where

$$\begin{aligned} F &= \mu_1(1 - r_1^{-3})(x + \mu_2) + \mu_2(1 - r_2^{-3})(x - \mu_1), \\ G &= \mu_1(1 - r_1^{-3})y + \mu_2(1 - r_2^{-3})y, \end{aligned}$$

and r_i is the distance of the particle from body i. Explain how these equations result in 5 equilibrium points for the test particle's motion.

(iii) The collinear L_3 equilibrium point is on the opposite side of M_1 from M_2 at a distance $r_1 = 1 + \beta$, where $|\beta| \ll 1$. Derive the leading order solution

$$\beta \approx \alpha \equiv -(7/12)\mu_2/\mu_1.$$

(iv) Show that the second order solution is $\beta \approx \alpha + (7/12)(\mu_2/\mu_1)^2$. If you wish you may use Lagrange's inversion method which says that if $b = a + ef(b)$ where $e < 1$ then

$$b = a + \Sigma_{j=1}^{\infty} \left(\frac{e^j}{j!} \right) \frac{\mathrm{d}^{j-1}}{\mathrm{d}a^{j-1}} [f(a)]^j.$$

(v) The particle is orbiting very close to L_3 at $y = Y$ and $x = x_{L_3} + X$, where $X \ll \alpha$ and $Y \ll \alpha$. Show that the equation of motion can be written in the form $\dot{\mathbf{X}} = A\mathbf{X}$, where the vector $\mathbf{X} = [x, y, \dot{x}, \dot{y}]$, and the matrix A should be given in terms of the derivatives of the functions F and G evaluated at L_3. Note that expressions for these derivatives are not needed at this stage.

(vi) Show that $\partial F/\partial y = \partial G/\partial x$ and that both are zero when evaluated at L_3.

4

(i) A planet of mass M_1 is on a circular orbit of radius a_1 around a star of mass $M_\star \gg M_1$. A planetesimal, which can be assumed to be massless, orbits in the same plane as the planet with semimajor axis $a > a_1$, eccentricity e and longitude of pericentre ϖ. The planetesimal is in $p + q : p$ mean motion resonance with the planet. Give the semimajor axis of the planetesimal, and describe with reference to a general form of the planetesimal's disturbing function why its motion is expected to be dominated by the term involving the resonant argument $\phi_1 = (p + q)\lambda - p\lambda_1 - q\varpi$, where λ and λ_1 are the mean longitudes of the planetesimal and planet, respectively.

(ii) Ignoring the precession of the planetesimal's pericentre, give the geometrical explanation for the quantities ϕ_1/p and ϕ_1/q, and an expression for the mean time between conjunctions in terms of the planetesimal's orbital period T. You may find it useful to consider the evolution of the planetesimal from a time when it is (a) at pericentre and the planet is at a longitude λ_0, and (b) at conjunction with the planet at longitude λ_c.

(iii) The planetesimal is in 8:5 resonance with the planet and is started at conjunction with the planetesimal at apocentre. Consider the path of the planetesimal in the frame rotating with the planet's mean motion. Determine how many planetesimal orbits are required before the pattern repeats. Note the locations on the rotating frame where the planetesimal could be at pericentre or apocentre and use this to sketch the path of the planetesimal in this frame.

(iv) By considering the perturbations the planetesimal experiences at conjunction, explain what value of ϕ_1 this resonant argument will librate about.

(v) The planetesimal's resonant argument ϕ_1 is librating about this equilibrium value with an amplitude of libration $\Delta\phi_1$. Give a constraint on the planetesimal's eccentricity for which close encounters with the planet might be possible, depending on the value of $\Delta\phi_1$.

(vi) Consider the parameter space of the planetesimal's libration amplitude and eccentricity (i.e., $\Delta\phi_1$ vs e). Describe without detailed calculation how you would determine the regions of parameter space on such a plot for which the planetesimal would be expected to have close encounters with the planet.

END OF PAPER