UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday, 3 June, 2011 1:30 pm to 4:30 pm

PAPER 61

PLANETARY SYSTEM DYNAMICS

Attempt no more than ***THREE*** *questions.*

There are ***FOUR*** *questions in total.*

The questions carry equal weight.

STATIONERY REQUIREMENTS
Cover sheet
Treasury Tag
Script paper

SPECIAL REQUIREMENTS
None

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Define a reference plane that is centred on a nearby star of mass $M_\star$, at distance d in parsec, with the Z-axis oriented along the line-of-sight. Consider an object of mass M orbiting the star with a semimajor axis a in AU and eccentricity e, and choose the X-axis to be the intersection line between the orbital plane and the sky plane in the direction of the node at which the motion of the object is toward the observer. Denote the inclination of the orbital plane with respect to the sky as I and the argument of pericentre measured from the X-axis as ω.

If the object's current true anomaly is f, then given $r^2\dot{f} = \sqrt{\mu a(1-e^2)}$, where $\mu = G(M_\star + M)$, show that its projected separation $R_{\rm sky}$ in arcsec and projected velocity $V_{\rm sky}$ in arcsec yr^{-1} are given by

$$\begin{aligned} R_{\rm sky} &= (r/d)[1-\sin^2 I \sin^2(\omega+f)]^{1/2}, \\ V_{\rm sky} &= (A/d)[1+e^2+2e\cos f - \sin^2 I[\cos(\omega+f)+e\cos\omega]^2]^{1/2}, \end{aligned}$$

where $r = a(1-e^2)(1+e\cos f)^{-1}$ is the distance of the object from the star, $A = 2\pi\sqrt{\frac{M_\star+M}{M_\odot a(1-e^2)}}$, and $M_\odot$ is the mass of the Sun.

Show that, for the orbit to remain bound, the combination $R_{\rm sky}V_{\rm sky}^2$ cannot exceed

$$max(R_{\rm sky}V_{\rm sky}^2) = \frac{8\pi^2}{d^3}\frac{M_\star+M}{M_\odot},$$

and describe the orbital configuration that this maximum value corresponds to.

A planet was recently discovered in orbit around the $2M_\odot$ star Fomalhaut at 8 parsec. The planet is just interior to a belt of dust that traces out an orbital plane with an inclination from edge-on of $I_{\rm belt} = 13^\circ$ and an eccentricity of $e_{\rm belt} \approx 0.13$. It is expected that the planet has a similar orbital plane and eccentricity to the belt. Show that to first order in eccentricities and inclinations from edge-on, this observable parameter $R_{\rm sky}V_{\rm sky}^2$ must be less than

$$max(R_{\rm sky}V_{\rm sky}^2) \approx \frac{8\pi^2}{3\sqrt{3}d^3}\frac{M_\star+M}{M_\odot}(1+\sqrt{3}e).$$

In fact the planet's motion is observed to be 70% faster than this. Discuss what this suggests about the orbit of the planet, including a consideration of how unknown perturbers may affect these conclusions, and of how the gravitational interaction between the planet and the belt can be used to constrain the planet's orbit further.

2

Consider a coplanar system in which two planets, of masses m_2 and m_3, orbit a central mass m_1. The reference frame (x', y') is centred on m_1 and rotates at angular velocity $n_0 = \sqrt{Gma_0^{-3}}$, where G is the gravitational constant, $m = m_1 + m_2 + m_3$, and a_0 is the mean distance of the centre of mass of m_2 and m_3 from m_1. Choose units of length, time and mass so that $a_0 = n_0 = G = 1$, and find the equations of motion for m_2 and m_3 in the (ξ, η) coordinate system, where $x'/a_0 = 1 + \mu^{1/3}\xi$ and $y'/a_0 = \mu^{1/3}\eta$, and $\mu = (m_2 + m_3)/m \ll 1$, retaining only terms up to lowest order in μ.

Hence show that the equation of relative motion can be written as

$$\begin{aligned} \ddot{\xi}_r - 2\dot{\eta}_r &= \partial U_H/\partial \xi_r, \\ \ddot{\eta}_r + 2\dot{\xi}_r &= \partial U_H/\partial \eta_r, \end{aligned}$$

where $\xi_r = \xi_3 - \xi_2$, $\eta_r = \eta_3 - \eta_2$, $U_H = (3/2)\xi_r^2 + \rho^{-1}$, and $\rho^2 = \xi_r^2 + \eta_r^2$.

Show that there is a constant of motion $\Gamma = 3\xi_r^2 + 2\rho^{-1} - \dot{\xi}_r^2 - \dot{\eta}_r^2$, and and that the L_1 and L_2 Lagrange equilibrium points correspond to motion for which this constant is $\Gamma_{L1} = 3^{4/3}$.

Solve the equations of relative motion in the limit $\rho \gg 1$ to find that

$$\begin{aligned} \xi_r &= D_1 \cos t' + D_2 \sin t' + D_3, \\ \eta_r &= -2D_1 \sin t' + 2D_2 \cos t' - (3/2)D_3 t' + D_4, \end{aligned}$$

where the D_i are constants of integration, and show that $\Gamma = (3/4)D_3^2 - D_1^2 - D_2^2$.

For $\Gamma > \Gamma_{L1}$ there is a region between the planets that is forbidden, since it would require $\dot{\xi_r}^2 + \dot{\eta_r}^2 < 0$. Consider that m_2 and m_3 are initially on circular orbits about m_1 with semimajor axes of a_2 and a_3. Show that the critical separation between the planets within which their orbits can cross is given by $|a_3 - a_2|/a_0 = 2.3^{1/6}\mu^{1/3}$.

3

First order theory gives for the secular evolution of the complex eccentricities of a system of N planets orbiting a star of mass $M_\star$:

$$\dot{z} = iAz, \tag{1}$$

where $z = [z_1, ..., z_N]^T$, $z_i = e_i \exp i\varpi_i$, and A is a matrix with elements

$$\begin{aligned} A_{ji} &= -(1/4)n_j(M_i/M_\star)\alpha_{ji}\bar{\alpha}_{ji}b^2_{3/2}(\alpha_{ji}), \\ A_{jj} &= -\sum_{i=1,i\neq j}^{N} A_{ji}\frac{b^1_{3/2}(\alpha_{ji})}{b^2_{3/2}(\alpha_{ji})}, \end{aligned}$$

where n_j is the mean motion of the j-th planet, M_j is its mass, a_j is its semimajor axis, and $\alpha_{ji} = \bar{\alpha}_{ji} = a_j/a_i$ for $a_i > a_j$ but $\alpha_{ji} = a_i/a_j$ and $\bar{\alpha}_{ji} = 1$ otherwise, and $b^1_{3/2}$ and $b^2_{3/2}$ are Laplace coefficients.

For a system of two planets where $a_2 > a_1$, show that the eigenvalues of A are

$$\begin{aligned} \lambda_1 &= -(A^\star/2)[f(L_1+L_2) + \sqrt{f^2(L_1-L_2)^2 + 4L_1L_2}], \\ \lambda_2 &= -(A^\star/2)[f(L_1+L_2) - \sqrt{f^2(L_1-L_2)^2 + 4L_1L_2}], \end{aligned}$$

where $L_i = M_i a_i^{1/2}$, $f = \frac{b^1_{3/2}(\alpha)}{b^2_{3/2}(\alpha)}$, $\alpha = a_1/a_2$, and $A^\star = -\frac{1}{4}\sqrt{\frac{G}{M_\star}}\frac{a_1^{1/2}}{a_2^{5/2}}b^2_{3/2}(\alpha)$. Show that these eigenvalues are real and comment on the sign and relative magnitude of λ_1 and λ_2 given $f > 1$.

For the case that at time $t = 0$ the eccentricities are $e_1(0) = 0$ and $e_2(0) \neq 0$, solve equation (1) for the evolution of complex eccentricities $z_1(t)$ and $z_2(t)$. Describe this evolution on a plot of $e_1(t)e_2(t)\sin[(\lambda_1 - \lambda_2)t]$ versus $e_1(t)e_2(t)\cos[(\lambda_1 - \lambda_2)t]$.

Show that the maximum eccentricity that planet 1 attains throughout its evolution is

$$max[e_1(t)/e_2(0)] = \left[\frac{L_1}{L_2} + \frac{1}{4}f^2\left(\frac{L_1}{L_2} - 1\right)^2\right]^{-1/2},$$

and describe how this depends on L_1/L_2 for $f = 1$, 2, and 10.

4

In 2010 a dust clump was discovered in the asteroid belt that had been created the previous year in a collision between two asteroids. This question considers the expected rate of collisional events that produce dust cross-sectional area above a given level σ_{lim}. The asteroid belt is assumed to have a mass M_{tot}, volume V, mean relative velocity of asteroid encounters v_{rel}, and to contain spherical asteroids of mean density ρ that follow a power law size distribution $n(D) \propto D^{-\alpha}$ between a maximum size D_{max} and minimum size $D_{min} \ll D_{max}$, where $3 < \alpha < 4$. The asteroids' dispersal threshold follows $Q_D^\star = K_Q D^{-a}$, where $a < 1$, and it may be assumed that relative velocities are large so that $v_{rel} \gg \sqrt{2Q_D^\star}$ and $v_{rel} \gg v_{esc}$ (where v_{esc} is the asteroids' escape velocity).

The size distribution is in steady state, which means that mass loss rate is independent of size in bins that are logarithmically spaced in size. For mass loss rates that are set by the catastrophic collision rate, show that $\alpha = (21 - a)/(6 - a)$.

When a target asteroid of size D and mass M is impacted by another of size D_{im}, the largest fragment from the target has a mass $f_{lr}M$, where $f_{lr} = 1 - 0.5Q/Q_D^\star$ (or zero for $Q > 2Q_D^\star$), and Q is the specific incident energy of the collision. The remainder of the target mass goes into fragments that have a range of sizes following a power law $n(D) \propto D^{-\alpha_r}$ between D_{minr} and a maximum size D_{maxr}, where $3 < \alpha_r < 4$. By scaling the fragment distribution so that the number of fragments larger than D_{maxr} that would have been in this distribution had it continued up to infinity is equal to one, show that $(D_{maxr}/D)^3 = (1 - f_{lr})(4 - \alpha_r)/(\alpha_r - 1)$.

Thus show that the total cross-sectional area created in collisions between objects of size D and D_{im} is given by $\sigma_{tot} \propto (D_{im}D^{a/3})^{\alpha_r - 1}$ for $D_{im} < D_x$ and $\sigma_{tot} \propto D^{\alpha_r - 1}$ for $D_{im} > D_x$, where $D_x = [2K_Q/v_{rel}^2]^{1/3} D^{1-a/3}$.

Hence find that the rate of collisions that produce dust cross-sectional area larger than σ_{lim} is $R_{col}(\sigma_{tot} > \sigma_{lim}) \propto \sigma_{lim}^{\gamma}$, where $\gamma = (3a - 18)(6 - a)^{-1}(\alpha_r - 1)^{-1}$.

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