Planetary System Dynamics Prof M. C. Wyatt
Part III Mathematics / Part III Astrophysics Michaelmas Term 2023

Examples Sheet 1

1. *Constants of motion*

(a) A system of N particles have masses $m_i (i = 1, 2, ..., N)$ and position vectors $\boldsymbol{r}_i (i = 1, 2, ..., N)$. If the masses move under their mutual gravitational attraction, derive an expression for the acceleration $\ddot{\boldsymbol{r}}_i$ on the i-th particle in terms of the mutual position vectors $\boldsymbol{r}_{ij} = \boldsymbol{r}_j - \boldsymbol{r}_i$.

(b) Show that the centre of mass of the system moves through space with a constant velocity.

(c) Show that the total angular momentum and the total energy of the system do not change with time.

(d) Are there any additional constants of motion for $N = 2$?

2. *Radial velocity planet detection / 3D orbital velocity*

One planet detection strategy involves measuring the radial velocity of the star relative to the motion of the barycentre of the star+planet system. A reference frame commonly used to measure exoplanet orbits is centred on the star and uses the plane of the sky (i.e., the plane tangent to the celestial sphere) as the $X - Y$ plane with the Z-axis directed along the line of sight. The X-axis is chosen to be the intersection line between the orbital plane and the sky plane in the direction of the node at which the motion of the planet is towards the observer.

(a) Sketch the geometry of an orbit within this frame which has a semimajor axis a and eccentricity e, and where I is the inclination of the orbital plane with respect to the sky and ω is the argument of the pericentre measured from the X-axis.

(b) Derive the velocity of the planet relative to the star along the line of sight, and show that the radial velocity of the star of mass $M_\star$ due to the gravity of a planet of mass $M_{\rm pl}$ can be written:

$$V_r = K[\cos(\omega + f) + e\cos\omega] + V_{r0}, \quad (1)$$

$$K = \frac{2\pi a}{t_{\rm per}\sqrt{1-e^2}} \frac{M_{pl}\sin I}{M_{pl} + M_\star} \quad (2)$$

where f is the planet's true anomaly, V_{r0} is a constant and $t_{\rm per}$ is the planet's orbital period.

(c) Assuming that exoplanet orbits are randomly oriented with respect to the Earth, what is the average amplitude of the radial velocity perturbation in $\mathrm{m\,s^{-1}}$ that would be measured due to an Earth-mass planet on a circular orbit around a Solar-mass star with a semimajor axis of 1 au? Compare this with the limits of current instrumentation which can detect signals down to around $1\,\mathrm{m\,s^{-1}}$.

(d) How high would the eccentricity of that planet's orbit have to be to double the maximum velocity offset from the motion of the barycentre of the system?

3. *Guiding centre / elliptic expansions*
An approximation that has many uses in planetary system dynamics is that of the guiding centre, in which the motion of a particle P moving on an elliptical orbit about a focus F is viewed in a reference frame that is centred on a point G, the guiding centre, that rotates about the focus on a circle of radius a equal to the particle's semimajor axis, with angular speed equal to the particle's mean motion n.

(a) Draw a diagram to show the relationship between polar coordinates (r, f) with origin at F to rectangular coordinates (x, y) with origin at G, with the x-axis directed along the line FG. Write down the expressions for the x and y coordinates of the particle in terms of r, a, f and mean anomaly M.

(b) Starting from $M = E - e\sin E$, $r = a(1 - e\cos E)$ and $h = na^2\sqrt{1-e^2}$, where E is eccentric anomaly, show (as concisely as possible) that

$$f - M \approx 2e\sin M + O(e^2), \tag{3}$$

$$r/a \approx 1 - e\cos M + 0.5e^2(1 - \cos 2M) + O(e^3). \tag{4}$$

(c) Show that to first order P moves about G in a centred ellipse defined by

$$\frac{x^2}{(ae)^2} + \frac{y^2}{(2ae)^2} \approx 1. \tag{5}$$

(d) Mark on your diagram g, the angle $PF'F$ where F' is the empty focus of the orbit in the non-rotating frame, and show that

$$\cos g = \frac{(1 - \frac{r}{a}) + e^2}{e(1 - \frac{r}{a}) + e}, \tag{6}$$

and so that $\cos g = \cos M + O(e^2)$. Comment on the implications for a satellite which, like the Moon, is in synchronous rotation.

(e) Derive an expression for the distance of the particle P from the geometric centre of the ellipse O, and hence show that to first order in eccentricity the distance OP is a constant.

4. *Perturbed orbits / radial perturbations and GR*

Consider a particle orbiting a central object that is also subject to a purely radial perturbation force that results in an acceleration of

$$d\boldsymbol{F} = \bar{R}\hat{\boldsymbol{r}}, \tag{7}$$

where $\bar{R}$ is a function of distance r.

(a) Write down the equations for the evolution of the particle's orbital elements $\dot{a}$, $\dot{e}$, $\dot{I}$, $\dot{\Omega}$ and $\dot{\varpi}$, and show that the particle's angular momentum is conserved with these equations. Argue why this should necessarily be the case, and discuss the constraints this places on the orbit.

(b) It is often helpful to consider the change in orbital elements averaged around the particle's orbit. Show that the average change in the element x (which could for example denote the particle's semimajor axis) is given by

$$\langle x \rangle = \frac{1}{2\pi a^2\sqrt{1-e^2}} \int_0^{2\pi} r^2 x df. \tag{8}$$

(c) Let $\bar{R} = Ar^\gamma$, where A and γ are constants. Show that

$$\langle \dot{a} \rangle = 2eA\sqrt{\frac{a^3}{\mu(1-e^2)}}\langle r^\gamma \sin f \rangle, \tag{9}$$

and find similar expressions for $\langle \dot{e} \rangle$ and $\langle \dot{\varpi} \rangle$. Discuss the evolution of the particle's semimajor axis and eccentricity.

(d) Show that $\langle r^{-2} \cos f \rangle = 0$ and use physical arguments to determine whether $\langle r^\gamma \cos f \rangle$ is positive or negative when $\gamma > -2$ and $\gamma < -2$, and so how the sign of $\langle \dot{\varpi} \rangle$ depends on A and γ.

(e) The effect of General Relativity (GR) can be considered to be a perturbing force of the form above with $\gamma = -3$ and $A = A'\mu^2$, where $\mu = GM_\star$. Given that GR causes the pericentre of Mercury's orbit (with $a = 0.387$ au, $e = 0.206$) to precess 0.43 arcsec per year faster than expected, estimate the pericentre precession rate due to GR for an exoplanet orbiting a solar mass star with $a = 0.05$ au and $e = 0.02$.

5. *Radiation pressure*

A comet orbits a star of mass $M_\star$ with a semimajor axis a and eccentricity e and pericentre orientation ϖ, and releases dust with a range of β at all points along its orbit.

(a) Show that the orbits of these dust grains (denoted with a dash) have a complex eccentricity that is given by

$$z' = \frac{z}{1-\beta} + \left(\frac{\beta}{1-\beta}\right) \exp i(\varpi + f), \tag{10}$$

where $z = e \exp i\varpi$ and f is the true anomaly of the comet when the grain was released. Draw a diagram showing the distribution of complex eccentricities of dust grains of a given β, noting how these compare with that of the comet, and how they depend on f.

(b) Find the critical value of β for which all particles of this size remain bound irrespective of the point in the orbit at which they were released, and likewise find the critical value for which all particles are unbound.