UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS   Part III

Thursday, 31 May, 2018   9:00 am to 12:00 pm

PAPER 316

PLANETARY SYSTEM DYNAMICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

(i) Assume that a comet orbits a star of mass $M_\star$ on a highly elliptical orbit for which its apocentre distance Q is much larger than its pericentre distance q. Its orbit is coplanar and prograde with that of a planet of mass $M_{\rm p} \ll M_\star$ that is on a circular orbit at a distance $a_{\rm p}$, where $q \ll a_{\rm p}$. Show that when the comet has a close encounter with the planet, their relative velocity is $v_{\rm rel} \approx v_{\rm p}[3 - 2(2q/a_{\rm p})^{1/2}]^{1/2}$, where $v_{\rm p}$ is the orbital velocity of the planet, and hence determine the range of impact velocities that the planet might experience when colliding with comets on coplanar orbits.

(ii) The comet's orbit is not near any mean motion resonances with the planet. Consider the path of the comet's orbit in the frame rotating with the planet. Describe the region of this phase space that the comet's orbit has covered after many orbits.

(iii) Determine the fraction of time that the comet spends at radii from r to $r + dr$, and hence show that the density of comets per cross-sectional area of the phase space from (ii) in the vicinty of the planet is $\pi^{-2}Q^{-1/2}a_{\rm p}^{-3/2}(1 - q/a_{\rm p})^{-1/2}$.

(iv) The planet is spherical and has constant density $\rho_{\rm p}$. Determine the constraint on the planet's mass for gravitational focussing to be ignored when considering interactions between the comet and the planet.

(v) Assuming that the constraint from (iv) is met, determine the mean time between collisions.

(vi) How does the collision time scale with Q, q, $M_{\rm p}$ and $a_{\rm p}$ if the comet is initially on an orbit that is inclined by $I \ll 1$ to the planet's orbit.

(vii) Close encounters between the comet and planet can also result in the comet becoming unbound from the star. For a hyperbolic encounter with the planet in which the relative velocity at large separations is v_∞ and the impact parameter is b, the change in the comet's velocity vector has a magnitude $\Delta v = 2v_\infty[1 + b^2v_\infty^4/(GM_{\rm p})^2]^{-1/2}$. Show that the time between encounters that eject the comet scales $\propto Q^{1/2}$.

(viii) Determine the condition on the comet's apocentre that determines whether it is more likely to be ejected by the planet or to collide with it, and hence sketch how the comet's lifetime depends on Q.

2

(i) Consider a dust particle orbiting a star of mass $M_\star$ for which the ratio of the radiation force to that of stellar gravity is β. Poynting-Robertson drag results in the particle's semimajor axis a and eccentricity e evolving according to

$$\begin{aligned} \dot{a} &= -A \frac{1}{a} \frac{2+3e^2}{(1-e^2)^{3/2}}, \\ \dot{e} &= -A \frac{5}{2a^2} \frac{e}{(1-e^2)^{1/2}}, \end{aligned}$$

where $A = \beta G M_\star / c$ and G and c are constants. Derive the result that $C = a(1-e^2)e^{-4/5}$ remains constant along the particle's trajectory.

(ii) Derive expressions for the evolution of the particle's pericentre distance q and apocentre distance Q, and hence show that $dQ/dq = (Q/q)^2(3q+5Q)/(5q+3Q)$.

(iii) The particle is started on an orbit with a pericentre distance q_0 and apocentre distance $Q_0 \gg q_0$. Quantifying wherever possible, sketch the trajectory of the particle's orbit on a plot of Q/q_0 versus q/q_0, explaining why this is very roughly described by two phases of evolution.

(iv) Derive the time for the particle started on a circular orbit at semimajor axis a_0 to reach the star.

(v) Derive a general expression for the time for the particle to reach the star from an orbit with an eccentricity e_0, and hence show that for $e_0 \to 1$, this time is $(4/5)A^{-1}Q_0^{1/2}q_0^{3/2}$. [*You may use the result that in this limit* $\int_0^{e_0} e^{3/5}(1-e^2)^{-3/2}de \approx (1/\sqrt{2})(1-e_0)^{-1/2}$].

(vi) Dust particles with a range of β are created in the break-up of planetesimals on nearly circular orbits in a ring at a radius r_b from the star. Derive expressions for the pericentre and apocentre distances of the particles' orbits immediately after they are created, and hence sketch these orbits relative to the planetesimal ring for a range of β and describe the resulting dust distribution.

(vii) Ignoring collisions between the dust grains, plot the lifetime of the dust grains as a function of β.

3

(i) Consider a test particle orbiting a star in the same plane as a coplanar system of N planets. The particle is on a low eccentricity orbit far from any of the planets and their mean motion resonances. Perturbations from the planets result in a disturbing function

$$\mathcal{R} = na^2 \left[\tfrac{1}{2} A e^2 + \sum_{j=1}^{N} A_j [e e_j \cos(\varpi - \varpi_j)] \right],$$

where n, a, e, ϖ are the mean motion, semimajor axis, eccentricity and longitude of pericentre of the particle's orbit, respectively, orbital elements with subscript j refer to those of planet j, A_j is a negative function of $\alpha_j = a/a_j$, a, and the planet to star mass ratio $M_j/M_\star$, and $A = -\sum_{j=1}^{N} A_j f(\alpha_j)$, where f is a function of α_j. Given Lagrange's planetary equations, $\dot{e} \approx -(na^2 e)^{-1} \partial\mathcal{R}/\partial\varpi$ and $\dot{\varpi} \approx (na^2 e)^{-1} \partial\mathcal{R}/\partial e$, show that the evolution of the particle's complex eccentricity $z = e \exp(i\varpi)$ is given by

$$\dot{z} = iAz + i \sum_{j=1}^{N} A_j z_j. \qquad (*)$$

(ii) Any evolution of the planet's orbits due to their mutual perturbations is encapsulated by $z_j = \sum_{i=1}^{N} e_{ji} \exp[i(g_i t + \beta_i)]$. Explain what conditions this implies for the planets' orbits, and the meaning of the constants e_{ji}, g_i and β_i.

(iii) Hence solve equation $(*)$ for the evolution of z, and describe this motion on an Argand diagram, explaining also the meaning of the terms proper and forced eccentricity.

(iv) Show that the forced eccentricity for $N = 1$ is $z_{\mathrm{f}} = z_1/f(\alpha_1)$.

(v) To first order in eccentricity e, an ellipse of semimajor axis a is a circle of radius a centred on the centre (rather than the focus) of the ellipse. Use this to describe, both quantitatively and with a sketch, the physical space covered by test particles, all of which have the same semimajor axis and proper eccentricity, but randomly distributed proper longitudes of pericentre.

(vi) Compare this with the space covered by particles which have a range of semimajor axes in the range $a \pm \Delta a$ and proper eccentricities that are much smaller than their forced eccentricities (i.e., where $e_{\mathrm{p}}/e_{\mathrm{f}} \ll \Delta a/a \ll 1$).

(vii) To determine the spatial distribution of particles within the space covered consider that the fraction of time a particle spends at different points along its orbit is inversely proportional to its orbital velocity. Show that to first order in eccentricity that the line density of particles on the same orbit is $\propto 1 - e\cos\theta$, where θ is the longitude relative to the pericentre direction.

(viii) Hence comment on the relative densities of particles near the direction of the forced pericentre and forced apocentre in the two scenarios described in (v) and (vi), and on whether observations of the particle distribution might be able to distinguish between these scenarios.

4

(i) A particle is on an elliptical orbit of semimajor axis a and eccentricity $e \ll 1$ about a central star. Explain with help of a diagram the meaning of the terms true anomaly (f), eccentric anomaly (E), mean anomaly (M) and mean motion (n).

(ii) Use your diagram to derive an expression that relates E to f and so show that the distance of the particle from the star is $r = a(1 - e\cos E)$.

(iii) Given Kepler's equation, $M = E - e\sin E$, show that to first order in eccentricity $r/a \approx 1 - e\cos M$ and $f - M \approx 2e\sin M$.

(iv) Consider the path of the particle in a frame centred on a point G that is offset from the star by a constant distance a_g which rotates about the star at a constant rate n_g, where $|(a - a_g)/a_g| \ll 1$ and $|(n - n_g)/n_g| \ll 1$. The x-axis of this frame points in the direction from the star to G, with the y-axis orthogonal in the orbital plane in the direction of motion. The particle is started at time $t = 0$ at the same longitude as G, at which time the pericentre direction is ahead of the particle by an angle ϖ. Make a new sketch showing the relation between the M, f, ϖ and the rotating frame centred on G.

(v) Keeping terms to first order in small quantities, derive the evolution of the particle's location in the rotating frame to be $x/a \approx (a - a_g)/a - e\cos(nt - \varpi)$ and $y/a \approx (n - n_g)t + 2e\sin(nt - \varpi) + 2e\sin\varpi$.

(vi) Consider now a coplanar system in which two planets, of masses m_1 and m_2, orbit a star of mass $m_\star$. The reference frame (x', y') rotates at angular velocity $n_0 = \sqrt{Gma_0^{-3}}$, where $m = m_\star + m_1 + m_2$ and a_0 is the mean distance of the centre of mass of the planets from the star. The coordinate system (ξ, η) is defined such that $x'/a_0 = 1 + \mu^{1/3}\xi$ and $y'/a_0 = \mu^{1/3}\eta$, where $\mu = (m_1 + m_2)/m \ll 1$. Retaining terms up to lowest order in μ, the resulting equations of motion for this three-body problem can be solved far from close encounters to find that for planet j: $\xi_j \approx D_{1j}\cos(n_0 t) + D_{2j}\sin(n_0 t) + D_{3j}$ and $\eta_j \approx -2D_{1j}\sin(n_0 t) + 2D_{2j}\cos(n_0 t) - \frac{3}{2}D_{3j}n_0 t + D_{4j}$, where D_{ij} are constants of integration. Show that this is consistent with the unperturbed motion derived in (v), assuming that $(a_j - a_0)/a_0 \ll 1$, and determine the orbital elements of the individual planets (a_j, e_j, ϖ_j) in terms of the constants D_{ij}.

END OF PAPER