UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday, 11 June, 2012 1:30 pm to 4:30 pm

PAPER 64

PLANETARY SYSTEM DYNAMICS

*Attempt no more than **THREE** questions.*

*There are **FOUR** questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Consider an asteroid in the interior $(p+1):1$ mean motion resonance with a planet that is on a circular orbit of semimajor axis $a_{\rm pl}$ around a star of mass $M_\star$. The relevant resonant argument is $\phi = (p+1)\lambda_{\rm pl} - \lambda - p\varpi$, where $\lambda_{\rm pl}$ and λ are the mean longitudes of the planet and asteroid respectively, ϖ is the asteroid's longitude of pericentre and p is a positive integer. Assume that resonant forces have caused the asteroid's eccentricity to be pumped up to a value approaching unity such that its pericentre is very close to the star at an offset $q \ll a_{\rm pl}$. Show that the angular motion of the asteroid is slower than the planet when it is further from the star than $r_{\rm x} \approx a_{\rm pl}(2q/a_{\rm pl})^{1/4}$, and estimate the true anomaly $f_{\rm x}$ of the asteroid at this point.

Show that, as the asteroid travels from pericentre to $r_{\rm x}$, the planet moves angularly by approximately $\frac{1}{3}2^{7/8}(q/a_{\rm pl})^{3/8}$ radians.

Describe the physical significance of the quantity $\phi/(p+1)$ and, by considering the longitude at which conjunctions occur, discuss which value of ϕ is likely to lead to stable libration for $p = 1$ or 2.

Draw the repeating pattern of the orbit of such an asteroid in the frame rotating with the planet for stable libration in the $p = 2$ resonance, highlighting one half of one of the orbits as the asteroid travels from pericentre to apocentre, and noting the location of $r_{\rm x}$ on the figure.

Assuming that the asteroid is in resonance, but that ϕ is unconstrained, make a crude estimate of the minimum number of encounters with the planet (of mass $M_{\rm pl}$) that are required to change the asteroid's pericentre distance by of order itself.

[You may assume that the asteroid's specific angular momentum $h = \sqrt{GM_\star a(1-e^2)}$, and that the rate of change of h is equal to the tangential component of any perturbing acceleration times the asteroid's distance from the star.]

2

Consider a planet of mass $M_{\rm pl}$ in orbit around a star of mass $M_\star$ for which the equation of relative motion is $\ddot{\boldsymbol{r}} + \mu\boldsymbol{r}/r^3$, where $\mu = G(M_\star + M_{\rm pl})$, $\boldsymbol{r}$ is the vector offset of the planet from the star, $r = |\boldsymbol{r}|$, and dots denote differentiation with respect to time. Show that there are two constants of motion

$$C = \frac{1}{2}v^2 - \frac{\mu}{r} \tag{1}$$

and

$$h = r^2\dot{\theta}, \tag{2}$$

where v is the speed of the planet relative to the star, and θ is the azimuthal angle in the planet's orbital plane relative to a fixed direction.

Given that $r = (h^2/\mu)[1 + e\cos(\theta - \varpi)]^{-1}$ describes the relative motion, where e (eccentricity) and ϖ (longitude of pericentre) are constants, show that C and h are related by

$$C = \frac{1}{2}\left(\frac{\mu}{h}\right)^2(e^2 - 1). \tag{3}$$

Consider now that the star is losing mass so that μ changes at a rate $\dot{\mu}$. Show that the osculating eccentricity varies at a rate

$$\dot{e} = -(\dot{\mu}/\mu)[e + \cos(\theta - \varpi)], \tag{4}$$

and that the rate of change of pericentre distance has the opposite sign to that of $\dot{\mu}$ at all times.

In the adiabatic approximation, all parameters are assumed to have negligible variation around the orbit. If the orbit averaged value of some quantity x is given by $< x >= \frac{1}{P}\int_0^P x dt$, where P is the time for θ to increase from 0 to 2π, show that, in the limit that the mass loss is adiabatic, $< \dot{e} >= 0$.

If the semimajor axis a is defined such that $h^2/\mu = a(1 - e^2)$, show that (μa) is constant in the adiabatic limit.

Determine the fraction of the instantaneous rate of increase in azimuthal angle that is due to pericentre precession, and so set constraints on the characteristic mass loss timescale for the adiabatic approximation to be valid.

3

Consider a test particle in a binary system comprised of two masses M_1 and M_2. The massive bodies orbit the centre of mass O at a constant separation a_{12}. Units are chosen so that $a_{12} = G(M_1 + M_2) = 1$, and the particle's equation of motion in the (x,y,z) rotating frame centred on O, for which $\hat{\boldsymbol{x}}$ points to M_2 and $\hat{\boldsymbol{z}}$ is parallel to the binary angular momentum vector, is

$$\ddot{x} - 2\dot{y} = \partial U/\partial x, \quad (1)$$

$$\ddot{y} + 2\dot{x} = \partial U/\partial y, \quad (2)$$

$$\ddot{z} = \partial U/\partial z, \quad (3)$$

where $U = \frac{1}{2}(x^2 + y^2) + \mu_1/r_1 + \mu_2/r_2$, $\mu_i = GM_i$ and r_i is the distance of the particle from M_i. Show that there is a constant of motion

$$C = 2\mu_1/r_1 + 2\mu_2/r_2 + (x^2 + y^2) - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2), \quad (4)$$

and also give this constant in inertial coordinates.

In the rest of the question assume that $M_2/M_1 \ll 1$ and consider the particle's motion as defined by its 2-body orbital elements about M_1: that is, its semimajor axis $a = (2\mu_1/r_1 - v_1^2)^{-1}$, where v_1 is the particle's speed relative to M_1, its eccentricity e and the inclination of its orbit with respect to the binary orbit I. For certain conditions, which should be specified, show that the combination of elements given by $T = a^{-1} + 2\sqrt{a(1 - e^2)}\cos I$, is a constant.

Consider a particle that is initially on an orbit, of semimajor axis $a = 1$, that is coplanar with the binary. The particle then has a close encounter with the body of mass M_2. Show that to be ejected the particle must encounter that body at a relative velocity in excess of $(\sqrt{2} - 1)/2$ (in the given units). Hence deduce that it can only be ejected in a single encounter if the initial eccentricity was above $\frac{1}{8}\sqrt{31 - 20\sqrt{2}}$.

Rewrite T in terms of pericentre distance q, and apocentre distance Q, and so show that the smallest pericentre $q_{\min}$ to which a particle of given T can be scattered following multiple close encounters with the body of mass M_2 is

$$q_{\min} = \frac{4 + 2T - T^2 - 4\sqrt{3 - T}}{T^2 - 8}. \quad (5)$$

[You may assume that the $\hat{\boldsymbol{z}}$-component of the particle's specific angular momentum is $\sqrt{\mu_1 a(1 - e^2)}\cos I$].

4

Consider a broad disk of planetesimals orbiting a star of mass $M_\star$. Their orbits are initially coplanar and circular, but each planetesimal orbit has a different semimajor axis a. The planetesimals can be considered to be massless. A planet of mass $M_{\rm pl}$ is introduced into the system orbiting in the disk-plane at a larger semimajor axis $a_{\rm pl} \gg a$ with an eccentricity of $e_{\rm pl} \ll 1$ and begins to perturb the orbits of the planetesimals through its secular perturbations. To first order the evolution of a planetesimal's complex eccentricity $z = e\exp(i\varpi)$, where e is its eccentricity, ϖ its longitude of pericentre relative to that of the planet and $i = \sqrt{-1}$, is given by

$$\dot{z} = iAz + iA_{\rm pl}e_{\rm pl}, \tag{1}$$

where $A = \frac{1}{4}n(M_{\rm pl}/M_\star)\alpha^2 b^1_{3/2}(\alpha)$, $A_{\rm pl} = -Ab^2_{3/2}(\alpha)/b^1_{3/2}(\alpha)$, n is the planetesimal's mean motion, $\alpha = a/a_{\rm pl}$, and $b^{1,2}_{3/2}(\alpha)$ are Laplace coefficients. Show that planetesimal orbits evolve as

$$z(t) = e_{\rm pl}\left(\frac{b^2_{3/2}(\alpha)}{b^1_{3/2}(\alpha)}\right)\left[1 - \exp(iAt)\right], \tag{2}$$

and sketch this evolution for a planetesimal at a reference semimajor axis of a_1 in the Argand plane, quantifying where possible.

To lowest order in α, the Laplace coefficients are

$$b^j_s(\alpha) \approx 2\frac{s.(s+1)...(s+j-1)}{1.2.3...j}\alpha^j. \tag{3}$$

Give $z(t)$ for $\alpha \ll 1$ in terms of α and planet parameters, and describe how the evolution of a planetesimal at a semimajor axis $a_2 = a_1 + \delta a$, where $\delta a/a_1 \ll 1$, differs from that sketched above.

By considering the shape of the orbits at a_1 and a_2 to lowest order in eccentricity, and the separation between the orbits as a function of longitude, show that the orbits overlap after a time $t_{\rm cross}$ where

$$t_{\rm cross} = \frac{32}{45}n_{\rm pl}^{-1}(M_\star/M_{\rm pl})e_{\rm pl}^{-1}(a_1/a_{\rm pl})^{-5/2}. \tag{4}$$

After this time collisions between the planetesimals can occur. If the collision velocity is of order the mean eccentricity times the Keplerian velocity, and the planetesimals are all of the same size with a dispersal threshold of $Q^\star_{\rm D}$, determine and comment on the constraint on $(a_1/a_{\rm pl})$ if the collisions are to be catastrophic.

END OF PAPER