Astrophysical Fluid Dynamics

What are fluids?
- things that can flow
- eg liquids (only on planet surfaces/interiors, incompressible)
  gases (usually compressible in astrophysics)
  not solids

Key concept
- ignore individual particles
- treat fluid as continuous medium made of large # of particles
- at each point fluid has well-defined macroscopic properties (eg. density, pressure)

Fluid element
- fluid approximation requires that it's possible to define a region of size l in which local variables (eg ρ, P) can be defined
- $\therefore \; l \ll q/|\nabla q|$ ie. small enough to ignore systematic variation in variable q
- $nl^3 \gg 1$ ie. large enough to ignore finite particle number (discreteness noise) ($n = \#/\text{volume}$)

Collisional fluid
- also $l \gg \lambda$ = mean free path $\sim (n\sigma)^{-1}$ (σ = cross-section of interaction), so that particles know about local conditions
- If particles interact, fluid attains a distribution of particle speeds that maximises entropy → that distrⁿ is well defined for given T, ρ
- Can define an equation of state, $P(T, \rho)$, that includes all the microphysical complexity
- Diverse fluids considered w/ same hydrodynamic eqⁿˢ, but different e.o.s.

Collisionless fluid
- distribution of speeds depends on initial conditions
- eg stars in galaxies, particles in Saturn's rings.

Two approaches to formulating fluid equations

Eulerian
- considers volume at fixed posⁿ $\underline{r}$ and specifies fluid properties as a fⁿ of time in that volume, eg $\rho(\underline{r}, t)$, $T(\underline{r}, t)$
- rate of change at that posⁿ is, eg, $\partial\rho/\partial t$
- eg viewing a river from river bank
- eg solve numerically using grid-based code such as AMR (Adaptive Mesh Refinement)

Lagrangian
- considers a fluid element $\underline{a}$ (eg coords at $t=0$) that comoves w/ fluid, and specifies fluid properties as a fⁿ of time in that element, eg $\rho(\underline{a}, t)$
- rate of change in that element is, eg, $D\rho/Dt$
- eg viewing a river from a boat adrift on it
- eg. solve numerically using ensemble of particles representing fluid elements and following their trajectories, such as SPH (Smoothed Particle Hydrodynamics)

Steady flows quantities at given posⁿ don't change $\therefore \partial/\partial t = 0$ everywhere → use Eulerian

Derivative following fluid motion

- Consider any quantity Q (scalar or vector) in fluid element at pos^n $\underline{r}$ at time t
- At time $t+\delta t$ the element is at $\underline{r}+\delta\underline{r}$, so:

$$DQ/Dt = \lim_{\delta t\to 0}\left[\frac{Q(\underline{r}+\delta\underline{r}, t+\delta t) - Q(\underline{r},t)}{\delta t}\right]$$

$$= \lim_{\delta t\to 0}\frac{1}{\delta t}\left[\underbrace{Q(\underline{r},t+\delta t) - Q(\underline{r},t)} + \underbrace{Q(\underline{r}+\delta\underline{r}, t+\delta t) - Q(\underline{r},t+\delta t)}\right]$$

Taylor expansion to 1st order in $\delta\underline{r}$ and δt

$$= \lim_{\delta t\to 0}\frac{1}{\delta t}\left[\delta t\,\partial Q(\underline{r},t)/\partial t + \delta\underline{r}\,.\,\underbrace{\nabla Q(\underline{r},t+\delta t)}_{\nabla Q(\underline{r},t) + \cancel{\delta t\,\partial\nabla Q/\partial t} + \ldots}\right]$$

0 to 1st order

- As fluid velocity $\underline{u} = \delta\underline{r}/\delta t$

$$DQ/Dt = \underbrace{\partial Q/\partial t}_{\text{Eulerian time derivative}} + \underbrace{\underline{u}\,.\,\nabla Q}_{\text{Convective derivative (because element has moved)}}$$

Convective derivative

Scalar Q: $\underline{u}.\nabla Q = u_x\partial Q/\partial x + u_y\partial Q/\partial y + u_z\partial Q/\partial z$ in Cartesian coords

$= u_i\partial_i Q$ using summation convention (sum over subscripts that appear twice) $\partial_i \equiv \partial/\partial x_i$ where x_i is i-th coord. variable

or $= u_r\partial Q/\partial r + \frac{u_\theta}{r}\partial Q/\partial\theta + \frac{u_\phi}{r\sin\theta}\partial Q/\partial\phi$ in Spherical polars

or $= u_R\partial Q/\partial R + u_z\partial Q/\partial z + \frac{u_\phi}{R}\partial Q/\partial\phi$ in cylindrical polars

Vector $\underline{Q} = [Q_x, Q_y, Q_z]$:

$$\underline{u}\,.\,\nabla\underline{Q} = [u_x\partial Q_x/\partial x + u_y\partial Q_x/\partial y + u_z\partial Q_x/\partial z,\ u_x\partial Q_y/\partial x + \ldots]$$

$$= u_i\partial_i Q_j$$

Kinematics = study of particle trajectories

- Consider velocity field defined everywhere in Eulerian coords $\underline{u}(\underline{r},t)$

(i) **Streamlines** – dir^n of local velocity is tangent to streamline i.e. $dy/dx = u_y/u_x$

- defined by $(dt =) \frac{dx}{u_x} = \frac{dy}{u_y} = \frac{dz}{u_z}$
- for spherical polars $\underline{u} = (\dot{r}, r\dot{\theta}, r\sin\theta\,\dot{\phi})$ so $(dt =) \frac{dr}{u_r} = \frac{r\,d\theta}{u_\theta} = \frac{r\sin\theta\,d\phi}{u_\phi}$
- for cylindrical polars $\frac{dR}{u_R} = \frac{dz}{u_z} = \frac{R\,d\phi}{u_\phi}$

(ii) **Particle paths** – particle motion instantaneously follows streamlines $d\underline{r}/dt = \underline{u}(\underline{r},t)$

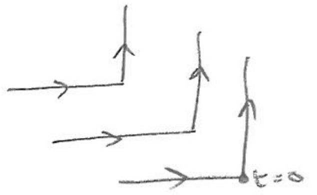
- but not identical if flow is time varying
- $\dot{x} = u_x,\ \dot{y} = u_y,\ \dot{z} = u_z$ cart.
- $\dot{r} = u_r,\ r\dot{\theta} = u_\theta,\ r\sin\theta\,\dot{\phi} = u_\phi$ sph pol.
- $\dot{R} = u_R,\ \dot{z} = u_z,\ R\dot{\phi} = u_\phi$ cyl. pol.

(iii) **Streaklines** – lines joining all particles which have ever passed through a point $\underline{r}_0$

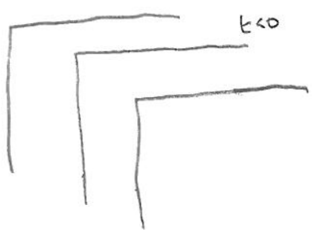
- think of adding dye to particles as they pass
- find $\underline{a}$ for which $\underline{r}(\underline{a},t) = \underline{r}_0$ then streakline is $\underline{r}(\underline{a},0)$

- For steady flows (i) = (ii) = (iii)
- If, say, $\underline{u} = (1,0,0)$ for $t<0$ and $\underline{u} = (0,1,0)$ for $t>0$ we get

streamlines



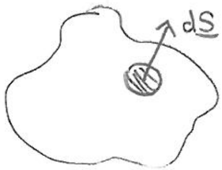
particle paths



streaklines

Fluid equations: conservation of mass (rate of change of mass in region = net inflow of mass)
" momentum (" momentum " is balanced by momentum flow and net force)
" energy (" energy " is determined by external energy gains/losses)

Conservation of mass



- Consider a fixed volume V whose surface S is patchwork of surface elements $d\underline{S}$ = vector normal (outward) to surface, $|d\underline{S}|$ = area
  Let ρ = mass density of fluid
  $\therefore$ Rate of change of mass in V is $\frac{\partial}{\partial t}\int_V \rho \, dV$
- Outward mass flow across $d\underline{S}$ is $\rho\underline{u}.d\underline{S}$ = density × length travelled in dir^{n} $\hat{d\underline{S}}$ per unit time × area
  Integrating over surface, rate of mass gain is $-\oint_S \rho\underline{u}.d\underline{S}$ - as $d\underline{S}$ is outward
  By divergence theorem is $-\int_V \nabla.(\rho\underline{u})dV$
- If no sources/sinks of mass $\frac{\partial}{\partial t}\int_V \rho dV = -\int_V \nabla(\rho\underline{u})\,dV$
  As true for all volumes $\underline{\partial\rho/\partial t + \nabla.(\rho\underline{u}) = 0}$ → Eulerian form of continuity eq.
- In Lagrangian form $D\rho/Dt = \partial\rho/\partial t + \underline{u}.\nabla\rho$
  $\therefore D\rho/Dt + \nabla.(\rho\underline{u}) - \underline{u}.\nabla\rho = 0$
  $\therefore \underline{D\rho/Dt + \rho\nabla.\underline{u} = 0}$ as $\nabla.(\rho\underline{u}) = \rho\nabla.\underline{u} + \underline{u}.\nabla\rho$

Incompressible $\equiv D\rho/Dt = 0$
$\therefore \nabla.\underline{u} = 0$

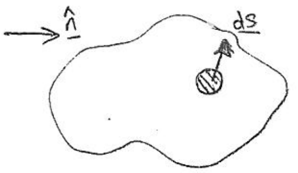
Pressure

Consider any surface element $d\underline{S}$ in fluid
- Pressure is the one-sided momentum flux associated w random motion of particles w.r.t. local rest frame of fluid due to their finite temperature
  [balanced by = and opposite flux through other side of $d\underline{S}$]
- If random motion isotropic: mom. flux is indep. of orientation of $d\underline{S}$ and components ||'l to $d\underline{S}$ cancel

$\therefore$ pressure force acting on $d\underline{S}$ is defined as $-p\,d\underline{S}$
(ie p is force/unit area)

ignoring viscosity

Inviscid momentum equations



- Consider fluid element subject to gravitational acceler^{n} $\underline{g}$ and pressure from surrounding fluid
- Consider components of forces in direction $\hat{\underline{n}}$

- Integrate pressure force over surface
$$F_n = \oint_S -p\hat{\underline{n}}.d\underline{S}$$
$$= -\int_V \nabla.(p\hat{\underline{n}})\,dV \quad \text{from divergence}$$
$$= -\int_V \hat{\underline{n}}.\nabla p + p\cancelto{0}{\nabla.\hat{\underline{n}}}\,dV \quad \text{from identity}$$

- Equation of motion for fluid element in $\hat{\underline{n}}$ dir^{n} is:
$$\left(\frac{D}{Dt}\left[\int_V \rho\underline{u}\,dV\right]\right).\hat{\underline{n}} = -\int_V \hat{\underline{n}}.\nabla p\,dV + \int_V \rho\underline{g}.\hat{\underline{n}}\,dV$$

- For small volume $\int_V dV \to \delta V$ as local variables constant across vol.
$$\therefore \left(\underline{u}\cancelto{0 \text{ as mass conserved in fluid element}}{\frac{D}{Dt}[\rho\delta V]} + \rho\delta V\frac{D\underline{u}}{Dt}\right).\hat{\underline{n}} = -\nabla p\,\delta V.\hat{\underline{n}} + \rho\delta V\underline{g}.\hat{\underline{n}}$$

- As holds for all $\hat{\underline{n}}$:
$$\underline{\rho D\underline{u}/Dt = -\nabla p + \rho\underline{g}} \quad \text{in Lagrangian form}$$
→ momentum of fluid element changes due to pressure gradients and grav. forces

$$\underline{\rho\partial\underline{u}/\partial t + \rho\underline{u}.\nabla\underline{u} = -\nabla p + \rho\underline{g}} \quad \text{in Eulerian form}$$
→ mom in grid cell changes due to pressure gradient and grav forces, and net influx of mom into cell.

- Consider (Eulerian) rate of change of momentum density ($\rho\underline{u}$)
$$\partial\rho\underline{u}/\partial t = \rho\partial\underline{u}/\partial t + \underline{u}\,\partial\rho/\partial t$$
$$= \underbrace{-\nabla p + \rho\underline{g} - \rho\underline{u}.\nabla\underline{u}}_{\text{from mom}} \underbrace{- \underline{u}\nabla.(\rho\underline{u})}_{\text{from mass}}$$

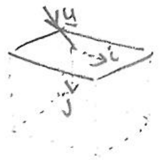
- In Cartesian coords, using summation convention, the component in ith dir^{n} is
$$\partial\rho u_i/\partial t = -\partial_j p\delta_{ij} + \rho g_i - \rho u_j\partial_j u_i - u_i\partial_j\rho u_j$$
where δ_{ij} = Kroenecker delta = 1 if $i=j$, 0 if not

NB $\underline{u}.\nabla\underline{u} = [u_x\partial u_x/\partial x + u_y\partial u_x/\partial y + u_z\partial u_x/\partial z, u_x\partial u_y/\partial x + \ldots]$ eg see hand-out

$\underline{u}\nabla.(\rho\underline{u}) = [u_x(\partial\rho u_x/\partial x + \partial\rho u_y/\partial y + \partial\rho u_z/\partial z), u_y(\partial\rho u_x/\partial x + \ldots]$

- Last two terms combine to $-\partial_j\rho u_i u_j$
$$\therefore \partial\rho u_i/\partial t = -\partial_j\sigma_{ij} + \rho g_i$$
where $\sigma_{ij} = \rho u_j u_i + p\delta_{ij}$ is <u>Stress tensor</u>

- More generally, σ_{ij} is the force per unit area in dir^n i acting on a surface whose normal is in dir^n j
  - <u>thermal pressure</u>, $p\delta_{ij}$, associated $\bar{w}$ random motions $\hat{\underline{u}}$ which are isotropic (diagonal elements, p is scalar)
  - <u>ram pressure</u>, $\rho u_j u_i$, associated $\bar{w}$ bulk motion $\underline{u}$ (3×3 matrix)



ρu_j = mass hitting surface $\bar{w}$ normal in j dir^n per unit time due to bulk flow

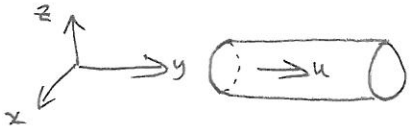
$\rho u_j u_i$ = momentum flux per unit time on that surface in the i dir^n

$\partial_j(\rho u_j u_i)$ = change in ith component of momentum due to mismatch in "i-momentum" carried through a cell in each of 3 orthogonal dir^{n}s

eg in x dir^n $= \frac{\partial}{\partial x}(\rho u_x^2) + \frac{\partial}{\partial y}(\rho u_x u_y) + \frac{\partial}{\partial z}(\rho u_x u_z)$

  - <u>other</u> any microscopic effects giving such a stress can be included as a pressure (eg. mag. fields).

<u>EG: flow down a pipe</u>



All surfaces experience momentum flux due to thermal pressure
But only surfaces normal to flow experience ram pressure

$$\sigma_{ij} = \begin{pmatrix} p & 0 & 0 \\ 0 & p+\rho u^2 & 0 \\ 0 & 0 & p \end{pmatrix}$$

$\therefore$ Pressure on sides of pipe is p

end of pipe is $p + \rho u^2$

Gravity

- Is a <u>conservative force</u> $\rightarrow$ work done on closed loop $\oint \underline{F} \cdot d\underline{l} = 0$
- By Stokes' theorem $\int_S \nabla \wedge \underline{F} \cdot d\underline{S} = 0$
- As "curl of grad is zero", conservative forces can be written $\underline{F} = \nabla \Phi$
  where Φ is a scalar potential whose gradient gives magnitude of force
- Define <u>gravitational potential</u> Ψ s.t. grav. accel. $\underline{g} = -\nabla \Psi$
  (-ve so that force acts in dir^n of diminishing potential)
- Consider work required to take unit mass from $\underline{r}$ to ∞ where $\underline{g}(\infty) \rightarrow 0$ as far from all masses

$$-\int_{\underline{r}}^{\infty} \underline{g} \cdot d\underline{l} = \int_{\underline{r}}^{\infty} \nabla \Psi \cdot d\underline{l} = \Psi(\infty) - \Psi(\underline{r})$$

independent of the path taken.
Note that only potential differences or potential gradients are important

- <u>EG</u> Let $\Psi = -GM/r$

$$\therefore \underline{g} = -\nabla \Psi = GM \nabla\left(\frac{1}{r}\right)$$
$$= -\frac{GM}{r^2}\left[\partial r/\partial x, \partial r/\partial y, \partial r/\partial z\right]$$

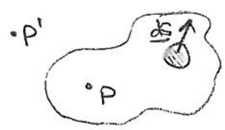
As $r^2 = x^2 + y^2 + z^2$, $\partial r/\partial x = x/r$ etc

$$\therefore \underline{g} = -\frac{GM}{r^3}[x, y, z] = -\frac{GM}{r^2}\hat{\underline{r}}$$

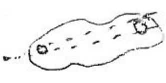
So given potential is that of a point mass M at origin

- For a point mass M at $\underline{r}'$: $\Psi = -GM/|\underline{r} - \underline{r}'|$
- For multiple point masses M_i each at $\underline{r}_i'$, since ∇ is a linear operator summing potentials $\rightarrow$ summing accelerations, so $\Psi = -\sum_i GM_i/|\underline{r} - \underline{r}_i'|$

Poisson's equation



- Consider a volume V bounded by surface S made of surface elements $d\underline{S}$
- Consider a point mass M at P that exerts grav. acc. $\underline{g} = -\frac{GM}{r^2}\hat{\underline{r}}$
- The quantity $\underline{g}\cdot d\underline{S} = -GM(d\underline{S}\cdot\hat{\underline{r}}/r^2) = -GM\,d\Omega$
  where $d\Omega = d\underline{S}\cdot\hat{\underline{r}}/r^2$ is solid angle subtended by $d\underline{S}$ at P
- Integrate over surface $\int_S \underline{g}\cdot d\underline{S} = -GM\Omega$
  where $\Omega = \int_S d\Omega$ depends on whether P is inside or outside surface



  – If P outside S: $\Omega = \int_S \hat{\underline{r}}\cdot d\underline{S}/r^2 = 0$ because every l.o.s. cuts surface twice with equal and opposite contributions
  $= \int_V \nabla\cdot\underbrace{(\hat{\underline{r}}/r^2)}_{0}\,dV = 0$ alternatively, use divergence theorem and use handout* to show this is zero

  – If P inside S, can't apply divergence theorem at P as singular, so split volume into small sphere around P of radius a, and the rest which has zero contribution by above arguments
  $\therefore\ \Omega = 4\pi a^2/a^2 = 4\pi$

- So $\int_S \underline{g}\cdot d\underline{S} = -4\pi GM$ if P is inside S, 0 if P is outside
- If there is mass distributed through space, it is only that within S that matters
  $\therefore\ \underline{\int_S \underline{g}\cdot d\underline{S} = -4\pi G\int_V \rho\,dV} = -4\pi G \times$ mass enclosed by Gaussian surface

- But $\int_S \underline{g}\cdot d\underline{S} = \int_V \nabla\cdot\underline{g}\,dV$ (divergence theorem)
  $\therefore\ \int_V \nabla\cdot\underline{g} + 4\pi G\rho\,dV = 0$
  $\therefore\ \nabla\cdot(-\nabla\Psi) + 4\pi G\rho = 0$ (as holds for all V)
  $\therefore\ \underline{\nabla^2\Psi = 4\pi G\rho}$

$*\ \nabla\cdot(A_r\hat{\underline{r}}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r)$

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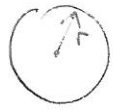
Eg: Spherically symmetric mass dist^n $\rho(r)$



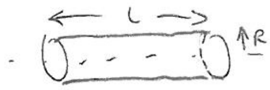
Choose Gaussian surface as sphere radius r

By symmetry $\underline{g}$ is radial w̄ same magnitude everywhere on surface $-|g|\hat{\underline{r}}$

$\therefore \int_S \underline{g}\cdot d\underline{S} = -4\pi r^2 |g|$ and this $= -4\pi G \int_0^r 4\pi \rho(r') r'^2 dr'$

$\therefore |g| = \frac{GM(r)}{r^2} = \frac{G}{r^2}\int_0^r 4\pi\rho(r') r'^2 dr'$ where $M(r)$ = enclosed mass

Eg Infinite cylindrically symmetric mass dist^n $\rho(R)$



Choose Gaussian surface as cylinder radius R, length L

By symmetry g is radial $= -|g|\hat{\underline{R}}$ on surface of cylinder

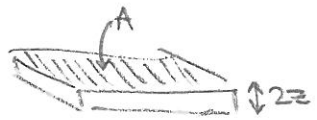
As $\underline{g}\cdot d\underline{S}$ is zero on flat ends $\int_S \underline{g}\cdot d\underline{S} = -2\pi R L |g|$

and this $= -4\pi G \int_0^R 2\pi R' L \rho(R') dR'$

$\therefore |g| = \frac{2G}{R}\int_0^R 2\pi R' \rho(R') dR'$

→ depends only on interior mass distribution

Eg: Infinite planar distribution symmetric about $z=0$, $\rho(z)$



Choose Gaussian surface as box of area A height $2z$

By symmetry $\underline{g}$ is vertical $= -|g|\hat{\underline{z}}$ on top and bottom

and $\underline{g}\cdot d\underline{S}$ is zero on sides $\therefore \int_S \underline{g}\cdot d\underline{S} = -2A|g|$

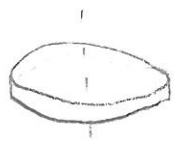
and this $= -4\pi G A \int_{-z}^{z} \rho(z') dz'$

$\therefore |g| = 4\pi G \int_0^z \rho(z') dz'$

Note that $|g|$ is constant w̄ height above z_{max} if $\rho(z > z_{max}) = 0$

Eg: Finite axisymmetric disk about $z=0$, $\rho(R,z)$

No surfaces where $\underline{g}\cdot d\underline{S}$ vanishes by symmetry eg.



$\therefore$ $\underline{g}$ locally does not just depend on enclosed mass.

→ galactic dynamics course

Spherically symmetric potential

$$\underline{g} = -\left(\frac{G}{r^2}\int_0^r 4\pi r'^2 \rho(r') dr'\right)\hat{\underline{r}} = -\nabla\Phi \overset{\text{from hand-out}}{=} -(\partial\Phi/\partial r)\,\hat{\underline{r}} = -(d\Phi/dr)\,\hat{\underline{r}}$$

$$\therefore \int_\infty^r d\Phi = \int_\infty^r \frac{G}{r''^2}\int_0^{r''} 4\pi r'^2 \rho(r') dr'\, dr''$$

Set $\Phi(\infty) = 0$ and integrate by parts

$$\therefore \Phi(r) = \Big[-\frac{G}{r''}\underbrace{\int_0^{r''} 4\pi r'^2 \rho(r') dr'}_{M(r'')}\Big]_\infty^r + \int_\infty^r 4\pi G r'' \rho(r'') dr''$$

$$= \cancelto{0}{\frac{GM(\infty)}{\infty}} - \frac{GM(r)}{r} + \int_\infty^r 4\pi G r'' \rho(r'') dr''$$

So, while mass outside r does not affect acceleration at r, it does affect potential at r, since we set $\Phi(\infty) = 0$ and adding more mass beyond r means more energy req. to move particle to ∞.

Gravitational potential energy

For a system of point masses, grav potential at $\underline{r}$ is $\Psi = -\sum_i \frac{GM_i}{|\underline{r}-\underline{r}_i|}$

Which is also $-$ energy req. to take unit mass from $\underline{r}$ to ∞

$\therefore$ Grav. pot. en. Ω = $-$energy req. to take all M_j to ∞

$$= \underbrace{-\frac{1}{2}}_{\text{only count each pair once}} \sum_{j\neq i} \underbrace{\sum_i \frac{GM_iM_j}{|\underline{r}_j-\underline{r}_i|}}_{\text{energy to take } M_j \text{ to } \infty}$$

$$= \frac{1}{2}\sum_j M_j \Psi_j \leftarrow \text{potential at } \underline{r}_j$$

So for distributed masses $= \frac{1}{2}\int \rho(\underline{r})\Psi(\underline{r})\,dV$

If spherically symmetric $= \frac{1}{2}\int_0^\infty 4\pi r'^2 \rho(r')\Psi(r')\,dr'$

Integrate by parts $= \frac{1}{2}\left[\cancelto{0 \text{ as } M(0)=\Psi(\infty)=0}{\left[M(r')\Psi(r')\right]_0^\infty} - \int_0^\infty M(r')\frac{d\Psi}{dr'}\,dr'\right]$

But $d\Psi/dr = \frac{G}{r^2}M(r)$ $\therefore \Omega = -\frac{1}{2}\int_0^\infty \frac{GM(r')^2}{r'^2}\,dr'$

Integrate by parts $= -\frac{1}{2}\left[\cancel{\left[-\frac{G}{r'}M(r')^2\right]_0^\infty} + \int_0^\infty \frac{G}{r'}2M(r')\frac{dM}{dr'}\,dr'\right]$

$$= -G\int_0^\infty \frac{M(r')}{r'}\,dM(r')$$

(as expected as PE of spherical shell is $dM(r') \times \frac{GM(r')}{r'}$)

Moment of inertia

- Consider motion of cloud of particles (atoms / ✱s / galaxies)
- If a particle of mass m at $\underline{r}$ is acted on by force $\underline{F}$ $\rightarrow m\underline{\ddot{r}} = \underline{F}$
- Consider 2nd derivative of moments of inertia of particle about origin

$$\frac{1}{2} d^2(mr^2)/dt^2 = m\, d/dt(\underline{r} \cdot d\underline{r}/dt) = m\underline{r} \cdot \underline{\ddot{r}} + m\underline{\dot{r}}^2$$

$$= m\underline{\dot{r}}^2 + \underline{r} \cdot \underline{F}$$

- Sum over all particles

$$\frac{1}{2} d^2 I/dt^2 = 2T + \sum \underline{r} \cdot \underline{F}$$

where $I = \sum mr^2$, T = total kinetic energy, $V = \sum \underline{r} \cdot \underline{F}$ = Virial

Virial

- If $\underline{F}_{ji}$ is force on m_j from m_i then $\underline{F}_{ij} = -\underline{F}_{ji}$

  $\therefore$ this pair contributes $\underline{F}_{ij} \cdot \underline{r}_{ij}$ to V where $\underline{r}_{ij} = \underline{r}_i - \underline{r}_j$ to avoid double-counting

  $\therefore$ in absence of external force field $V = \frac{1}{2}\sum_{j \neq i} \sum_i \underline{F}_{ij} \cdot \underline{r}_{ij}$

- If ideal gas laws apply s.t. coll= processes occur at $\underline{r}_i = \underline{r}_j$, only gravity matters for which $\underline{F}_{ij} = -(Gm_i m_j / r_{ij}^3)\underline{r}_{ij}$

$$\therefore V = -\frac{1}{2}\sum_{j \neq i} \sum_i Gm_i m_j / r_{ij}$$

$$= \Omega \quad \text{grav. pot. en. of cloud}$$

Virial theorem

In steady state I is constant.

$$\therefore 2T + \Omega = 0$$

Note that may need to decompose T (total kinetic energy) into that of the bulk flow (T_K) and that stored in particle motions in rest frame of fluid element, i.e. the thermal, or internal, energy (ε).

Equation of state

- To solve mom. eq., need to know pressure in fluid.
  e.o.s. is rel$^=$ betw. p and other thermodynamic properties that can be defined for collisional fluid

Ideal gases

- Most fluids in astronomy are approx. ideal
  - → at microscopic level well described by kinetic theory where point-like particles have isotropic random motions set by fluid temperature
  - → particles interact, but infrequently s.t. internal energy is (mostly) in kinetic energy (rather than PE. of interparticle interactions)
  - → internal energy / unit mass $\underline{\varepsilon = \varepsilon(T)}$ (only a f$^=$ of temperature)
  - → e.o.s.: $\underline{p = \frac{R_*}{\mu} \rho T}$

    where R_* = modified gas constant $= 8300\ \mathrm{J/K/kg} = k/m_H$ ← Boltzmann's const, ← Mass of H atom

    μ = mean molecular weight

    NB $p = nkT$ s.t. $\rho = n\mu m_H$ ← #/vol.
- Breaks down at high ρ or low T s.t. energy of interparticle interactions or finite particle size important and $\varepsilon = \varepsilon(\rho, T)$ and e.o.s. modified, eg. giant planet interiors
- Also high ρ in neutron ✱s, WDs ⇒ dist$^=$ of particle energies set by quantum mechanics ∴ ideal gas e.o.s. not appropriate

Barotropic e.o.s.

- Generally to get p need T from energy equ$^=$ that determines internal energy of fluid by considering heat gain / loss mechanisms.
- But energy equ$^=$ not needed if e.o.s. is barotropic → $p = p(\rho)$

(i) Isothermal e.o.s

- T is const. ∴ $p \propto \rho$ (for ideal gas)
- Requires heating and cooling processes to control T to narrow range and for thermal $=^m$ (heating balances cooling) to be attained on timescales shorter than flow timescales

(ii) Adiabatic e.o.s.

- Fluid elements that are thermally isolated from surroundings undergoing reversible changes have an e.o.s. $p = K\rho^\gamma$ where K = const
- NB this is different to an isentropic fluid in which same e.o.s. applies, but all elements in fluid have same K.

Thermodynamics review

Reversible changes = energy is conserved (i.e., no viscous or dissipative processes converting kinetic energy into heat

1st law of thermodynamics $\underbrace{đQ}_{\text{heat absorbed from surroundings}} = \underbrace{d\mathcal{E}}_{\text{change in internal energy}} + \underbrace{p\,dV}_{\text{work done in changing vol by } dV}$ } per unit mass — for reversible changes (< if not)

NB $đ$ is Pfaffian operator meaning change dep. on route through thermodynamic phase space, and $V = 1/\rho$

For ideal gas $\mathcal{E} = \mathcal{E}(T) \therefore d\mathcal{E} = \left(\frac{d\mathcal{E}}{dT}\right) dT$

Define $C_\alpha = (dQ/dT)_{\alpha = \text{const}}$ = specific heat capacity at const. α

$\therefore C_V = d\mathcal{E}/dT$

Subst. e.o.s. too

$$đQ = C_V\,dT + \frac{R_*}{\mu}\rho T\, d\left(\frac{1}{\rho}\right)$$

Adiabatic changes = no heat transfer across boundary $\therefore đQ = 0$

$\therefore$ for an ideal gas undergoing a reversible adiabatic change:

$$\frac{C_V}{R_*/\mu}\frac{dT}{T} = \frac{d\rho}{\rho}$$

$$\therefore \quad \rho \propto T^{C_V/(R_*/\mu)}$$

and $\quad p \propto T^{1 + C_V/(R_*/\mu)}, \quad p \propto \rho^{1 - \frac{R_*/\mu}{C_V}}$

Rewrite using C_p

ideal gas e.o.s $\Rightarrow p\,dV + V\,dp = \frac{R_*}{\mu}dT$

so reversible 1st law $\Rightarrow đQ = \underbrace{\left(C_V + \frac{R_*}{\mu}\right)}_{C_p} dT - V\,dp$

Defining $\gamma = C_p/C_V = 1 + (R_*/\mu)/C_V$

$$\rho \propto T^{\frac{1}{\gamma - 1}}, \quad p \propto T^{\frac{\gamma}{\gamma-1}}, \quad p \propto \rho^\gamma$$

What is γ? depends on # of ways fluid can store kinetic energy

$C_V = N\frac{1}{2}(R_*/\mu) \quad \therefore \gamma = 1 + 2/N$

where N = # of degrees of freedom of fluid

eg. monatomic gas $N = 3$ (one for each orthogonal direction)

diatomic gas $N = 5$ (" + 2 rotational modes)

though depends on T as at low T fluid can be below threshold for excitation of particular rotation and vibration modes

Other ideal gas relations

$$\mathcal{E} = C_V T = \frac{1}{\gamma - 1}\frac{p}{\rho}$$

Energy equation

- In absence of dissipative proc., 1st law of therm can be written

$$D\mathcal{E}/Dt = DW/Dt - dQ/dt$$

rate of change of internal energy = work done − energy lost by cooling to surroundings | per unit mass

where $DW/Dt = -p\, D(\frac{1}{\rho})/Dt = \frac{p}{\rho^2} D\rho/Dt$

- Define a cooling function $\dot{Q}_{cool}$ (per unit mass)

$$\therefore D\mathcal{E}/Dt = \frac{p}{\rho^2} D\rho/Dt - \dot{Q}_{cool}$$

- Define $E = \rho(\underbrace{\tfrac{1}{2}u^2}_{\text{kinetic}} + \underbrace{\mathcal{E}}_{\text{internal}} + \underbrace{\Psi}_{\text{gravitational potential energy}})$ = total energy per unit volume

$$\therefore \overset{①}{DE/Dt} = \overset{②}{\frac{E}{\rho} D\rho/Dt} + \rho(\overset{③}{\underline{u}\cdot D\underline{u}/Dt} + \overset{④}{D\Psi/Dt} + \overset{⑤}{\frac{p}{\rho^2} D\rho/Dt} - \dot{Q}_{cool})$$

①: $\partial E/\partial t + \underline{u}\cdot\nabla E$ (by defn of D/Dt)

②: $-\frac{E}{\rho}\rho\nabla\cdot\underline{u}$ (by cons. of mass)

③: $\underline{u}\cdot(-\nabla p - \rho\nabla\Psi)$ (by cons. of momentum) cancel

④: $\rho\,\partial\Psi/\partial t + \rho\underline{u}\cdot\nabla\Psi$ (by defn of D/Dt)

⑤: $-\rho\frac{p}{\rho^2}\rho\nabla\cdot\underline{u} = -p\nabla\cdot\underline{u}$ (by cons. of mass)

$$\therefore \partial E/\partial t + \underbrace{\underline{u}\cdot\nabla E + E\nabla\cdot\underline{u}}_{\nabla\cdot(E\underline{u})} + \underbrace{\underline{u}\cdot\nabla p + p\nabla\cdot\underline{u}}_{\nabla\cdot(p\underline{u})} = -\rho\dot{Q}_{cool} + \rho\,\partial\Psi/\partial t$$

$$\therefore \underline{\partial E/\partial t + \nabla\cdot((E+p)\underline{u}) = -\rho\dot{Q}_{cool} + \rho\,\partial\Psi/\partial t} \quad \text{— usually } 0$$

Just need $E(\rho, T$ etc) and $\dot{Q}_{cool}$

Heating / cooling mechanisms

• **Cosmic rays** high E particles (protons) ionise atoms at rate / vol $\propto \rho \times$ cosmic ray flux → free electrons → heat

• **Conduction** particles' random motion allows internal energy to be transferred from hot to cold regions (by warm ptcles colliding w cooler ones)

heat flux / area $= -K\nabla T$

K = thermal conductivity $\sim C_V (mkT/3)^{1/2}/\sigma$

rate of change of energy / vol. $\propto K\nabla^2 T$

(generally small, but equalises temperature damping thermal instability, unless prevented by mag fields)

• **Convection** (eg. stars) is instability set up by gravity and temperature gradient

large scale fluid motion, but net effect can be modelled as transfer hot → cold.

• **Radiation** energy carried by photons

– **Optically thick** emitted photons scattered and reabsorbed / scattered locally

diffusion problem w heat transfer hot → cold until photons reach photosphere then escape to ∞

– **Optically thin** energy loss per unit vol per time by

+ **recombination** $n_e n_p kT \underbrace{\beta(H^0,T)}_{\text{combin}^= \text{ of recombin}^= \text{ cross section and vel. distr}^=}$ (of free electrons onto H)

+ **free-free rad$^=$** (from e^- accelerated by ptcle of charge Z

$\propto Z^2 n_e n_p T^{1/2}$

+ **collisionally excited atomic line rad$^=$** e^- coll$^=$s of atoms in ground state → excite to low lying en. level, returning to ground → emission of photon w energy χ

$\propto n_{ion} n_e e^{-\chi/kT} \chi T^{-1/2}$

n_{ion}: sometimes not enough en. to excite H (O^+, N^+)

determine from balance betw. ionis$^=$ (from photons or coll$^=$s) and recombin$^=$

+ **coll$^=$ betw molecules** (KE → rot$^=$ or internal vibr$^=$, important $< 100K$, CO, O_2)

So... $\dot{Q}_{cool} = A\rho T^{\alpha} - H$

ρ: density enhancements precipitate cooling

H: cosmic rays

Hydrostatic equilibrium

Static $\rightarrow$ $\underline{u} = 0$ everywhere

Equilibrium $\rightarrow \partial/\partial t = 0$

Basic equations

Continuity: $\partial\rho/\partial t + \nabla.(\rho\underline{u}) = 0$ trivial

Momentum: $\rho\partial\underline{u}/\partial t + \rho\underline{u}.\nabla\underline{u} = -\nabla p - \rho\nabla\Phi$

$\rightarrow \underline{\frac{1}{\rho}\nabla p = -\nabla\Phi}$

Poisson's eq: $\underline{\nabla^2\Phi = 4\pi G\rho}$

Readily solved for barotropic eqn of state.

$\boxed{\text{Eg}}$ Isothermal slab

Consider static isothermal slab, infinite in x and y and symmetrical about $z=0$ supported by gas pressure, under its own self gravity, no external forces

Isothermal $\rightarrow p = \left(\frac{R_*}{\mu}T\right)\rho = A\rho$

Geometry $\rightarrow \nabla = \partial/\partial z$ and $\rho = \rho(z)$, $\Phi = \Phi(z)$

$\therefore$ Mom eq $\rightarrow \frac{1}{\rho}A\,d\rho/dz = -d\Phi/dz$

$\therefore A[\ln\rho]_{\rho_0}^{\rho} = [-\Phi]_{\Phi_0}^{\Phi}$ where $\rho_0 = \rho|_{z=0}$, $\Phi_0 = \Phi|_{z=0}$

$\therefore \Phi = \Phi_0 - A\ln(\rho/\rho_0)$

and $\underline{\rho = \rho_0 e^{-[(\Phi-\Phi_0)/A]}}$

$\therefore$ Poisson's eq $\rightarrow d^2\Phi/dz^2 = 4\pi G\rho_0 e^{-[(\Phi-\Phi_0)/A]}$

Let $X = -(\Phi-\Phi_0)/A$ and $Z = \sqrt{\frac{2\pi G\rho_0}{A}}\,z$

$\therefore d^2X/dZ^2 = -2e^{X}$ w b.c. $X = dX/dZ = 0$ @ $Z=0$

Multiply by dX/dZ s.t. $\frac{d}{dZ}\left[\frac{1}{2}(dX/dZ)^2\right] = \frac{d}{dZ}\left[-2e^{X}\right]$

Integrate to get $(dX/dZ)^2 = 4(1 - e^{X})$ from b.c.

$\therefore \underline{2Z} = I = \int_0^X (1-e^{X})^{-1/2}\,dX$

Let ~~$s = e^{X}$ $I = \int_1^s \frac{2}{s(1-s^2)^{1/2}}\,ds$~~

$e^{X} = \sin^2\theta$ $\quad = \int_{\pi/2}^{\theta}\frac{2}{\sin\theta}\,d\theta$

$t = \tan\theta/2$ $\quad = \int_1^t \frac{2}{t}\,dt = \underline{2\ln t}$

$\therefore \rho = \rho_0 e^{X} = \rho_0\sin^2\theta = \rho_0\frac{4t^2}{(1+t^2)^2} = \rho_0\left[\frac{2}{e^{-Z}+e^{Z}}\right]^2 = \underline{\rho_0/\cosh^2\left[\sqrt{\frac{2\pi G\rho_0}{A}}\,z\right]}$

Eg2 Isothermal atmosphere

Consider Earth's atmos. as plane in const $\underline{g} = -g\hat{\underline{z}}$

Isothermal $\Rightarrow p = \left(\frac{R_*}{\mu}T\right)\rho$

Geometry $\Rightarrow \nabla = \partial/\partial z$ and $\rho = \rho(z)$

$\therefore$ Mom. eq. $\Rightarrow \frac{1}{\rho}\left(\frac{R_*}{\mu}T\right) d\rho/dz = -g$

$$\therefore \underline{\rho = \rho_0 e^{-\left(\frac{\mu g}{R_* T}\right)z}} \qquad \text{where } \rho_0 = \rho|_{z=0}$$

As $\mu \simeq 28$, $T \simeq 300$K then $\rho/\rho_0 \simeq e^{-z/h}$ where $h = 9$ km

eg. at observatories at 5000 m, density of atmos is ~60% that at sea level.

Eg3 Stars as self gravitating polytropes

Consider spherical system (eg. no rotⁿ to break symmetry)

Geometry $\Rightarrow \nabla = d/dr$, $p = p(r)$, $\Psi = \Psi(r)$

$\therefore$ Mom. eq $\Rightarrow dp/dr = -\rho\, d\Psi/dr$

As $\rho > 0$, p is a monotonic function of Ψ

$\therefore p = p(\Psi)$ (ie. unique value of Ψ corresponds to unique value of p)

$$\therefore dp/dr = (dp/d\Psi)(d\Psi/dr)$$

$$\therefore \rho = -(dp/d\Psi) = \rho(\Psi)$$

Thus $p = p(\rho)$ and non-rotating stars are <u>barotropes</u> and surfaces of constant p, ρ and Ψ coincide.

<u>Polytropes</u> parametrise barotropic e.o.s. as $p = K\rho^{1+\frac{1}{n}}$ where n ~ polytropic index

NB: This is only an approximation to $p(\rho)$ which is valid over some range of radii (and reasonable for entire interior)

$\therefore 1 + \frac{1}{n}$ is not necessarily equal to $\gamma = C_p/C_v$, unless star was isentropic

$$\text{Mom. eq.} \Rightarrow -d\Psi/dr = \frac{1}{\rho}\,dp/dr = \frac{1}{\rho}\frac{dp}{d\rho}\frac{d\rho}{dr} = \frac{K(n+1)}{n}\rho^{\frac{1}{n}-1}\frac{d\rho}{dr}$$

Integrate from surface where $\rho = 0$, $\Psi = \Psi_T$

$$\therefore \rho = \left[(\Psi_T - \Psi)/(n+1)K\right]^n$$

If $\Psi = \Psi_c$, $\rho = \rho_c = \left[\frac{\Psi_T - \Psi_c}{(n+1)K}\right]^n$ at centre

$$\underline{\rho = \rho_c\left[\frac{\Psi_T - \Psi}{\Psi_T - \Psi_c}\right]^n}$$

Poisson's eq $\Rightarrow \nabla^2\Psi = 4\pi G\rho_c\left[\frac{\Psi_T - \Psi}{\Psi_T - \Psi_c}\right]^n$

Let $\theta = \left(\frac{\Psi_T - \Psi}{\Psi_T - \Psi_c}\right) \Rightarrow \nabla^2\theta = -\frac{4\pi G\rho_c}{\Psi_T - \Psi_c}\theta^n$

Now $\nabla^2\theta = \frac{1}{r^2}\frac{d}{dr}(r^2\, d\theta/dr)$

Let $\xi = \left[\frac{4\pi G\rho_c}{\Psi_T - \Psi_c}\right]^{1/2} r \Rightarrow \underline{\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{d\theta}{d\xi}\right] = -\theta^n}$

This is Lane-Emden eqⁿ of index n which has b.c. of $\theta = 1$ and $d\theta/d\xi = 0$ @ $\xi = 0$

(ie., zero gravitational accelerⁿ at centre, ok unless there's a point mass at centre)

Solutions to the Lane-Emden equation $\left(\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \text{ w } \theta=1, d\theta/d\xi = 0 \text{ @ } \xi=0\right)$

$\boxed{n=0}$ $\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -1$

$\therefore \xi^2\frac{d\theta}{d\xi} = -\xi^3/3 - \cancel{C}$ ↗ 0 from b.c.

$\therefore \theta = \cancel{D}^{\,1 \text{ from b.c.}} - \xi^2/6$

As surface is at $\theta = 0$, this occurs at $\xi = \sqrt{6}$

$\boxed{n=1}$ Rewrite eqn w $\chi = \theta\xi$

$\therefore d\theta/d\xi = \frac{1}{\xi}d\chi/d\xi - \frac{1}{\xi^2}\chi$

$\therefore \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi\frac{d\chi}{d\xi} - \chi\right) = \frac{1}{\xi}\frac{d^2\chi}{d\xi^2}$

$\therefore \frac{d^2\chi}{d\xi^2} = -\xi(\chi/\xi)^n$

For $n=1$ $\quad = -\chi$

$\therefore \chi = A\,\text{SIN}(\xi + \cancel{\beta})$ ↗ 0 from b.c.

$\therefore \theta = \frac{\cancel{A}^{\,1 \text{ from b.c.}}}{\xi}\text{SIN}\,\xi$

So surface is at $\xi = \pi$

$\boxed{n=5}$ Rewrite eqn w $x = \xi^{-1}$

$\therefore \frac{1}{\xi^2}\frac{d}{d\xi}\left(\underbrace{\xi^2}_{x^{-2}}\underbrace{\frac{d\theta}{d\xi}}_{-x^2 d\theta/dx}\right) = x^4\, d^2\theta/dx^2 = -\theta^n$

Letting $t = \ln x$ and $\theta = (x/2)^{1/2} z$

$\therefore d^2\theta/dx^2 = \left(\frac{1}{2}\right)^{1/2}\left[x^{1/2}d^2z/dx^2 + x^{-1/2}dz/dx - \frac{1}{4}zx^{-3/2}\right]$

$= \left(\frac{1}{2}\right)^{1/2}x^{-3/2}\left[d^2z/dt^2 - \frac{1}{4}z\right]$

So the eqn for $n=5$ becomes $d^2z/dt^2 = \frac{1}{4}z(1-z^4)$ [NB x's cancel

Multiply by dz/dt $\therefore \frac{d}{dt}\left[\frac{1}{2}(dz/dt)^2\right] = \frac{1}{4}z(1-z^4)dz/dt$

$\therefore \frac{1}{2}(dz/dt)^2 = \frac{1}{8}z^2 - \frac{1}{24}z^6 + \cancel{D}$ ↗ 0 from b.c.

$\therefore \int\frac{dz}{z(1-z^4/3)^{1/2}} = -\frac{1}{2}\int dt$

Letting $\frac{1}{3}z^4 = \text{SIN}^2\phi$ gives $\text{TAN}\,\phi/2 = Ce^{-t}$

$\therefore \theta = \left(1 + \frac{1}{3}\xi^2\right)^{-1/2}$

So surface is at $\xi \to \infty$

$\boxed{n=\infty}$ $p = K\rho$ → isothermal gas sphere, $K = \left(\frac{R^*}{\mu}\right)T$

Need to rederive eqs:

Mom eq → $d\Phi/dr = -K\frac{1}{\rho}d\rho/dr$ the soln to which is $\ln\rho/\rho_c = \frac{1}{K}[\Phi_c - \Phi]$

Poisson's eq → $\frac{1}{r^2}\frac{d}{dr}\left[r^2\left(-K\frac{1}{\rho}\frac{d\rho}{dr}\right)\right] = 4\pi G\rho$

Letting $\rho = \rho_c e^{-\psi}$ (ie $\Phi_c = 0$, $\psi = \Phi/K$) and $r = a\xi$ where $a = \left[\frac{K}{4\pi G\rho_c}\right]^{1/2}$

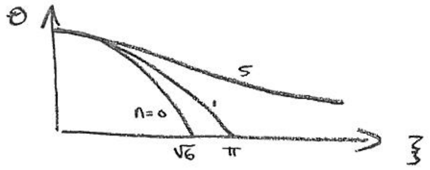
$\to \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi}{d\xi}\right) = e^{-\psi}$ w b.c. $\psi = d\psi/d\xi = 0$ at $\xi = 0$

NB letting $\rho \propto r^{-x}$ → $x = -2$

At large r, soln tends to $\rho \propto r^{-2}$, so mass in isothermal sphere → ∞ as $r \to \infty$ and must be truncated at some radius, and to exist in hydrostatic eq^m must be embedded in external medium of appropriate pressure

→ Bonnor-Ebert sphere

Scaling relations for polytropes



All stars w same "n" have same $\theta(\xi)$ profile
However different stars have different ρ_c or K and so different radii and masses, since these determine how dimensionless variables translate into physical quantities

- Remember $\rho = \rho_c \theta^n$

$$r = \left[\frac{(n+1)K}{4\pi G \rho_c^{1-1/n}}\right]^{1/2} \xi$$

- Total mass $M = \int_0^{r_{max}} 4\pi r^2 \rho \, dr$

$$= 4\pi\rho_c \left[\frac{(n+1)K}{4\pi G \rho_c^{1-1/n}}\right]^{3/2} \int_0^{\xi_{max}} \theta^n \xi^2 \, d\xi$$

Radius $R = \left[\frac{(n+1)K}{4\pi G \rho_c^{1-1/n}}\right]^{1/2} \xi_{max}$ — same for stars w same "n"

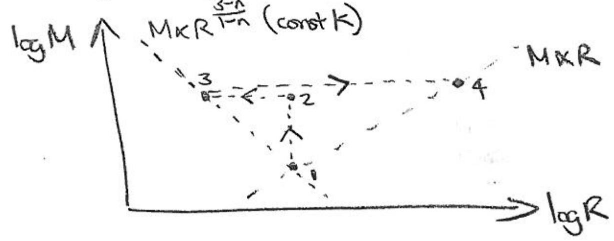
- If K = same for all stars (w same n), then $M \propto \rho_c^{\frac{1}{2}(\frac{3}{n}-1)}$, $R \propto \rho_c^{\frac{1}{2}(\frac{1}{n}-1)}$ $\therefore M \propto R^{\frac{3-n}{1-n}}$

Might expect this to be true for fully convective stars for which e.o.s. is nearly adiabatic
But for monatomic gas $\gamma = 5/3 \rightarrow n = 3/2$ $\therefore$ predicts $M \propto R^{-3}$ ie more massive stars are smaller! whereas $M \propto R$ is observed

- If T_c = same for all stars then:
  as $T_c = \frac{\mu}{R_*} P_c/\rho_c = \frac{\mu}{R_*} K \rho_c^{1/n}$
  $\rightarrow K \propto \rho_c^{-1/n}$
  $\therefore M \propto \rho_c^{-1/2}$, $R \propto \rho_c^{-1/2}$ and $M \propto R$

Response of star to mass gain

- Timescale to reach thermal equilibrium is the energy contents of the star divided by its luminosity
  ie, $t_{th} \sim GM^2/RL \sim 30\,\mathrm{Myr}$
- Timescale to reach hydrostatic equilibrium is the time for sound wave to cross the star
  ie, $t_h \sim R/c_s \sim 1\,\mathrm{day}$
- So response of star to mass gain is to evolve at const K on timescale t_h
  Then readjust "K" on timescale t_{th} so as to achieve thermal =^{m}



1 = initial state ($t=0$)
2 = state after perturbation ($t=0$)
3 = intermediate state ($t = t_h$)
4 = final state ($t = t_{th}$)

eg, for $n = \frac{3}{2}$, star shrinks then expands.

$\boxed{\text{EG 1}}$ • Consider star rotating at angular velocity Ω

Drop non-rotating mass ΔM onto star – what is new ang vel $\Omega + \Delta\Omega$?

• Angular momentum $J \propto MR^2\Omega$ is conserved

If $M \propto R^{\frac{3-n}{1-n}}$ then $J \propto M^{1+2(1-n)/(3-n)}\,\Omega$

$$\propto M^{\frac{5-3n}{3-n}}\,\Omega$$

$$\therefore \Delta J \propto \left(\frac{5-3n}{3-n}\right) M^{\frac{2-2n}{3-n}}\,\Omega\,\Delta M + M^{\frac{5-3n}{3-n}}\,\Delta\Omega = 0$$

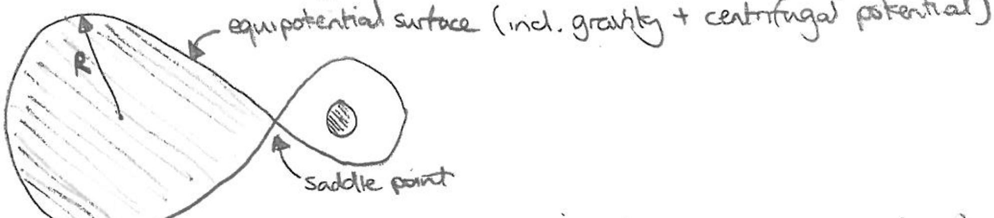
$$\therefore \Delta\Omega/\Delta M = -(\Omega/M)\left[\frac{5-3n}{3-n}\right]$$

So stars spins up if ~~3n~~ $n > 5/3$ and spins down if $n < 5/3$

(Physically larger $n \rightarrow$ squashier e.o.s. $\rightarrow$ star shrinks more $\rightarrow$ spins up to conserve J)

NB on longer timescales star spins down as $M \propto R \Rightarrow \Delta\Omega/\Delta M = -3\,\Omega/M$

$\boxed{\text{EG 2}}$ • Consider star in binary losing mass to companion through Roche Lobe overflow



• If star is larger than critical surface, mass flows onto companion through saddle point

eg. late in life as star expands as red giant, or if binary orbit shrinks

• Mass loss affects stellar radius s.t. if $M \propto R^{\frac{3-n}{1-n}}$

$$\Delta R/\Delta M \propto \frac{1-n}{3-n}\,M^{-2/(3-n)} \text{ – unimportant}$$

• So, if $1 < n < 3$ the star expands as it loses mass

Moreover mass loss means critical surface gets closer to star

$\rightarrow$ runaway process

Sound waves

Principal mechanism by which disturbances propagate in fluids
[eg. talking, shock waves, mass added to star – exception is Alfvenic waves in magnetised media]

(a) • Consider uniform medium w/ no external forces

$$\partial\rho/\partial t + \nabla.(\rho\underline{u}) = 0$$
$$\partial\underline{u}/\partial t + \underline{u}.\nabla\underline{u} = -\frac{1}{\rho}\nabla p$$

(b) Unperturbed state of fluid in equilibrium ($\partial/\partial t = 0$) in $p = p_0$, $\rho = \rho_0$, $\underline{u} = 0$

(c) • Consider a Lagrangian perturbation to this equilibrium

$$p = p_0 + \Delta p \quad , \quad \rho = \rho_0 + \Delta\rho \quad , \quad \underline{u} = \Delta\underline{u}$$

(i.e. a perturbation to fluid elements)

• The Eulerian pert^n, which is what should be substituted into above eqn^s, is

$$\delta X = \Delta X - \underline{\xi}.\nabla X = \Delta X - \underline{\xi}.\nabla X_0$$

i.e. the quantity X at fixed point P changes both because pert^n has changed local fluid element and because the pert^n may have moved a fluid element w/ different unperturbed value of X to point P

$\underline{\xi}$ = small displacement of fluid element at P due to pert^n

[NB derivation is analogous to that for D/Dt]

• Here, as unperturbed quantities are uniform ($\nabla X_0 = 0$) $\Rightarrow \delta X = \Delta X$

(d) • So to 1st order:

$$\partial\Delta\rho/\partial t + \rho_0\nabla.\Delta\underline{u} = 0$$
$$\partial\Delta\underline{u}/\partial t = -\frac{1}{\rho_0}\nabla\Delta p$$

If fluid is barotropic $\Delta p = \frac{dp}{d\rho}\Delta\rho$

$$\partial\Delta\underline{u}/\partial t = -\left(\frac{dp}{d\rho}\right)\frac{1}{\rho_0}\nabla\Delta\rho$$

(e) $\partial/\partial t \Rightarrow \partial^2\Delta\rho/\partial t^2 + \rho_0\nabla.\partial\Delta\underline{u}/\partial t = 0$

$$\underline{\partial^2\Delta\rho/\partial t^2 = \left(\frac{dp}{d\rho}\right)\nabla^2\Delta\rho}$$ which is a <u>wave equation</u>

(f) • Solution: $\Delta\rho = \Delta\rho_0 e^{i(kx - \omega t)}$ (ie. in 1D, $kx \Rightarrow \underline{k}.\underline{x}$ in higher dimensions)

Substituting into wave eqn $\Rightarrow \omega^2/k^2 = dp/d\rho$

As points of constant phase ($kx - \omega t = $ const) propagate at a speed ω/k

the wave travels at speed $c_s = \omega/k$

$$= \sqrt{dp/d\rho} \quad = \quad \underline{\text{Sound speed}}$$

k = wavenumber = $2\pi/\lambda$

ω = angular frequency = $2\pi\nu$

• All pert^ns obey same wave eqn^n, eg. substituting sol^n into mom. eq.

$$\rightarrow \Delta u = \left(\frac{\Delta\rho_0}{\rho_0}\right)\frac{\omega}{k}e^{i(kx-\omega t)} = \left(\frac{\Delta\rho}{\rho_0}\right)c_s$$

As $\Delta\rho/\rho_0 \ll 1$ the disturbance propagates faster than the speed of individual fluid elements
[eg. disturbance produced by my voice does not make air move at 800 mph!]

• Sound waves are longitudinal waves that propagate because a density pert^n → pressure gradient → acceler^n of fluid elements → fluid velocities that induce density pert^ns [longitudinal because pressure forces act ⊥ to surfaces in fluid]

• Stiff e.o.s. (high $dp/d\rho$) → large restoring force → rapid propagation

Isothermal vs adiabatic

- $c_s^2 = dp/d\rho|_T = \left(\frac{R_g}{\mu}\right)T$ occurs if density pert^ns pass heat to each other (eg by conduction or rad^n) faster than oscill^n timescale $\frac{1}{\omega}$ for fluid elements

- $c_s^2 = dp/d\rho|_S = \gamma\left(\frac{R_g}{\mu}\right)T$ if no heat exchange between fluid elements (ie heat transfer timescale $\gg \frac{1}{\omega}$) s.t. compressions heat up by pdV work

- thermal behaviour of sound waves not necessarily same as unperturbed medium eg sound waves in air behave adiabatically in isothermal atmosphere.

NB in this example sound waves are non-dispersive → c_s is not a function of ω
ie all frequencies propagate at same rate.
[the shape of a packet of waves of diff. freqs. is preserved as it propagates]

Stratified atmosphere

Same eqs as before but include const. gravity in z dir^n
x and y components of eq^{n}s unaffected, so horizontal sound waves unaffected

(a) $\rightarrow$ $\partial\rho/\partial t + \partial/\partial z\,(\rho u) = 0$
$\partial u/\partial t + u\,\partial u/\partial z = -\frac{1}{\rho}\,\partial p/\partial z - g$

(b) Equilibrium sol^n, assuming isothermal $p = A\rho$ where $A = R_* T/\mu$
$u_0 = 0$
$\rho_0(z) = \tilde{\rho} e^{-z/H}$ where $H = A/g$

(c) Perturb with Lagrangian pert^n, but need to I/P Eulerian perts into eq^{n}s
$\rho = \rho_0 + \delta\rho = \rho_0 + \Delta\rho - \underline{\xi}\cdot\nabla\rho_0$ [where $\underline{\xi}$ is Lagrangian displacement of fluid element due to pert^n]
$= \rho_0 + \Delta\rho - \xi_z\,\partial\rho_0/\partial z$ [since only considering z var^{n}s here]
$= \rho_0(1 - \xi_z/H + \Delta\rho/\rho_0)$ $\leftarrow$ Similarly $p = p_0 + \Delta p - \xi_z\,\partial p_0/\partial z$
$\underline{u} = \underline{u}_0 + \delta\underline{u} = \Delta\underline{u} - \underline{\xi}\cdot\nabla\underline{u}$ $\qquad = A\rho_0 + \Delta p - A\xi_z\,\partial\rho_0/\partial z$
$= \Delta u_z - \xi_z\,\partial\Delta u_z/\partial z$ $\qquad = A\rho_0(1 - \xi_z/H) + \Delta p$
$= \Delta u_z$

(d) Continuity $\rightarrow$ $\partial[\overset{\text{steady } =^m}{\rho_0} + \Delta\rho - \xi_z\,\partial\rho_0/\partial z]/\partial t + \partial/\partial z\,[(\rho_0 + \overset{\text{2nd order}}{\Delta\rho - \xi_z\,\partial\rho_0/\partial z})\,\Delta u_z] = 0$
$\therefore\ \partial\Delta\rho/\partial t - (\partial\xi_z/\partial t)(\partial\rho_0/\partial z) + \Delta u_z\,\partial\rho_0/\partial z + \rho_0\,\partial\Delta u_z/\partial z = 0$

__But__ $\Delta\underline{u} = D\underline{\xi}/Dt = \partial\underline{\xi}/\partial t + \underline{u}\cdot\nabla\underline{\xi}$
$\simeq \partial\underline{\xi}/\partial t$ [to first order as $u_0 = 0$]

This means above terms cancel
$\therefore\ \partial\Delta\rho/\partial t + \rho_0\,\partial\Delta u_z/\partial z = 0$

__Momentum__ $\rightarrow$ $\partial\Delta u_z/\partial t = -\rho_0^{-1}[1 - \xi_z/H + \Delta\rho/\rho_0]^{-1}\frac{\partial}{\partial z}[A\rho_0(1 - \xi_z/H) + \Delta p] - g$
$= -\rho_0^{-1}[1 + \xi_z/H + \Delta\rho/\rho_0]\left[-\frac{A}{H}\rho_0 + \frac{\partial\Delta p}{\partial z} + \frac{\xi_z}{H}\frac{A\rho_0}{H} - \frac{A\rho_0}{H}\frac{\partial\xi_z}{\partial z}\right] - g$
$= -\rho_0^{-1}\left[\frac{\partial\Delta p}{\partial z} + \left(\frac{A\rho_0}{H}\right)\left(-\underset{①}{1} + \underset{②}{\frac{\xi_z}{H}} - \underset{③}{\frac{\partial\xi_z}{\partial z}} - \underset{②}{\frac{\xi_z}{H}} - \underset{③}{\frac{\Delta\rho}{\rho_0}}\right)\right] - \underset{①}{g}$

① cancels because this is $=^m$ sol^n (when pert^n is zero)
② cancels straightforwardly
③ cancels from continuity as $\partial\Delta\rho/\partial t + \rho_0\,\partial(\partial\xi_z/\partial t)/\partial z = 0$
$\therefore\ \Delta\rho/\rho_0 = -\partial\xi_z/\partial z$ (multiplying through by ∂t)

$\therefore\ \partial\Delta u_z/\partial t = -\rho_0^{-1}\,\partial\Delta p/\partial z = -\frac{c_u^2}{\rho_0}\,\partial\Delta\rho/\partial z$ where $c_u^2 = \frac{dp}{d\rho}$ assumes fluid elements obey barotropic eq^n of state

(e) __Wave eq^n__ $\partial/\partial t$ (cont) is $\partial^2\Delta\rho/\partial t^2 + \rho_0\,\partial(\partial u_z/\partial t)/\partial z = 0$
$\therefore\ \partial^2\Delta\rho/\partial t^2 - c_u^2\,\partial^2\Delta\rho/\partial z^2 + \frac{c_u^2}{\rho_0}(\overset{-\rho_0/H}{\partial\rho_0/\partial z})(\partial\Delta\rho/\partial z) = 0$
$\therefore\ \underline{\partial^2\Delta\rho/\partial t^2 - c_u^2\,\partial^2\Delta\rho/\partial z^2 - \frac{c_u^2}{H}\,\partial\Delta\rho/\partial z = 0}$

(f) Dispersion relation

Let $\Delta\rho \propto e^{i(kz-\omega t)}$

then $-\omega^2 + c_u^2 k^2 - c_u^2 ik/H = 0$ is the dispersion rel^{n} betw angular freq. ω and wavenumber k

$\therefore\ k^2 - \frac{i}{H}k - \frac{\omega^2}{c_u^2} = 0$ ie. quadratic in k

$\therefore\ k = \frac{i}{2H} \pm \sqrt{\frac{\omega^2}{c_u^2} - \frac{1}{4H^2}}$

For $\omega > c_u/2H$

$Im(k) = \frac{1}{2H}$

$Re(k) = \pm\sqrt{\frac{\omega^2}{c_u^2} - \frac{1}{4H^2}} = \pm k_r$

$\therefore\ \Delta\rho \propto e^{-z/2H}\, e^{i[k_r z - \omega t]}$

Although this is exponentially decaying w z, note that $\rho_0 \propto e^{-z/H}$ so fractional var^{n} $\Delta\rho/\rho_0$ grows w height until $\Delta\rho/\rho_0 \to 1$, linear treatment breaks down and shocks forms (or more realistically energy is dissipated and wave is damped)

This looks like standard wave eqn, but k_r is a f^{n} of ω, so lines of const phase propagate at speed $V_{ph} = \frac{\omega}{k_r} = c_u\left[1 + \left(\frac{1}{2k_r H}\right)^2\right]^{1/2}$

$\therefore$ ~~different~~ frequencies travel at diff speeds $\rightarrow$ dispersive

eg. consider two frequencies $\omega + \delta\omega$ w corresponding $k_r + \delta k_r$ and $\omega - \delta\omega$, $k_r - \delta k_r$

the superposed amplitude is $\sin[kz - \omega t + (\delta k z - \delta\omega t)] + \sin[kz - \omega t - (\delta k z - \delta\omega t)]$

$= 2\sin(kz - \omega t)\cos(\delta k_r z - \delta\omega t)$

wave propagating at V_{ph} — modulation of wave amplitude propagating at $\delta\omega/\delta k_r$ = group velocity

$$V_{gr} = \frac{d\omega}{dk_r} = c_u\left[1 + \left(\frac{1}{2k_r H}\right)^2\right]^{-1/2}$$

NB $V_{gr} = V_{ph}$ in non-dispersive medium ($H \to \infty$)

For $\omega < c_u/2H$

$Re(k) = 0$ and $Im(k) > 0$

$\therefore\ \Delta\rho \propto e^{-kz}\, e^{i\omega t}$

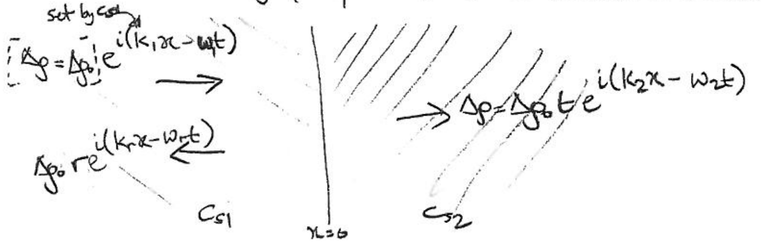
$\rightarrow$ all points oscillate in phase with an amplitude that depends on z
= standing wave

NB wavelength of travelling wave is same order as H, so change in properties of atmosphere over wavelength are significant.

see handout for general approach to wave propagation problems.

Transmission at interfaces [previously considered $c_s = \text{const}$]

Consider non dispersive medium w boundary at $x=0$ and sound wave travelling L to R & ang. freq. ω_1. Some fraction is transmitted, some reflected:



At boundary ($x=0$), accelerations are finite, so $\omega_2 = \omega_r = \omega_1 = \omega$ $\left[\text{NB } -\omega_1^2 e^{-i\omega_1 t} - r\omega_r^2 e^{-i\omega_r t} = -t\omega_2^2 e^{-i\omega_2 t}\right]$

Reflected wave travels at c_{s1} in negative x dir^n, so $k_r = -k_1$

Amplitude at ($x=0$) is continuous, so $1 + r = t$

Derivative at ($x=0$) is continuous, so $k_1(1-r) = k_2 t$

$$\therefore \quad t = 2k_1/(k_1+k_2)$$
$$r = (k_1-k_2)/(k_1+k_2)$$

NB $k_i = \omega/c_{si}$

- So if $c_{s2} > c_{s1}$, $k_2 < k_1$, so $r > 0$ and reflected wave is in phase w incident wave
  $c_{s2} \ll c_{s1}$, $k_2 \gg k_1$, so $t \to 0$ ($\therefore$ difficult to excite disturbances in cold GMC through sound waves)

- Kinetic energy flux in wave is $\propto \rho(\delta u)^2 c_s$
  Approximate $\sim \frac{p}{c_s}(\delta u)^2$
  Since p is constant across boundary
  $KE_1 \propto \frac{1}{c_{s1}} \times 1^2 \propto k_1$
  $KE_t \propto \frac{1}{c_{s2}} t^2 \propto k_2 (2k_1)^2/(k_1+k_2)$
  $KE_r \propto \frac{1}{c_{s1}} r^2 \propto k_1(k_1-k_2)^2/(k_1+k_2)$

$\rightarrow KE_t + KE_r = KE_1$

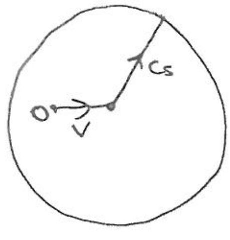
Shocks

Disturbances propagate at speed c_s relative to fluid (as pert^n was Lagrangian)

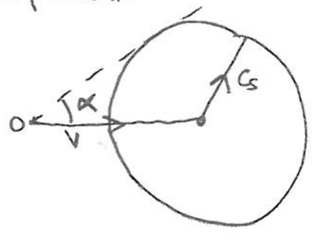
Consider observer O at source of spherical disturbance into fluid flowing at v

The velocity of disturbance relative to observer:

Subsonic



Supersonic



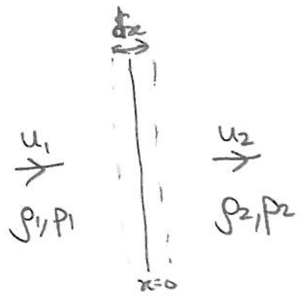
Disturbance can propagate 4π sr if subsonic

" " " into Mach cone defined by α if supersonic

$\therefore$ disturbances cannot propagate upstream from obstacle in supersonic flow, so flow is undisturbed until it reaches obstacle where properties change discontinuously in shock

Rankine-Hugoniot relations

Consider shock at $x=0$, width δx, normal in x-dir^n



(RH1) Continuity $\boxed{\hat{x}}$ $\partial\rho/\partial t + \partial/\partial x(\rho u_x) = 0$

Integrate over layer of thickness dx

$$\partial/\partial t \int \rho\, dx + (\rho u_x)_{dx/2} - (\rho u_x)_{-dx/2} = 0$$

As mass doesn't accumulate $\underline{\rho_1 u_1 = \rho_2 u_2}$

NB $\partial(\rho u_i)/\partial t = -\partial_j(\rho u_j u_i + p\,\delta_{ij}) + \rho\,\partial_i \Psi$

(RH2) Momentum $\boxed{\hat{x}}$ $\partial(\rho u_x)/\partial t = -\partial/\partial x(\rho u_x^2 + p) - \rho\,\partial\Psi/\partial x$

Integrate over dx, noting that Ψ is continuous and that continuity means LHS = 0

$$\therefore\ \underline{p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2}$$

ie sum of ram and thermal pressures is constant

NB $\boxed{\hat{y}}$ component, if there had been an additional u_{y1}, u_{y2} components, is $\rho_1 u_1 u_{y1} = \rho_2 u_2 u_{y2} \rightarrow u_{y1} = u_{y2}$ (and likewise for z)

so easiest to do problem (as above) in frame moving at u_y, u_z.

(RH3) Energy If shock is adiabatic, so $\dot{Q}_{cool} = 0$

$$\partial E/\partial t + \partial/\partial x\,[(E+p)u_x] = \rho\,\partial\Psi/\partial t$$

Integrate over dx, noting Energy doesn't accumulate

$$(E_1 + p_1)u_1 = (E_2 + p_2)u_2$$

As $E = \rho(\tfrac{1}{2}u^2 + \mathcal{E} + \Psi)$ (then $\div$ continuity, noting Ψ is continuous)

$$\therefore\ \underline{\tfrac{1}{2}u_1^2 + \mathcal{E}_1 + p_1/\rho_1 = \tfrac{1}{2}u_2^2 + \mathcal{E}_2 + p_2/\rho_2}$$

[showing how KE can be converted into enthalpy]

$\hookrightarrow = \mathcal{E} + p/\rho$

For ideal gas $\mathcal{E} = C_V T = \frac{C_V}{(R_0/\mu)} P/\rho = \frac{1}{\gamma-1} P/\rho$

So for an adiabatic shock: $\frac{1}{2} u_1^2 + \left(\frac{\gamma}{\gamma-1}\right) p_1/\rho_1 = \frac{1}{2} u_2^2 + \left(\frac{\gamma}{\gamma-1}\right) p_2/\rho_2$

Combine to get ρ_2/ρ_1

If (RH1) $\rho_1 u_1 = j$ then mom (RH2) is $p_1 + j^2/\rho_1 = p_2 + j^2/\rho_2$

$$\therefore j^2 = (p_2 - p_1)\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)^{-1}$$

energy (RH3) is $\frac{1}{2} j^2/\rho_1^2 + \left(\frac{\gamma}{\gamma-1}\right)\frac{p_1}{\rho_1} = \frac{1}{2} j^2/\rho_2^2 + \left(\frac{\gamma}{\gamma-1}\right)\frac{p_2}{\rho_2}$

$$\therefore \frac{1}{2}(p_2 - p_1)\underbrace{\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)^{-1}\left(\frac{1}{\rho_1^2} - \frac{1}{\rho_2^2}\right)}_{\frac{1}{\rho_1} + \frac{1}{\rho_2}} = \left(\frac{\gamma}{\gamma-1}\right)\left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right)$$

$$\therefore \frac{1}{\rho_2}\left[p_2\left(\frac{1}{2} - \frac{\gamma}{\gamma-1}\right) - \frac{1}{2}p_1\right] = \frac{1}{\rho_1}\left[p_1\left(\frac{1}{2} - \frac{\gamma}{\gamma-1}\right) - \frac{1}{2}p_2\right]$$

$$\therefore \rho_2/\rho_1 = u_1/u_2 = \frac{(\gamma+1)p_2 + (\gamma-1)p_1}{(\gamma+1)p_1 + (\gamma-1)p_2}$$

So in limit for strong shocks (for which $p_2 \gg p_1$ (neglect upstream pressure)

$$\rightarrow \rho_2/\rho_1 = \frac{\gamma+1}{\gamma-1} = 4 \text{ for } \gamma = 5/3$$

This is the maximum possible density contrast (as a larger thermal pressure behind the shock prevents it from being compressed too much).

Rewrite in terms of Mach # $M = u/c_s$

As adiabatic $c_s^2 = \gamma p/\rho$

(RH1) $\rightarrow \rho_1 M_1 c_{s1} = \rho_2 M_2 c_{s2}$

(RH2) $\rightarrow \rho_1 c_{s1}^2/\gamma + \rho_1 M_1^2 c_{s1}^2 = \rho_2 c_{s2}^2/\gamma + \rho_2 M_2^2 c_{s2}^2$ [just substituting for p and u^2]

$\therefore M_1^{-1} c_{s1}\left[M_1^2 + \frac{1}{\gamma}\right] = M_2^{-1} c_{s2}\left[M_2^2 + \frac{1}{\gamma}\right]$ [eliminate ρ by dividing by (RH1) (LHS by LHS, RHS by RHS)]

(RH3) $\rightarrow \cancel{\tfrac{1}{2}} M_1^2 c_{s1}^2 + \left(\frac{2}{\gamma-1}\right) c_{s1}^2 = \cancel{\tfrac{1}{2}} M_2^2 c_{s2}^2 + \left(\frac{2}{\gamma-1}\right) c_{s2}^2$ ["]

$\therefore \left(M_1^2 + \frac{2}{\gamma-1}\right) M_1^2 \left(M_1^2 + \frac{1}{\gamma}\right)^{-2} = \left(M_2^2 + \frac{2}{\gamma-1}\right) M_2^2 \left(M_2^2 + \frac{1}{\gamma}\right)^{-2}$ [eliminate c_s by dividing by (RH2)^2 "]

Expanding out:

$$(M_1^2 - M_2^2)\left[2M_1^2 M_2^2\left(\frac{1}{\gamma-1} - \frac{1}{\gamma}\right) - \frac{1}{\gamma^2}(M_1^2 + M_2^2) - \frac{2}{\gamma^2(\gamma-1)}\right] = 0$$ [after some effort]

Thus either $M_1 = M_2$ and there's no shock

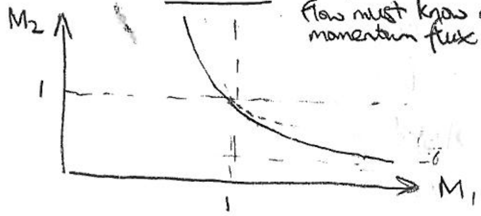
or $$M_2^2 = \frac{2 + (\gamma-1)M_1^2}{2\gamma M_1^2 - (\gamma-1)}$$

Rewrite, letting $B = 2\gamma M_1^2 - (\gamma-1) = 1 + \gamma(2M_1^2 - 1)$

$$\rightarrow M_2^2 = \frac{1}{B}\left[B + (1+\gamma)(1 - M_1^2)\right]$$

So, if $M_1 > 1$ (and $\gamma > 0$)

$\rightarrow M_2 < 1$ and post shock flow is subsonic, which is expected as post shock flow must know about conditions at the shock to set conditions like mass and momentum flux rates.



As $M_1 \rightarrow \infty$, $M_2 \rightarrow \frac{\gamma-1}{2\gamma}$

The shock itself

- The RH cond$^{\underline{n}}$s only specify cond$^{\underline{n}}$s on either side of shock where inviscid, adiabatic equ$^{\underline{n}}$s are valid
- Within the shock it must be a different story as entropy of post-shock fluid has changed

$$dS = C_v \, d \ln(p/\rho^\gamma) \neq 0^{*}$$

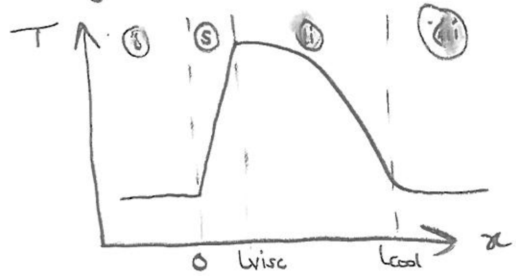
[not specified as such, just that $p = K\rho^\gamma$ outside shock, not what K is]

- As entropy can only increase application of the RH rel$^{\underline{n}}$s shows that the supersonic flow must be upstream of shock (ie. $M_1 > 1$)

  ie shocks convert cold fast fluid into hot slow fluid, not the other way around
- How? by an irreversible process such as viscosity converting KE into heat, and shock thickness is set by length over which viscosity converts mechanical energy into heat, L_{visc}

Isothermal shocks

- Adiabatic assumption may be ok just after shock, but $\dot{Q} \neq 0$ and shocked gas may cool over some length scale L_{cool}, perhaps back to original temperature



(i) = pre shock
(s) = shock (viscous)
(ii) = adiabatic
(iii) = isothermal

- To work out cond$^{\underline{n}}$s in isothermal region note that first two RH rel$^{\underline{n}}$s still hold

$$\text{(RH1)} \quad \rho_1 u_1 = \rho_3 u_3$$

$$\text{(RH2)} \quad p_1 + \rho_1 u_1^2 = p_3 + \rho_3 u_3^2$$

- Third doesn't as gas loses heat, but $T_3 = T_1$

$$\therefore c_{s3} = c_{s1} = (p_1/\rho_1)^{1/2}$$

NB assuming pre-shock gas also behaves isothermally

So $\rho_1 c_{s1}^2 + \rho_1 u_1^2 = \rho_3 c_{s3}^2 + \rho_3 u_3^2$ [putting p into (RH2)]

$\therefore (c_{s1}^2 + u_1^2) u_3 = (c_{s1}^2 + u_3^2) u_1$ [dividing by (RH1) LHS by LHS etc]

$\therefore (u_3 - u_1)(c_{s1}^2 - u_1 u_3) = 0$

- Thus either $u_3 = u_1$ ie no shock

  or $u_1 u_3 = c_{s1}^2$

$$\therefore \rho_3/\rho_1 = u_1/u_3 = (u_1/c_{s1})^2 = M_1^2$$

So density ratio can be high in high Mach # isothermal shock and $M_3 < M_1$ as before

* NB $đQ = T\,dS = C_v \, dT - \frac{p}{\rho^2} d\rho$

$\therefore dS = C_v \, d\ln T - \frac{R_*}{\mu} d\ln\rho$

$= C_v \, d \ln T/\rho^{(R_*/\mu)/C_v}$

$= C_v \, d \ln \frac{\mu}{R_*} p/\rho^{1+(R_*/\mu)/C_v} \rightarrow \gamma$

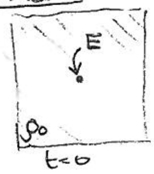
$= C_v \, d \ln p/\rho^\gamma$

Blast waves

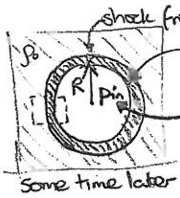
eg. SN explodes depositing energy E in ISM of uniform density ρ_0

Shocked medium expands sweeping up gas → ISM has swiss-cheese-like structure

Simple model



t=0



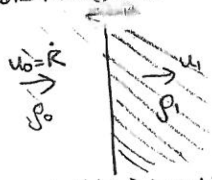
Some time later

shock front has reached radius R

shocked layer of density ρ_1, thickness D

momentum of shell provided by internal pressure

In frame moving with shock



NB ($\dot{R}\leftarrow$) in inertial frame

- Ignore thermal pressure from medium (ie strong shock, $p_1 >> p_0$),
  → if adiabatic $\rho_1/\rho_0 = \frac{\gamma+1}{\gamma-1}$
- Mass conservation (RH1) $u_1/u_0 = \rho_0/\rho_1 = \frac{\gamma-1}{\gamma+1}$
  → velocity of shocked layer in inertial frame $U = \dot{R} - u_1 = u_0 - u_1 = \left(\frac{2}{\gamma+1}\right)\dot{R}$
- Mass in shocked layer is sum of swept up medium $\frac{4\pi}{3}\rho_0 R^3$
  NB if D/R is small thus is $4\pi R^2 D\rho_1 \rightarrow D/R = \frac{1}{3}\left(\frac{\gamma-1}{\gamma+1}\right) << 1$ justifying assumption of thin shell
- Rate of change of momentum of shocked layer is $\frac{d}{dt}\left[\frac{4\pi}{3}\rho_0 R^3 \left(\frac{2}{\gamma+1}\right)\dot{R}\right]$
- Assume provided by internal pressure that is some fraction (α) of pressure in shocked layer

$$p_{in} = \alpha p_1$$

where (RH2) $p_1 + \rho_1 u_1^2 = \underbrace{p_0}_{\text{assumed negligible}} + \rho_0 u_0^2$

$$\therefore p_1 = \rho_0 \dot{R}^2\left[1 - \left(\frac{\gamma-1}{\gamma+1}\right)\right] = \left(\frac{2}{\gamma+1}\right)\rho_0 u_0^2$$

- Thus $\left(\frac{4\pi}{3}\frac{2}{\gamma+1}\rho_0\right)\frac{d}{dt}(R^3\dot{R}) = 4\pi R^2 \alpha \left(\frac{2}{\gamma+1}\right)\rho_0 \dot{R}^2$ ie pressure × area

$$\therefore \frac{d}{dt}(R^3\dot{R}) = 3\alpha R^2 \dot{R}^2$$

Let $R \propto t^b \rightarrow b(4b-1)t^{4b-2} = 3\alpha b^2 t^{4b-2}$

$$\therefore b = \frac{1}{4-3\alpha}$$

$\therefore$ $\underline{R \propto t^{\frac{1}{4-3\alpha}}}$

- If blast wave is adiabatic, energy E is conserved and must be taken up in
  - KE of shell $= \frac{1}{2}\left(\frac{4\pi}{3}\rho_0 R^3\right)U^2$
  - Internal energy: As internal energy/unit vol is $\frac{p}{\gamma-1}$ [NB our simple assumptions are ρ, p are const over vol]

$$= \frac{4\pi}{3}R^3 \frac{p_{in}}{\gamma-1} + 4\pi R^2 D \frac{p_1}{\gamma-1}$$

$$= \frac{4\pi}{3}R^3 p_1\left[\frac{\alpha}{\gamma-1} + \frac{1}{\gamma+1}\right]$$

$$\therefore E = \frac{4\pi}{3}R^3\left[\frac{1}{2}\rho_0\left(\frac{2}{\gamma+1}\right)^2\dot{R}^2 + \left[\frac{1}{\gamma+1} + \frac{\alpha}{\gamma-1}\right]\left(\frac{2}{\gamma+1}\right)\rho_0\dot{R}^2\right]$$

$$\propto R^3\dot{R}^2$$

$$\propto t^{\frac{6\alpha-3}{4-3\alpha}}$$

So for $\dot{E}=0$, $\alpha = 1/2$ and $\underline{R \propto t^{2/5}}$

Dimensional arguments

Problem defined by $E \quad (ML^2T^{-2})$
$\rho_0 \quad (ML^{-3})$

So no natural length scale, but at time t can define a length dimension $\lambda = \left(\frac{Et^2}{\rho_0}\right)^{1/5}$,
and can assume sol$^=$ is self-similar s.t. E, ρ_0

$$X(r,t) = X_1(r,t,\text{dimensional values of problem})\, X'(\xi) \quad \text{where } \xi = r/\lambda \text{ and } X_1 = K \times$$ combination that gives appropriate dimensions

(ie distr of X at given time is a scaled version of that at other times)

Thus shock front will occur at some constant $\xi_0 \therefore R = \xi_0 \lambda = \xi_0 \left(\frac{E}{\rho_0}\right)^{1/5} t^{2/5}$

$\boxed{\text{eg}}$ if $\xi_0 \simeq 1$ then as SN ejecting $1M_\odot$ at 10^4 km/s $(E = 10^{44}\,J)$ into ISM of $\rho_0 \simeq 10^{-21}$ kg/m^3 $\rightarrow R = 10^{13} t^{2/5}$
or in astronomers units $R \simeq 0.3\,t^{2/5}$ pc (t in yr), $\dot{R} \simeq 10^5\, t^{-3/5}$ km/s

NB, sol$^=$ only valid for $t > 100$ yr (for $\dot R < 10^4$ km/s and for swept up mass > SN ejecta)
$t < 10^5$ yr (after which energy losses important)

Sedov Similarity Solution

Define dimensionless variables inside the shock scaled s.t. $\rho'(\xi_0), p'(\xi_0) = u'(\xi_0) = 1$

Thus $\rho(r,t) = K_1 \rho_0 \rho'(\xi)$ [as physical quantities inside shock only depend on ξ]

From (RH1) $K_1 = \left(\frac{\gamma+1}{\gamma-1}\right)$

The velocity in inertial frame requires additional (r/t) factor (being only combin$^=$ of fundamental quantities giving units of velocity)

$$u(r,t) = K_2 (r/t)\, u'(\xi)$$

As $\dot R = \frac{2}{5} R/t$ and $u(R,t) = \left(\frac{2}{\gamma+1}\right)\dot R \rightarrow K_2 = \frac{4}{5(\gamma+1)}$

The pressure requires $\rho_0 (r/t)^2$ factor as it has units of velocity squared

$$p(r,t) = K_3 \rho_0 (r/t)^2 p'(\xi)$$

As (RH2) gave $p(R,t) = p_1 = \left(\frac{2}{\gamma+1}\right)\rho_0 u_0^2 = \left(\frac{2}{\gamma+1}\right)\rho_0 (\dot R)^2 \rightarrow K_3 = \frac{8}{25(\gamma+1)}$

- Continuity equ$^=$ (sph. symm.) is $\partial\rho/\partial t + \frac{1}{r^2}\partial/\partial r\,(r^2\rho u) = \overset{①}{\partial\rho/\partial t} + \overset{②}{2\rho u/r} + \overset{③}{\partial\rho u/\partial r} = 0$

But $\partial/\partial t|_r = (\partial\xi/\partial t)_r \partial/\partial\xi|_r = -\frac{2}{5}\frac{\xi}{t}\,\partial/\partial\xi|_r$ ← NB $\xi = \rho_0^{1/5} E^{-1/5} r\, t^{-2/5}$

$\partial/\partial r|_t = (\partial\xi/\partial r)_t \partial/\partial\xi|_t = (\xi/r)\,\partial/\partial\xi|_t$

So ① is $-\frac{2}{5}\frac{\xi}{t} K_1 \rho_0 \partial\rho'/\partial\xi$

② is $2K_1 K_2 \rho_0 \frac{1}{t}\rho' u'$

③ requires some care as need to do differential at constant t, so substitute $r = \xi\left(\frac{E}{\rho_0}\right)^{1/5} t^{2/5}$ to get

$$\frac{\xi}{r}\frac{\partial}{\partial\xi}\Big|_t \left[K_1 \rho_0 \rho' K_2 \xi \left(\frac{E}{\rho_0}\right)^{1/5} t^{-3/5} u'\right] = \frac{K_1 K_2 \rho_0}{t}\frac{\partial}{\partial\xi}(\rho' u' \xi) = \frac{K_1 K_2 \rho_0}{t}\left[\xi\, \partial\rho' u'/\partial\xi + \rho' u'\right]$$

Combining to $\xi\, \partial\rho'/\partial\xi - \left(\frac{2}{\gamma+1}\right)\left[3\rho' u' + \xi\, \partial(\rho' u')/\partial\xi\right] = 0$

Note that the "r" and "t" cancelled leaving this only in terms of ξ, so similarity sol$^=$ is poss.

- Momentum $\partial u/\partial t + u\,\partial u/\partial r = -\frac{1}{\rho}\partial p/\partial r$

$$\rightarrow -u' - \frac{2}{5}\xi\, du'/d\xi + \frac{4}{5(\gamma+1)}(u'^2 + u'\xi\, du'/d\xi) = -\frac{2}{5}\frac{\gamma-1}{\gamma+1}\frac{1}{\rho'}(2p' + \xi\, dp'/d\xi)$$

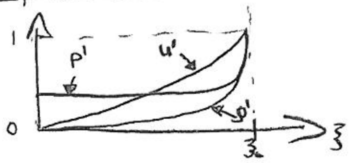
- Energy $D/Dt\,(p/\rho^\gamma) = 0 \rightarrow \xi\frac{d}{d\xi}(\ln p'/\rho') = \frac{5(\gamma+1) - 4u'}{2u' - (\gamma+1)}$

- Boundary conditions $p'(\xi_0) = \rho'(\xi_0) = u'(\xi_0) = 1$

and to find ξ_0, use $E = \int_0^R \left(\frac{1}{2}\rho u^2 + \frac{p}{\gamma-1}\right) 4\pi r^2\, dr$

or in dimensionless variables $\frac{32\pi}{25(\gamma^2-1)}\int_0^{\xi_0} (p' + \rho' u'^2)\,\xi^4\, d\xi = 1$

Solution for $\gamma = 7/5$



Thus our simple model of most mass in shell, $p_{in}/p_1 \sim$ const, $u_1 \sim$ const was ok.

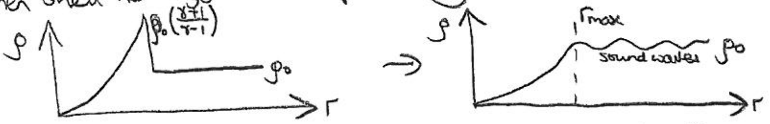
Maximum radius

This sol^n only applicable if no counter pressure outside shock (s.t. post-shock state is scale free) $\rightarrow$ breaks down when

$$p_1 = \left(\frac{2}{\gamma+1}\right)\rho_0 \dot{R}^2 \sim p_0$$

As $c_{s0}^2 = \gamma p_0/\rho_0$ this is when $\dot{R}/c_{s0} = \sqrt{\frac{\gamma+1}{2\gamma}}$

ie when shell no longer moves supersonically wrt medium at which pt sound waves propagate ahead of blast wave



To estimate r_{max}, revisit simple model c $\alpha = \frac{1}{2}$ $\rightarrow E = \frac{4\pi}{3}R^3\rho_0 \dot{R}^2 \frac{5\gamma-3}{(\gamma^2-1)(\gamma+1)}$

So at point where $p_1 \sim p_0$

$$= \underbrace{\left[\frac{4\pi}{3} R_{max}^3 \rho_0 \frac{c_{s0}^2}{\gamma(\gamma-1)}\right]}_{\text{total thermal energy inside } R_{max}} \frac{5\gamma-3}{2(\gamma+1)}$$

As $\varepsilon_0 = \frac{p_0}{\rho_0(\gamma-1)} = c_{s0}^2/\gamma(\gamma-1) =$ thermal energy/unit mass,

R_{max} is point at which energy of explosion $\sim$ total thermal energy inside R_{max}

Dimensional arguments revisited

If medium has finite temperature problem also defined by c_{s0} (LT^{-1})

so can define a characteristic lengthscale $\left(\frac{E}{\rho_0 c_{s0}^2}\right)^{1/3}$ that is of order R_{max} and timescale $\frac{1}{c_{s0}}\left(\frac{E}{\rho_0 c_{s0}^2}\right)^{1/3}$

$\boxed{\text{eg}}$ for $T_0 \sim 10^4$ K, $R_{max} \sim 100$pc, $t_{max} \sim 10$ Myr

- As SN rate/vol is $\sim 10^{-7}$ Myr^{-1} pc^{-3} we'd expect a fraction $\frac{4\pi}{3}R_{max}^3 t_{max}$ SN$_{rate} \sim 10$ (ie all) of ISM to be in SN driven bubbles and heated to high temp ($\sim 10^6$K)
- This isn't the case as gas behind shock cools after 0.1Myr (20pc) and bubble grows more slowly b/c
- Finite scale height of galactic disk $\rightarrow$ bubble blows out of plane and cavity is depressurised

NB this means that only 1% of SN energy is deposited in ISM, rest is radiated away, put into galactic halo etc.

Bernoulli's equation

Applies to steady barotropic (inviscid) flows

Derivation:

Remember inviscid mom. eq.

$$\underbrace{\partial \underline{u}/\partial t}_{0 \text{ if steady}} + \underbrace{\underline{u}.\nabla\underline{u}}_{\nabla(\frac{1}{2}u^2) - \underline{u}\wedge(\nabla\wedge\underline{u}) \text{ by identity (see EG1.5)}} = \underbrace{-\frac{1}{\rho}\nabla p}_{-\nabla\int\frac{dp}{\rho} \text{ for a barotropic e.o.s.}} - \nabla\Psi$$

Define vorticity $\underline{w} = \nabla\wedge\underline{u}$ [which is a measure of angular velocity of fluid]

$\therefore \quad \underline{u}\wedge\underline{w} = \nabla\left[\frac{1}{2}u^2 + \int\frac{dp}{\rho} + \Psi\right]$

Take $\underline{u}.$ this eqn gives

$\underline{u}.\nabla H = 0$ [ie convective derivative of H is zero]

where $\underline{H = \frac{1}{2}u^2 + \int\frac{dp}{\rho} + \Psi}$ = Bernoulli's const

= constant along streamlines

$DH/Dt = \underbrace{\cancel{\partial H/\partial t}}_{0 \text{ as steady}} + \underbrace{\cancel{\underline{u}.\nabla H}}_{0} = 0$

Meaning

If $p=0$, just says that KE + PE = const.

If Ψ = const, kinetically it describes how KE is converted betw. random molecular motion (p) and bulk flow (u)

hydrodynamically it reflects fact that pressure differences are req. to accelerate the flow.

EG Aeroplane wing: curved → u higher on top → p lower → lift

Shower curtain: downward vel of air when turned on → p lower

Water draining from tank etc.

Irrotational flows → $\underline{w} = 0$ (also called "curl-free")

In this case $\nabla H = 0$, so H is constant everywhere [not just on streamlines]

Also means it's possible to write $\underline{u} = -\nabla\Phi_u$ [as $\nabla\wedge\nabla\Phi_u = 0$ for all Φ]

[eg Stokes theorem $\int_S \nabla\wedge\underline{u}.d\underline{S} = \oint\underline{u}.d\underline{l}$

So $\oint\underline{u}.d\underline{l} = 0$ for irrotational flow, and a uniform flow is, but a rotating flow is not.]

Incompressible flows → $\nabla.\underline{u} = 0$ (from 2nd lecture)

This means it's possible to write $\underline{u} = \nabla\wedge\phi$ [as $\nabla.\nabla\wedge\underline{u} = 0$ for all $\underline{u}$]

eg for 2D flow $\underline{u} = -\nabla\wedge[\phi(x,y)\hat{\underline{e}}_z]$

$\therefore u_x = -\partial\phi/\partial y$ and $u_x = \partial\phi/\partial x$

[and for an irrotational incompressible flow: $\underline{\nabla^2\Phi_u = 0}$]

De Laval Nozzle (linear application of Bernoulli's eqn)

Consider steady barotropic (inviscid) flow in z dirn down tube of variable cross-section $A(z)$, and ignore gravity.

Continuity $\rightarrow$ $\rho u A = \text{const} = \dot{M}$ (mass flow rate)

Momentum $\rightarrow$ $\cancel{\partial \underline{u}/\partial t}^{\,0 \text{ as steady}} + \underline{u}.\nabla\underline{u} = -\frac{1}{\rho}\nabla p - \cancel{\nabla\Phi}^{\,0 \text{ as ignore gravity}}$

$$= -c_s^2 \frac{1}{\rho}\nabla\rho \qquad \text{(as barotropic)}$$
$$= -c_s^2 \nabla \ln\rho$$

Flow is uniform so irrotational $\therefore$ $\underline{u}.\nabla\underline{u} = \nabla(\tfrac{1}{2}u^2) - \underline{u}\wedge(\cancel{\nabla\wedge\underline{u}})^{0}$
$$= u\nabla u = u^2\nabla\ln u$$

But continuity $\Rightarrow$ $\ln\rho + \ln u + \ln A = \ln\dot{M}$
$$\therefore \nabla\ln\rho = -\nabla\ln u - \nabla\ln A$$

Combining $\rightarrow$ $\underline{(u^2 - c_s^2)\nabla\ln u = c_s^2\nabla\ln A}$

$\therefore$ gas can only make a sonic transition (eg sub- to super-sonic) at a max or min in A

but a max or min in A could also correspond to a max or min in u *

Bernoulli's eq w/o gravity: $\frac{1}{2}u^2 + \int\frac{dp}{\rho} = \frac{1}{2}u^2 + \int\frac{c_s^2}{\rho}d\rho = \text{const.}$

<u>If isothermal</u> $\quad c_s^2 = \frac{R_*}{\mu}T = \text{const}$

$\therefore$ $\frac{1}{2}u^2 + c_s^2\ln\rho = \text{const.}$

If there's a sonic transn at A_m, that const is $c_s^2(\frac{1}{2} + \ln\rho_m)$

$\therefore$ $u^2 = c_s^2\left[1 + 2\ln(\rho_m/\rho)\right]$
$$\underline{= c_s^2\left[1 + 2\ln\left(\frac{uA}{c_s A_m}\right)\right]} \quad \text{thus giving } u(z) \text{ and } \rho(z) \text{ for given } A(z), c_s, \dot{M}$$

<u>If polytropic</u> $\quad p = K\rho^{1+\frac{1}{n}}$

$$c_s^2 = \left(\frac{n+1}{n}\right)K\rho^{\frac{1}{n}}$$
$$\therefore \int\frac{c_s^2}{\rho}d\rho = nc_s^2$$

If there's a sonic transn at A_m, $\dot{M} = \rho_m c_{sm} A_m = K^{1/2}\left(\frac{n+1}{n}\right)^{1/2} A_m \rho_m^{1+\frac{1}{2n}}$

$$\therefore \rho_m = \left[\left(\frac{\dot{M}}{A_m}\right)^2\frac{n}{K(n+1)}\right]^{\frac{n}{2n+1}}$$

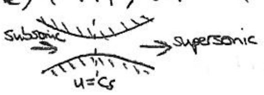
Put into Bernoulli's eq $\rightarrow$ $\frac{1}{2}\left(\frac{\dot{M}}{\rho A}\right)^2 + (n+1)K\rho^{\frac{1}{n}} = c_{sm}^2\left(\frac{1}{2}+n\right)$
$$= \left(\tfrac{1}{2}+n\right)\left(\frac{n+1}{n}\right)K\rho_m^{1/n}$$

giving $\rho(z)$ and $u(z)$ for given $\dot{M}, A(z), K, n$

* In subsonic regime, if $A\downarrow$, $u\uparrow$ (eg rivers flowing through narrows)

supersonic regime, if $A\uparrow$, $u\uparrow$ (as supersonic flows are compressible $\rightarrow$ $\rho\downarrow$ and $u\uparrow$ to keep $\dot{M}$ const)

$\quad \hookrightarrow$ NB from above $\nabla\ln\rho/\nabla\ln u = -u^2/c_s^2$

eg: Jet Engine 

$\boxed{\text{EG}}$ Jets from AGN or PPDs.

Disk gas $\rightarrow$ easiest to escape $\perp$ to disk $\rightarrow$ acts like pipe, but <u>confinement is through pressure balance</u>

(Though real jets are time variable and include shocks, and probably involve mag. fields for their acceletn and confinement)

Spherical Accretion (3D application analogous to DL nozzle)

Consider point mass star, mass M, accreting gas spherically symmetrically in a steady barotropic inviscid flow.
Gas is accelerated from rest (subsonic) to freefall (supersonic) → must be sonic transition

Continuity → $\cancel{\partial\rho/\partial t}^{\,0} + \nabla.(\rho\underline{u}) = 0$

$\therefore \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho u) = 0$ (as spherically symmetric, NB $\underline{u} = +u\hat{\underline{r}}$)

$\therefore 4\pi r^2\rho u = -\dot{M}$

Momentum → $\cancel{\partial\underline{u}/\partial t}^{\,0} + \underline{u}.\nabla\underline{u} = -\frac{1}{\rho}\nabla p - \nabla\Psi$ where $\Psi = -GM/r$

$\therefore u^2 d\ln u/dr = -c_s^2 d\ln\rho/dr - GM/r^2$ (for same reasons as DL nozzle, noting that $\nabla \to \frac{d}{dr}$ and $\nabla\wedge\underline{u} = 0$ for spherically symmetry)

From continuity $\ln\rho + \ln u + \ln r^2 = -\ln\dot{M}/4\pi$

$d\ln\rho/dr = -d\ln u/dr - 2/r$

$\therefore (u^2 - c_s^2)\,d\ln u/dr = \frac{2c_s^2}{r}\left[1 - \frac{GM}{2c_s^2 r}\right]$

$\therefore$ gas can only make a sonic transⁿ at $r_s = \frac{GM}{2c_s^2}$
but this radius could also correspond to a min or max in u

Bernoulli's eq: $\frac{1}{2}u^2 + \int\frac{c_s^2}{\rho}d\rho - \frac{GM}{r} = \text{const}$

If isothermal (Bondi accretion)

Given T, know c_s (which is const) and so, given M, r_s
But not ρ_s or $\dot{M}$ that are related through $\dot{M} = 4\pi r_s^2\rho_s c_s$ (assuming u is $-$ve)
Bernoulli → $\frac{1}{2}u^2 + c_s^2\ln\rho - GM/r = \frac{1}{2}c_s^2 + c_s^2\ln\rho_s - GM/r_s$

$\therefore \underline{u^2 = 2c_s^2[\ln\rho_s/\rho - 3/2] + 2GM/r}$

As $r \to 0$, $u \to \sqrt{2GM/r}$ (free-fall)
$r \to \infty$, we know $u \to 0$ so $\rho_s = \rho_\infty e^{3/2}$ and $\dot{M} = \pi(GM)^2\rho_\infty e^{3/2}/c_s^3$
So problem well defined given T, M, ρ_∞

EG. $1M_\odot$ in $200K$, 10^6 (mean MW density) H atoms$/m^3$ ISM → $r_s \approx 4\times10^{13}\,m$, $\dot{M} = 3\times10^{18}\,kg/yr$ ($0.01M_\odot/10Gyr$)
→ Stars accrete from ISM that is $10^6\times$ denser where self gravity important

If polytropic

Bernoulli → $\frac{1}{2}u^2 + nc_s^2 - \frac{GM}{r} = c_{ss}^2\left[\frac{1}{2} + n - 2\right]$ (evaluating at sonic transⁿ)

$\therefore \frac{1}{2}u^2 + (n+1)K\rho^{\frac{1}{n}} - \frac{GM}{r} = (n - \frac{3}{2})c_{ss}^2$ (substituting in $c_s^2 = \left(\frac{n+1}{n}\right)K\rho^{1/n}$)

As $r \to \infty$, $u \to 0$, so $(n - \frac{3}{2})c_{ss}^2 = (n+1)K\rho_\infty^{1/n}$

$\therefore$ If $n < 3/2$ the sonic point is never reached (because gas too incompressible and pressure gradient prevents flow being accelerated)

Now $\dot{M} = 4\pi r_s^2\rho_s c_{ss} = \pi(GM)^2\rho_s c_{ss}^{-3}$ (just substituting in for r_s)
but $\rho_s^{1/n} = \left(\frac{n}{n+1}\right)K^{-1}c_{ss}^2$ (from same $c_s^2 = \ldots$ equⁿ above)
$\therefore \dot{M} = \pi(GM)^2\left(\frac{n}{n+1}\right)^n K^{-n}c_{ss}^{2n-3}$ (and so $\dot{M}$ is given in terms of M, K, n, ρ_∞)

NB, If $n = 3/2$, which is adiabatic flow of monatomic gas, $c_{ss} \to \infty$ and so $r_s \to 0$
but $\dot{M} \to \pi(GM)^2\left(\frac{3}{5}\right)^{3/2}K^{-3/2}$

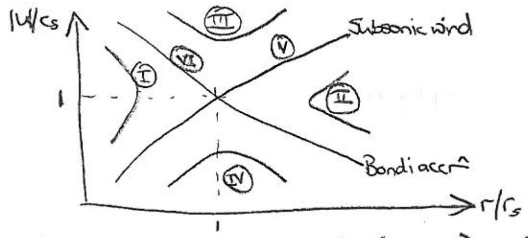
Finally $\frac{1}{2}\left[\frac{\dot{M}}{4\pi r^2\rho}\right]^2 + (n+1)K\rho^{\frac{1}{n}} = (n+1)K\rho_\infty^{\frac{1}{n}} + GM/r$ gives ρ vs r

Stellar winds

Eqn^s are exactly same as for accretion, except u is +ve and $\dot{M}$ is −ve
But here b.c. are set at inner boundary

If isothermal ⇒ $(u^2 - c_s^2)\, d\ln u/dr = \frac{2c_s^2}{r}\left[1 - \frac{GM}{2c_s^2 r}\right]$

$$\therefore \left(\frac{u}{c_s}\right)^2 - \ln\left(\frac{u}{c_s}\right)^2 = 4\ln\left(\frac{r}{r_s}\right) + 4\left(\frac{r_s}{r}\right) + C$$



Nature of sol^s dep on C:

I and II are unphysical (two values of u at each r)
(flow is such "C" would readjust to new st. st.)

III and IV are always sub or supersonic

V and VI have $C = -3$

NB sol^n is steady, but winds can be variable (eg CME), and note that interacts w ISM in shock

Astrophysical importance:

- source of mechanical energy to ISM $\sim \frac{1}{2}|\dot{M}_{wind}|V_\infty^2$ ← terminal velocity
  (typically comparable to SN input, but dominates in <2Myr clusters)
- returns products of nucleosynthesis to ISM

Drivers

OB★s : rad^n absorbed by metallic ions
10^{-6} to 10^{-4} $M_\odot$/yr, $V_\infty \sim 1000$ km/s

AGB : rad^n absorbed by dust that forms in wind flow
10^{-8} to 10^{-4} $M_\odot$/yr, $V_\infty \sim 20$ km/s

Vorticity evolution

Helmholtz's equation

Rewrite barotropic inviscid mom. eq:

$$\partial \underline{u}/\partial t = \underline{u} \wedge \underline{\omega} - \nabla H$$

Take curl of this eq

$$\partial \underline{\omega}/\partial t = \nabla \wedge (\underline{u} \wedge \underline{\omega})$$ (as $\nabla \wedge \nabla H = 0$)

So an irrotational flow ($\underline{\omega} = 0$) remains as such ... (NB $\partial^2 \underline{\omega}/\partial t^2 = \nabla \wedge (\partial \underline{u}/\partial t \wedge \underline{\omega} + \underline{u} \wedge \partial \underline{\omega}/\partial t)$)

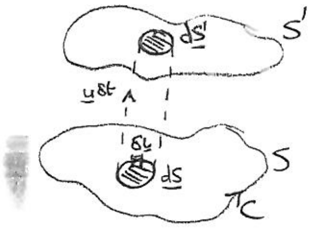
... unless flow is viscous in which case vorticity can be created

Kelvin's vorticity theorem

Consider vorticity linked to surface S inside a fluid $\int_S \underline{\omega} . d\underline{S}$

This can change due to intrinsic change in $\underline{\omega}$ or change in surface S caused by flow

$$\therefore D \int_S \underline{\omega} . d\underline{S} / Dt = \int_S \partial \underline{\omega}/\partial t . d\underline{S} + \int_S \underline{\omega} . D d\underline{S}/Dt$$



Consider area element that changed from $d\underline{S}$ to $d\underline{S}'$ in time δt, and the volume with $d\underline{S}$ and $d\underline{S}'$ at ends.

If $\delta \underline{l}$ is length element on curve bounding $d\underline{S}$, the vector area of side of this volume is $-\oint \delta t \, \underline{u} \wedge \delta \underline{l}$

But $d\underline{S}' - d\underline{S} - \oint \delta t \, \underline{u} \wedge \delta \underline{l} = 0$ (sum of outwardly pointing vector areas is 0)

$$\therefore D d\underline{S}/Dt = \oint \underline{u} \wedge \delta \underline{l}$$

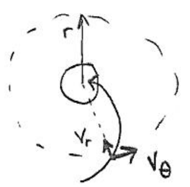
Thus
$$\begin{aligned}
\int_S \underline{\omega} . D d\underline{S}/Dt &= \int_S \oint \underline{\omega} . (\underline{u} \wedge d\underline{l}) \\
&= \int_S \oint (\underline{\omega} \wedge \underline{u}) . d\underline{l} && \text{(as } \underline{a}.(\underline{b} \wedge \underline{c}) = \underline{c}.(\underline{a} \wedge \underline{b})\text{)} \\
&= \oint_C (\underline{\omega} \wedge \underline{u}) . d\underline{l} && \text{(as all inner components cancel)} \\
&= \int_S \nabla \wedge (\underline{\omega} \wedge \underline{u}) . d\underline{S} && \text{(by Stokes' theorem)}
\end{aligned}$$

$$\begin{aligned}
\therefore D \int_S \underline{\omega} . d\underline{S} / Dt &= \int_S d\underline{S} . [\partial \underline{\omega}/\partial t - \nabla \wedge (\underline{u} \wedge \underline{\omega})] \\
&= 0 && \text{(by Helmholtz' eq)}
\end{aligned}$$

ie flux of vorticity is conserved and moves with fluid (that is barotropic and inviscid)

$$\therefore D \oint_C \underline{u} . d\underline{l} / Dt = 0$$

eg. bath tub vortex



circulation $\Gamma = 2\pi r v_\theta = \text{const}$

$\therefore v_\theta \propto 1/r$

- tornado has same effect, noting also that as $v_\theta \uparrow$ $p \downarrow$

Fluid instabilities

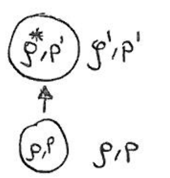
Consider a steady fluid ($\partial/\partial t = 0$)

If small pert^ns grow w time, that configuration is said to be unstable w.r.t. pert^ns (unlikely found in nature)

A stable config. → pert^ns diminish, or oscillate (eg. sound waves)

Convective instability

Consider perfect gas in hydrostatic eq^m in uniform grav. field in $-\hat{z}$ dir^n



Take fluid element at same ρ, p as surroundings

Displace by δz where local cond^ns are $p' = p + \frac{dp}{dz}\delta z$, $\rho' = \rho + \frac{d\rho}{dz}\delta z$

Pressure imbalances quickly removed by acoustic waves, but heat exchange takes longer → gas changes adiabatically to new p', ρ^*

As $p/\rho^\gamma = p'/\rho^{*\gamma} \rightarrow \rho^* = \rho(p'/p)^{\frac{1}{\gamma}} = \rho + \left(\frac{\rho}{\gamma p}\right)\frac{dp}{dz}\delta z$ (to 1st order)

Archimedes' principle → if $\rho^* < \rho'$ gas continues to rise (but sinks if $\rho^* > \rho'$)

Schwarzschild criterion → instability if $\left(\frac{\rho}{\gamma p}\right)\frac{dp}{dz} < \frac{d\rho}{dz}$

eg. if surrounding were adiabatic (p/ρ^γ const) → neutrally stable
but if p/ρ^γ decreases w z → unstable

Alternatively, as $p = \frac{R_0}{M}\rho T \rightarrow d\rho/dz = \frac{\rho}{p}\frac{dp}{dz} - \frac{\rho}{T}\frac{dT}{dz}$

$\therefore \rho^* - \rho' = \left[-\left(1-\frac{1}{\gamma}\right)\frac{\rho}{p}\frac{dp}{dz} + \frac{\rho}{T}\frac{dT}{dz}\right]\delta z$

Then as dp/dz and dT/dz have same sign (from ideal gas)
and $dp/dz < 0$ (from momentum eq. – pressure supports gravity)
→ system stable if $\left|dT/dz\right| < \left(1-\frac{1}{\gamma}\right)\frac{T}{p}\left|dp/dz\right|$

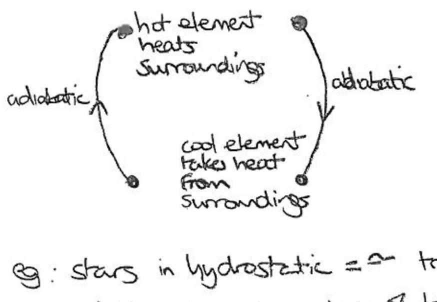
If stable → internal gravity waves

e.o.m. of fluid element: $\rho^*\frac{d^2}{dt^2}\delta z = -(\rho^* - \rho')g$

$\therefore \frac{d^2}{dt^2}\delta z + \frac{g}{T}\left[dT/dz - \left(1-\frac{1}{\gamma}\right)\frac{T}{p}dp/dz\right]\delta z = 0$

→ SHM w angular freq. $\sqrt{\frac{g}{T}\left[dT/dz - \left(1-\frac{1}{\gamma}\right)\frac{T}{p}dp/dz\right]}$

If unstable → convection cells



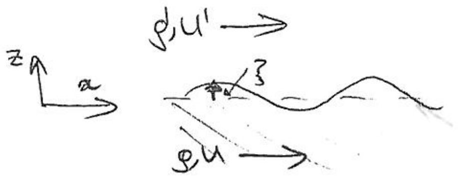
size of convective cell set by scale elements cease to be adiabatic (ie exchange heat w surroundings)
→ convection transports heat upwards in displaced elements.

eg: stars in hydrostatic eq^m transport energy by rad^n unless convectively unstable which occurs in regions of large temperature gradients such as envelopes of low mass stars (caused by high opacity of partially ionised composition due to low T)

Rayleigh-Taylor Instability

An unstably stratified config. of fluid layers turns over due to gravity

(i) Consider 2 barotropic, inviscid fluids in uniform (vertically down) grav. field w/ interface at $z = 0$, both w/ uniform flow in x dir^n



Assume flow is irrotational → can write $\underline{u} = -\nabla\Phi$ (ie. $\Phi = -Ux$)

∴ Eulerian mom. eq. $\partial(-\nabla\Phi)/\partial t + \nabla(\tfrac{1}{2}u^2) = -\nabla\int\frac{dp}{\rho} - \nabla\Psi$

Also assume incompressible, and note $\Psi = gz$

$$\therefore -\nabla\partial\Phi/\partial t + \nabla\tfrac{1}{2}u^2 + \nabla\tfrac{p}{\rho} + \nabla gz = 0$$

Integrate ⇒ $\underline{-\partial\Phi/\partial t + \tfrac{1}{2}u^2 + p/\rho + gz = F(t)}$ (const in space, but not nec. in t)

(Note that uniform flow is steady sol^n w/ F as Bernoulli's const
and that there's a similar eq^n w/ dashes for top fluid)

(ii) Consider a pert^n to interface $\xi(x,t)$
w/ corresponding velocity pert^n $\quad \Phi = -Ux + \phi \quad$ and $\quad \Phi' = -U'x + \phi'$
s.t. $\quad u^2 = |-\nabla\Phi|^2 = U^2 - 2U\partial\phi/\partial x$ (to first order)

Use mom. eq. to determine pressure at interface ($z = \xi$) which is same top and bottom

$$\to p|_{z=\xi} = \rho F(t) - \rho[-\partial\phi/\partial t|_{z=0} + \tfrac{1}{2}U^2 - U\partial\phi/\partial x|_{z=0} + g\xi] = p'|_{z=\xi}$$

$$\therefore \rho[-\partial\phi/\partial t|_{z=0} + \cancel{\tfrac{1}{2}U^2} - U\partial\phi/\partial x|_{z=0} + g\xi] - \rho'[-\partial\phi'/\partial t|_{z=0} + \cancel{\tfrac{1}{2}U'^2} - U'\partial\phi'/\partial x|_{z=0} + g\xi] = \cancel{K(t)} \quad (A)$$

Cancellations arise because the constants are same w/o pert^n (eg. pert^{n}s vanish at $z = \pm\infty$)

Also remember $\nabla^2\Phi = 0$ for irrotational incompressible fluid

$$\therefore \nabla^2\phi = \nabla^2\phi' = 0 \quad (B)$$

And consider vertical velocity of fluid element just below interface (to 1st order)

$$-\partial\phi/\partial z|_{z=0} = D\xi/Dt = \partial\xi/\partial t + U\partial\xi/\partial x \quad (C)$$

and $\quad -\partial\phi'/\partial z|_{z=0} = \partial\xi/\partial t + U'\partial\xi/\partial x$

(iii) Seek solutions of form $\quad \xi = Ae^{i(kx-\omega t)}$

$\phi = Ce^{i(kx-\omega t)+jz} \quad$ and $\quad \phi' = C'e^{i(kx-\omega t)+j'z}$

Sub into

(B) $\quad -k^2 + j^2 = -k^2 + j'^2 = 0$

For $\phi \to 0$ at $z \to -\infty$ → $j = +k$

Likewise $j' = -k$

(C) $\quad -kC = i(kU - \omega)A$

$kC' = i(kU' - \omega)A$

(A) $\quad \rho(-i(kU-\omega)C + gA) = \rho'(-i(kU'-\omega)C' + gA)$

(iv) 3 eqns, 3 unknowns (A, C, C')

∴ eliminate unknowns to get dispersion rel^n

$$\therefore \rho(kU-\omega)^2 + \rho'(kU'-\omega)^2 = kg(\rho-\rho')$$

$$\therefore (\omega/k)^2(\rho+\rho') - 2\left(\frac{\omega}{k}\right)(\rho U+\rho' U') + \left[\rho U^2 + \rho' U'^2 + \frac{g}{k}(\rho-\rho')\right] = 0$$

∴ phase velocity of wave $\omega/k = \left[\frac{\rho U+\rho' U'}{\rho+\rho'}\right] \pm \sqrt{\left(\frac{g}{k}\right)\frac{\rho-\rho'}{\rho+\rho'} - \frac{\rho\rho'(U-U')^2}{(\rho+\rho')^2}}$

If $U=U'=0$

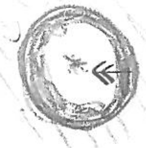
and $\rho'<\rho$ $\rightarrow$ $\omega/k = \pm\sqrt{\frac{g}{k}\frac{\rho-\rho'}{\rho+\rho'}}$

Surface gravity waves w/ vel. that dep on k (ie. dispersive)

$\rho'>\rho$ $\rightarrow$ $\omega/k = \pm i\sqrt{\frac{g}{k}\frac{\rho'-\rho}{\rho+\rho'}}$

∴ $\xi = Ae^{ikx}e^{\sqrt{gk(\rho'-\rho)/(\rho+\rho')}\,t}$ ie exponentially growing mode (RT instab.)

NB a light fluid accelerating into a heavy fluid (w/o gravity) gives same instab.

eg  Blast wave → thin shell that is decelerating
→ in the frame of interface the fluids experience outward acceler^n → dense (shell) on top of tenuous (ISM)
→ RT instab → filaments in shell

If $U\neq 0, U'\neq 0$ and $\rho>\rho'$ (→ RT stable) → Kelvin Helmholtz instability

Unstable if term in square root is −ve (as → imaginary part of ω → exponential growth)

$$\Rightarrow \rho\rho'(U-U')^2 > \left(\frac{g}{k}\right)(\rho^2-\rho'^2)$$

or $\quad k > (\rho^2-\rho'^2)g/\rho\rho'(U-U')^2$

→ Interface betw two fluids wrinkles if they move at diff speeds and pert^s w/ large enough k (small enough wavelength) are always unstable (unless damped by surface tension)

eg. jet moving w.r.t. ISM or IGM (esp as $g\neq 0$)

Gravitational (Jeans) Instability

Same analysis as for sound waves in uniform medium, but including self gravity (→ extra term in mom eq, add Poisson's eq, and include $\Delta\Psi$)

(a) $\partial\rho/\partial t + \nabla.(\rho \underline{u}) = 0$

$\partial \underline{u}/\partial t + \underline{u}.\nabla\underline{u} = -\frac{1}{\rho}\nabla p - \nabla\Psi$

$\nabla^2\Psi = 4\pi G\rho$

(b) Equilibrium $\underline{u}=0$, $p=p_0$, $\rho=\rho_0$, $\Psi=\Psi_0$?

Not quite: hydrostatic and uniform → $\frac{1}{\rho_0}\nabla p_0 = -\nabla\Psi_0$ and $\nabla^2\Psi_0 = 4\pi G\rho_0$

this is zero ... so, so is this ... and this ... so a static Universe is empty!

But ok estimate → Jean's swindle (NB, should perturb hydrostatic isothermal slab sol^n)

(c) Perturb: $\underline{u}=\Delta\underline{u}$, $p=p_0+\Delta p$, $\rho=\rho_0+\Delta\rho$, $\Psi=\Psi_0+\Delta\Psi$

Uniform, so $\delta X = \Delta X$

(d) $\partial\Delta\rho/\partial t + \rho_0\nabla.\Delta\underline{u} = 0$

$\partial\Delta\underline{u}/\partial t = -c_s^2\frac{1}{\rho_0}\nabla\Delta\rho - \nabla\Delta\Psi$ ($c_s^2 = dp/d\rho$ ie. assuming barotropic)

$\nabla^2\Delta\Psi = 4\pi G\Delta\rho$

(f) Let $\Delta\rho = \rho_1 e^{i(\underline{k}.\underline{x}-\omega t)}$, $\Delta\Psi = \Psi_1 e^{i(\underline{k}.\underline{x}-\omega t)}$, $\Delta\underline{u} = \underline{u}_1 e^{i(\underline{k}.\underline{x}-\omega t)}$

$\therefore -\rho_1\omega + \rho_0\underline{k}.\underline{u}_1 = 0$

$-\omega\underline{u}_1 = -c_s^2\frac{\rho_1}{\rho_0}\underline{k} - \Psi_1\underline{k}$

$-k^2\Psi_1 = 4\pi G\rho_1$

3 equ^{n}s, 3 unknowns $(\underline{u}_1, \Psi_1, \rho_1)$ to be eliminated to get dispersion rel^n:

$$\omega^2 = c_s^2(k^2 - k_J^2)$$

where $k_J^2 = 4\pi G\rho_0/c_s^2$

$\therefore$ unstable if $k_J^2 > k^2$ (as then ω is imaginary → exponentially growing mode)

[In normal sound waves a compression is counteracted by pressure; here the self gravity of the compression makes density & compression increase]

$\therefore$ Unstable if $\lambda \left(= \frac{2\pi}{k}\right) > 2\pi/k_J = \sqrt{\frac{\pi c_s^2}{G\rho_0}}$ = Jean's length

* The mass within this wavelength $\sim \rho_0(2\pi/k_J)^3 \sim \frac{\pi^{3/2}c_s^3}{G^{3/2}\rho_0^{1/2}}$ = Jean's mass

As long wavelength that evolves slowly → wave isothermal $\therefore c_s^2 \sim \frac{R_*}{\mu}T$

→ Jean's mass is $\sim \left(\frac{\pi(R_*/\mu)T}{G}\right)^{3/2}\rho_0^{-1/2}$

* NB This lengthscale is that over which sound crossing time L/c_s is equal to free fall time due to self gravity $L/\sqrt{G\rho_0 L^3/L}$

[→ sound waves are slower than collapse and so can't set up pressure gradients to counteract it]

Astrophysically explains why Universe not smooth

→ star form^n (in GMCs $T\sim 10K$, $n=10^{11}m^{-3}$ → $M_J \sim 1M_\odot$)

galaxy form^n (densities too low in current IGM → galaxies formed when Universe younger and denser)

planet form^n (maybe)

Thermal Instability

Runaway heating (or cooling) following pert^n ΔT from initial state of thermal $=^m$

Simple analysis: If gas heats up at const p, then $\dot{Q} \to \dot{Q} + \left(\frac{\partial \dot{Q}}{\partial T}\right)_p \Delta T$

So unstable if $(\partial\dot{Q}/\partial T)_p < 0$ (NB $\dot{Q}$ is rate of cooling)

Full pert^n analysis

Assume uniform medium, ignore gravity, so perturbed mass and momentum eqs are:

$$\partial\Delta\rho/\partial t + \rho_0 \nabla\cdot\Delta \underline{u} = 0$$

$$\partial\Delta\underline{u}/\partial t = -\frac{1}{\rho_0}\nabla\Delta p$$ (NB have kept Δp here)

To get energy equ^n here, first write $p = K\rho^\gamma$ where K is a variable (ie. not necessarily adiabatic)

$$\therefore dp = \rho^\gamma dK + \gamma\frac{p}{\rho}d\rho = \frac{p}{\rho}d\rho + \left(\frac{R_*}{M}\right)\rho\, dT$$ (from ideal gas law)

$$\therefore \rho^\gamma dK = \frac{p}{\rho}(1-\gamma)d\rho + \left(\frac{R_*}{M}\right)\rho\, dT$$

$$\therefore dK = \rho^{1-\gamma}(1-\gamma)\underbrace{\left[\frac{p}{\rho^2}d\rho - \overbrace{\left(\frac{R_*/M}{\gamma-1}\right)}^{C_v}dT\right]}_{\text{From 1st law of thermodynamics, this is } -\bar{d}Q}$$ ($\bar{d}Q$ = heat absorbed)

$$\therefore dK/dt = (1-\gamma)\rho^{1-\gamma}\dot{Q}$$ (NB similar equ^n derived to consider entropy change across shock)

So for perturbations:

$$d\Delta K/dt = -(\gamma-1)\rho_0^{1-\gamma}\left[(\partial\dot{Q}/\partial p)_\rho \Delta p + (\partial\dot{Q}/\partial\rho)_p\Delta\rho\right]$$
$$= -A^*\Delta p - B^*\Delta\rho$$

And $$\Delta p = \rho_0^\gamma \Delta K + (\gamma p_0/\rho_0)\Delta\rho$$ (Note if we assume adiabatic s.t. $\Delta K = 0$, or isothermal $dT=0$, we get normal sound wave sol^n)

- Now let $\Delta\rho = \rho_1 e^{i(\underline{k}\cdot\underline{x})+qt}$, $\Delta\underline{u} = \underline{u}_1 e^{i\underline{k}\cdot\underline{x}+qt}$, $\Delta K = K_1 e^{i\underline{k}\cdot\underline{x}+qt}$, $\Delta p = p_1 e^{i\underline{k}\cdot\underline{x}+qt}$

  → unstable if q is real and +ve (if imaginary get oscillatory sol^{n}s)

$$\therefore q\rho_1 + \rho_0 i\underline{k}\cdot\underline{u}_1 = 0$$
$$q\underline{u}_1 = -\frac{1}{\rho_0}i\underline{k}p_1$$
$$qK_1 = -A^* p_1 - B^*\rho_1$$
$$p_1 = \rho_0^\gamma K_1 + (\gamma p_0/\rho_0)\rho_1$$

4 equ^{n}s, 4 unknowns ($\rho_1, p_1, K_1, \underline{u}_1$) to be eliminated to get dispersion rel^n:

$$\to E(q) \equiv q^3 + (A^*\rho_0^\gamma)q^2 + (k^2\gamma p_0/\rho_0)q - B^*k^2\rho_0^\gamma = 0$$ (ie a polynomial in q)

$= (\gamma-1)\rho_0^{1-\gamma}\left(\frac{\partial\dot{Q}}{\partial\rho}\right)_p$

As $E(\infty) = \infty$ and $E(0) = -B^*k^2\rho_0^\gamma$, there is a real +ve root (→ unstable) if $B^* > 0$

ie if $(\partial\dot{Q}/\partial\rho)_p = -\left(\frac{T}{\rho}\right)(\partial\dot{Q}/\partial T)_p > 0$

→ $(\partial\dot{Q}/\partial T)_p < 0$ → Field criterion (for instability)

this is sufficient, but not necessary, cond^n for instab.

Same as simple analysis! But, could also be unstable if $B^* < 0$ depending on A^* (must be large and −ve), though Field criterion is good determinate of thermal stability

*

- Eg. If $\dot{Q} = A\rho T^\alpha - H$ (cooling processes; eg const heating by cosmic rays)

  $= (ApM/R_*)T^{\alpha-1} - H$

  then Field criterion says unstable if $\alpha < 1$ (such as optically thin thermal Bremsstrahlung ($\alpha = \frac{1}{2}$))

- Explains temp structure of ISM, as warm 10^4 K and cold 10^2 K neutral atomic phases coexist (along w/ hotter phase ionised by SN), which is only poss. if gas is thermally unstable

* NB growth timescale just given by $1/q$ and depends on k, and in various limits is just balance betw two of the terms. Eg for short wavelength (large k) it's last 2 terms so $q \sim B^*\rho_0^\gamma/(\gamma p_0/\rho_0)$.
If cooling mostly at const p, $B^*\rho_0^\gamma \propto \Delta\dot{Q}$ and $\gamma p_0/\rho_0 \propto E_0$ → this is the thermal timescale

Viscous flows

Remember: Eulerian eqn for rate of change of momentum density in component form:

$$\partial \rho u_i/\partial t = -\partial_j \sigma_{ij} + \rho g_i$$

where σ_{ij} = stress tensor

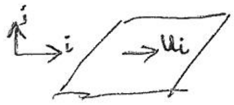
= force per unit area in i dirn acting on surface w normal in j dirn

$$= \underbrace{\rho u_i u_j}_{\text{ram}} + \underbrace{p\,\delta_{ij}}_{\text{thermal}} - \underbrace{\sigma'_{ij}}_{\text{viscous stress tensor}}$$

(momentum advected w fluid) (force due to pressure differentials)

Here, add term due to differential motion of neighbouring fluid elements

Linear shear flow



Consider flow w ||'l streamlines in i dirn

but velocity gradient ⊥ to streamlines in j dirn

Thermal motion → particles have non-zero velocities in j dirn: $v_j \sim \alpha\sqrt{\frac{kT}{m}}$

- appropriate scaling factor
- rms ptcl velocity
- boltzmann's const
- temperature
- ptcle mass

∴ momentum flux across surface w normal in j dirn: $\rho u_i v_j$ (both in $+j$ and $-j$ directions)

That momentum is redistributed in particle interactions over scale of mean free path

$$\delta l \sim \frac{1}{\sigma n} = \frac{m}{\pi a^2 \rho}$$ (for hard spheres radius a)

∴ there is also momentum flux across surface from fluid δl below the surface of

$$\rho(u_i - \delta l\,\partial_j u_i)v_j$$ (and similarly in the $-j$ direction from that δl above the surface)

and net mom flux is: $-\rho\,\partial_j u_i\,\alpha\sqrt{\frac{kT}{m}}\frac{m}{\pi a^2 \rho} = -\eta\,\partial_j u_i$

where $\eta = \left(\frac{5\sqrt{\pi}}{64\pi a^2}\right)\sqrt{mkT}$ = shear viscosity coefficient

∴ $\underline{\sigma'_{ij} = \eta\,\partial_j u_i}$ (as $\partial_j(-\eta\,\partial_j u_i)$ is change in ith component of momentum due to mismatch in i momentum carried through a cell in each of 3 orthogonal dirns)

Note: η is independent of ρ (as although $\rho\uparrow$ → mom flux ↑, also $\delta l\downarrow$)

but does depend on T (as $T\uparrow \rightarrow v_j\uparrow$ so mom flux ↑)

More general form of σ'_{ij} (for molecular viscosity)

zeta

Let $\sigma'_{ij} = \eta\left(\partial_j u_i + \partial_i u_j - \frac{2}{3}\delta_{ij}\partial_k u_k\right) + \zeta\,\delta_{ij}\partial_k u_k$

$$= \begin{pmatrix} \zeta\partial_k u_k & \eta(\partial_1 u_2 + \partial_2 u_1) & \eta(\partial_1 u_3 + \partial_3 u_1) \\ \eta(\partial_2 u_1 + \partial_1 u_2) & \zeta\partial_k u_k & \eta(\partial_2 u_3 + \partial_3 u_2) \\ \eta(\partial_3 u_1 + \partial_1 u_3) & \eta(\partial_3 u_2 + \partial_2 u_3) & \zeta\partial_k u_k \end{pmatrix}$$

Note
- σ'_{ij} is symmetric ($\sigma'_{ij} = \sigma'_{ji}$) s.t. force on jth face of infinitessimally small cube in i dir is balanced by that on ith face in j dir → cube not torqued up (though note that a large cube can be torqued up if velocity field varies across cube)
- diagonal terms are equal, to $\zeta\,\nabla.\underline{u}$ (ie $\propto$ local rate of change of density)
  → if fluid is squashed, normal forces resisting this are same in all dirs
  → fluid is isotropic
- ζ = coefficient of bulk viscosity, and this term is assoc. w̄ mom. transfer due to bulk compression of flow (cf. η term assoc. w̄ mom transfer in shear flows)

Navier-Stokes eqn

Mom eq: $\partial \rho u_i/\partial t = -\partial_j \rho u_i u_j - \partial_j p\delta_{ij} + \partial_j\left[\eta\left(\partial_j u_i + \partial_i u_j - \frac{2}{3}\delta_{ij}\partial_k u_k\right) + \zeta\delta_{ij}\partial_k u_k\right] + \rho g_i$

Using contn and removing δ_{ij}: $\rho\left(\partial u_i/\partial t + u_j\partial_j u_i\right) = -\partial_i p + \rho g_i + \partial_j\left[\eta(\partial_j u_i + \partial_i u_j)\right] - \frac{2}{3}\partial_i(\eta\,\partial_k u_k) + \partial_i\zeta\partial_k u_k$

Bulk viscosity is important in shocks as causes decln in dirn normal to shock front, but is of limited relevance elsewhere, so assume $\zeta \simeq 0$

Also assume $\eta \simeq$ const (eg. isothermal)

$\rightarrow \quad \rho\left(\partial u_i/\partial t + u_j\partial_j u_i\right) = -\partial_i p + \rho g_i + \eta\partial_j\partial_j u_i + \frac{1}{3}\eta\partial_i\partial_j u_j$ (NB terms combine)

$$\partial\underline{u}/\partial t + \underline{u}.\nabla\underline{u} = -\frac{1}{\rho}\nabla p - \nabla\Phi + \nu\left[\nabla^2\underline{u} + \frac{1}{3}\nabla(\nabla.\underline{u})\right]$$

where $\nu \equiv \eta/\rho$ = kinematic viscosity

*

Evolution of vorticity

Take curl of NS, noting $\underline{\omega} = \nabla\wedge\underline{u}$, assuming fluid barotropic

$$\partial\underline{\omega}/\partial t - \nabla\wedge(\underline{u}\wedge\underline{\omega}) = \nabla\wedge\left[\nu\nabla^2\underline{u} + \tfrac{1}{3}\nu\cancel{\nabla}(\nabla.\underline{u})\right]$$

$$= \nu\nabla\wedge\left[\nabla(\nabla.\underline{u}) - \nabla\wedge(\nabla\wedge\underline{u})\right] + \nabla\nu\wedge\left[\nabla^2\underline{u} + \tfrac{1}{3}\nabla(\nabla.\underline{u})\right]$$

(as div curl = 0)

If $\nabla\nu \ll \nu$

$$\simeq -\nu\left[\nabla(\cancel{\nabla}.\underline{\omega}) - \nabla^2\underline{\omega}\right] + O(\nabla\nu)$$

$$= \nu\nabla^2\underline{\omega} + O(\nabla\nu)$$

→ by Kelvin's vorticity theorem, the flux of vorticity is not conserved for viscous fluid

* If $\eta \neq$ const, extra terms on RHS: $\underbrace{(\partial_j\eta)(\partial_j u_i}_{(\nabla\eta.\nabla)\underline{u}} + \partial_i u_j) - \underbrace{\tfrac{2}{3}(\partial_i\eta)(\partial_k u_k)}_{-\frac{2}{3}\nabla\eta\,\nabla.\underline{u}}$

As $\partial_i\left[(\partial_j\eta)u_j\right] = (\partial_j\eta)\partial_i u_j + u_j\partial_i\partial_j\eta$

$\therefore (\partial_j\eta)\partial_i u_j = \nabla(\nabla\eta.\underline{u}) - (\underline{u}.\nabla)\nabla\eta$

$\therefore$ Extra terms on RHS are: $\frac{1}{\rho}\left[\nabla\eta.\nabla\right]\underline{u} - \frac{2}{3}\frac{1}{\rho}\nabla\eta\,\nabla.\underline{u} + \frac{1}{\rho}\nabla(\nabla\eta.\underline{u}) - \frac{1}{\rho}(\underline{u}.\nabla)\nabla\eta$

Energy dissipation

Consider an incompressible fluid (to make our lives easier!)
The rate of change of KE is:

$$\partial \tfrac{1}{2}\rho u^2/\partial t = u_i\, \partial \rho u_i/\partial t = -u_i \partial_j \sigma_{ij} + u_i \rho g_i$$
$$= -u_i \partial_j \rho u_i u_j - u_i \partial_j \delta_{ij} p + u_i \partial_j \sigma'_{ij} + \rho u_i \partial_i \Psi$$
$$= -u_i \partial_j \rho u_i u_j - u_i \partial_i p + \partial_j u_i \sigma'_{ij} - \sigma'_{ij} \partial_j u_i + \rho u_i \partial_i \Psi$$

Incompressible $\Rightarrow \partial_i u_i = 0$, and swap indices where both are repeated:

$$= -\partial_j \rho u_j \tfrac{1}{2} u_i . u_i - \partial_i p u_i + \partial_i u_j \sigma'_{ij} - \sigma'_{ij} \partial_j u_i + \partial_i \rho u_i \Psi$$
$$= -\partial_i \left[\rho u_i \left(\tfrac{1}{2}u^2 + p/\rho + \Psi \right) - u_j \sigma'_{ij} \right] - \sigma'_{ij} \partial_j u_i$$

Integrate over a volume inside fluid (applying divergence theorem $\int \nabla . \underline{A}\, dV = \int \underline{A} . d\underline{S}$)

$$\partial \int \tfrac{1}{2}\rho u^2 dV/\partial t = \partial E_k/\partial t = -\int [\underbrace{\rho \underline{u} \left(\tfrac{1}{2}u^2 + p/\rho + \Psi\right)}_{\text{advection of KE and PE and internal pressure}} - \underbrace{\underline{u} . \underline{\sigma}'}_{\text{viscous stresses integrated over surface}}] . d\underline{S} - \underbrace{\int \sigma'_{ij} \partial_j u_i\, dV}_{\text{rate of viscous dissipation in vol.}}$$

The first moves energy to neighbouring volumes, but the latter is dissipated as heat within vol.

eg choose volume encompassing whole fluid

$$\partial E_{tot}/\partial t = -\int \sigma'_{ij} \partial_j u_i\, dV$$
$$= -\tfrac{1}{2} \int \sigma'_{ij} (\partial_j u_i + \partial_i u_j)\, dV \quad (\text{as } \sigma'_{ij} = \sigma'_{ji})$$
$$= -\tfrac{1}{2} \int \eta (\partial_j u_i + \partial_i u_j)^2\, dV \quad (\text{as } \sigma'_{ij} = \eta(\partial_j u_i + \partial_i u_j) \text{ for incompressible})$$

$\therefore \eta$ is +ve from 2nd law of thermo.

Eg Incompressible steady flow down circular pipe

Continuity and symmetry → velocity is only a function of R, not z or ϕ

$$\text{N-S} \rightarrow \underbrace{\cancel{\partial \underline{u}/\partial t}}_{\text{steady}} + \underbrace{\cancel{\underline{u}\cdot\nabla\underline{u}}}_{\text{by symmetry}} = -\frac{1}{\rho}\nabla p - \underbrace{\cancel{\nabla\Psi}}_{\text{ignore gravity}} + \nu\left[\nabla^2\underline{u} + \frac{1}{3}\underbrace{\cancel{\nabla(\nabla\cdot\underline{u})}}_{\text{incompressible}}\right]$$

$$\therefore \quad \frac{1}{\rho}\nabla p = \nu\nabla^2\underline{u}$$

$\hat{R}/\hat{\phi}$ components → $\partial p/\partial R = \partial p/\partial\phi = 0$

$\hat{\underline{z}}$ component → $\frac{1}{\rho}dp/dz = \nu\frac{1}{R}\frac{d}{dR}(R\,dU/dR) = \text{const.}$

If pressure changes by Δp over length L: $= \Delta p/\rho L$

Integrating: $U = -\frac{\Delta p}{4\rho\nu L}R^2 + a\ln R + b$

b.c. of $U \neq \infty$ @ $R=0 \rightarrow a=0$

$U=0$ @ $R=R_0 \rightarrow \underline{U = \frac{\Delta p}{4\nu\rho L}(R_0^2 - R^2)}$

So mass flow rate $Q = \int_0^{R_0} 2\pi\rho U R\,dR$

$$\underline{= \frac{\pi\Delta p}{8\nu L}R_0^4}$$

Turbulence

- → flow is irregular in space and time
- → break variables into a mean and a fluctuating part
- → transition from laminar to turbulent flow depends on Reynold's number, Re
- → Re is dimensionless combination of ν (L^2T^{-1}), U (LT^{-1}) and lengthscale associated with flow L (L)

$\therefore \quad Re = LU/\nu$

(transition is at $Re \approx 3000$)

Accretion discs = circular shear flows

eg: circumplanetary, circumstellar, circum-super massive black hole

origin: gas w non zero ang. mom. bound to central object settles into plane defined by mean ang. mom. of gas supply (residual motion damped by shocks)

shear: centrifugal forces balance grav. attraction

$\Rightarrow \Omega^2 R \simeq GM/R^2$

$\therefore \Omega = \sqrt{GM/R^3} \equiv$ Keplerian motion

As $d\Omega/dR \neq 0$ viscosity $\rightarrow$ ang. mom. is transferred outward from faster inner regions causing inner material to move in

Equations

Use cylindrical coords and assume axisymmetric $(\partial/\partial\phi = 0)$ and negligible vertical motion $(u_z = 0)$

ie $\underline{u} = (u_R, u_\phi, 0)$

NB/ $\nabla\Psi = (\partial\Psi/\partial R)\hat{\underline{e}}_R + (\partial\Psi/\partial z)\hat{\underline{e}}_z$

$\nabla.\underline{A} = \frac{1}{R}\partial/\partial R[RA_R] + \partial A_z/\partial z$

$\nabla^2\Psi = \frac{1}{R}\partial/\partial R[R\partial\Psi/\partial R] + \partial^2\Psi/\partial z^2$

and $(\underline{A}.\nabla)\underline{B}$ for $\partial/\partial\phi = \partial/\partial z = 0$

$= (A_R\partial B_R/\partial R - A_\phi B_\phi/R)\hat{\underline{e}}_R + (A_R\partial B_\phi/\partial R + A_\phi B_R/R)\hat{\underline{e}}_\phi + A_R\frac{\partial B_z}{\partial R}\hat{\underline{e}}_z$

and $\nabla^2\underline{u} = (\partial^2 u_R/\partial R^2 + \frac{1}{R}\partial u_R/\partial R - u_R/R^2)\hat{\underline{e}}_R + (\partial^2 u_\phi/\partial R^2 + \frac{1}{R}\partial u_\phi/\partial R - u_\phi/R^2)\hat{\underline{e}}_\phi$

Continuity $\partial\rho/\partial t + \nabla.(\rho\underline{u}) = 0$

$\therefore \partial\rho/\partial t + \frac{1}{R}\partial/\partial R[R\rho u_R] = 0$

Define $\Sigma = \int\rho\, dz$ = surface density, and integrate over z

$\therefore \partial\Sigma/\partial t + \frac{1}{R}\partial/\partial R[R\Sigma u_R] = 0$ (1)

NS $\partial\underline{u}/\partial t + \underline{u}.\nabla\underline{u} = -\frac{1}{\rho}\nabla p - \nabla\Psi + \nu\nabla^2\underline{u} + \frac{1}{3}\nu\nabla(\nabla.\underline{u}) + \frac{1}{\rho}(\nabla\eta.\nabla)\underline{u} + \frac{1}{\rho}\nabla(\nabla\eta.\underline{u}) - \frac{1}{\rho}(\underline{u}.\nabla)\nabla\eta - \frac{2}{3}\frac{1}{\rho}\nabla\eta\nabla.\underline{u}$

For $\partial/\partial\phi = u_z = 0$, and neglecting vertical variation & u_ϕ, the ϕ component is

$\therefore \partial u_\phi/\partial t + u_R\partial u_\phi/\partial R + u_R u_\phi/R = \nu[\partial^2 u_\phi/\partial R^2 + \frac{1}{R}\partial u_\phi/\partial R - u_\phi/R^2] + \frac{1}{\rho}\left(\frac{\partial\eta}{\partial R}\right)\left[\frac{\partial u_\phi}{\partial R} - \frac{u_\phi}{R}\right]$

Integrate over z and define $\langle\nu\rangle = \int\rho\nu\, dz / \int\rho\, dz = \int\eta\, dz/\Sigma$

$\therefore \Sigma(\partial u_\phi/\partial t + u_R\partial u_\phi/\partial R + u_\phi u_R/R) = \langle\nu\rangle\Sigma[\partial^2 u_\phi/\partial R^2 + \frac{1}{R}\partial u_\phi/\partial R - u_\phi/R^2]$
$+ \partial[\langle\nu\rangle\Sigma]/\partial R\,[\partial u_\phi/\partial R - u_\phi/R]$

Tidying up:

$$\Sigma\left(\partial u_\phi/\partial t + \frac{u_R}{R}\partial Ru_\phi/\partial R\right) = \frac{1}{R^2}\frac{\partial}{\partial R}\left[\nu\Sigma R^3\, d\Omega/dR\right] \quad (2)$$

where $\Omega = u_\phi/R$ and we've dropped $\langle\rangle$ around ν

Combine Ru_ϕ(1) + R(2),

$$\therefore \underbrace{\partial[R\Sigma u_\phi]/\partial t}_{\text{Eulerian rate of change of ang. mom / unit area of annulus at } R} + \underbrace{\frac{1}{R}\frac{\partial}{\partial R}[\Sigma R^2 u_\phi u_R]}_{\text{Net rate of ang. mom. loss from this unit area due to advection of ang. mom. w radial flow}} = \underbrace{\frac{1}{R}\frac{\partial}{\partial R}[\nu\Sigma R^3\, d\Omega/dR]}_{\text{Net viscous torques per unit area}} \quad (3)$$

NB viscous stress tensor $\Rightarrow \sigma_{R\phi} = \eta R\, d\Omega/dR$ = force/area on surface w normal in radial dir^n in tangential dir^n (ie from annuli rubbing against e.o.)

$\therefore$ torque $G = R \times \sigma_{R\phi} \times 2\pi R 2H$

$= 2\pi\nu\Sigma R^3\, d\Omega/dR$ (as $\Sigma = 2H\rho$)

Different combination of ① and ②

Expect $\partial u_\phi/\partial t = 0$ (as set by balance of centrifugal and gravitational forces), so ② gives:

$$u_R = \frac{\partial}{\partial R}\left[\nu\Sigma R^3 d\Omega/dR\right] \Big/ R\Sigma\, \partial(R^2\Omega)/\partial R \qquad ④$$

For keplerian disk $\Omega = \sqrt{GM/R^3}$

$$u_R = -3\,\partial\left[\nu\Sigma R^{1/2}\right]/\partial R \Big/ \Sigma R^{1/2} \qquad ④K$$

Substitute into ① to get

$$\partial\Sigma/\partial t = \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}\left[\nu\Sigma R^{1/2}\right]\right] \qquad ⑤ \equiv \text{viscous diffusion eqn}$$

Solution for constant viscosity

Rewrite ⑤ with $s = 2R^{1/2}$

∴ $\partial\Sigma/\partial t = \frac{12}{s^3}\frac{\partial^2}{\partial s^2}\left[\nu\Sigma s\right]$

If ν = const, then (as $\partial s/\partial t = 0$)

$$\partial s\Sigma/\partial t = \frac{12\nu}{s^2}\frac{\partial^2}{\partial s^2}(s\Sigma)$$

∴ can write $s\Sigma = S(s)T(t)$ and

$$\frac{1}{T}dT/dt = \frac{12\nu}{s^2}\frac{1}{S}d^2S/ds^2 = -\lambda^2$$ (ie have to be a constant to be indep of t and s)

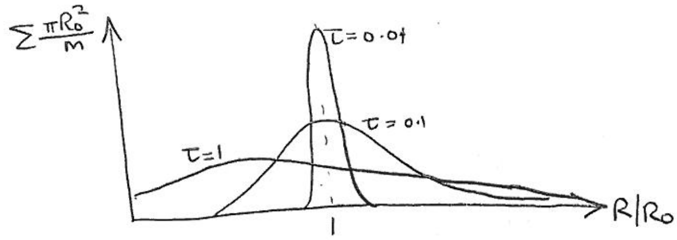
→ T is exponential

S is a Bessel function

eg If $\Sigma(R,0) = \frac{m}{2\pi R_0}\delta(R-R_0)$ (ie. start with all mass m at R_0)

then $\Sigma(x,\tau) = \left(\frac{m}{\pi R_0^2\tau x^{1/4}}\right)e^{-\frac{(1+x^2)}{\tau}} I_{1/4}(2x/\tau)$ – modified Bessel function

where $x = R/R_0$, $\tau = 12\nu t/R_0^2$ are dimensionless variables



- see part II CATAM project 23.8!
- most of the mass moves in, but outer parts move out (see ④)
- most ang mom ends up in small mass at large R.

Note:
- characteristic spreading timescale is $t_{visc} \sim R_0^2/\nu$ (also = momentum in annulus / torque on that annulus)
- as Reynolds # $Re = Ru_\phi/\nu$ $t_{visc} \sim \frac{R}{u_\phi}Re \sim t_{orb}Re$
- for molecular viscosity $\nu \sim c_s l$ ∴ $Re = Ru_\phi\sigma n/c_s$
  → as MMSN had $n \sim 10^{22}\ m^{-3}$, $\sigma \sim 10^{-20}\ m^2$, $T \sim 300\ K \rightarrow c_s \sim 1600\ m/s$ @ 1.5×10^{11} m ($u_\phi \sim 3\times10^4$ m/s)
  ∴ $Re \sim 10^{14}$
  → molecular viscosity is negligible
- but flow is turbulent → $\nu_{turb} \sim v_{turb}l_{turb}$ which can be significant (also MHD turbulence)

Solution for steady thin disk

Assume some supply of fluid maintaining steady ($\partial/\partial t$) accⁿ rate $\dot{m}$

(1) $\rightarrow \frac{1}{R}\frac{\partial}{\partial R}\left[R\Sigma u_R\right] = 0$

$\therefore\ R\Sigma u_R = \text{const} = -\dot{m}/2\pi$

(4R) for Keplerian disk

$\rightarrow u_R = -\frac{\dot{m}}{2\pi}(R\Sigma)^{-1} = -3\,\partial\left[\nu\Sigma R^{1/2}\right]/\partial R \,/\, \Sigma R^{1/2}$

$\therefore\ \frac{\dot{m}}{3\pi}\left[R^{1/2}\right] = \left[\nu\Sigma R^{1/2}\right]$

Use b.c. that $\nu\Sigma = 0$ at $R = R_*$ (since star rotation $\Omega_* < \Omega_K$ – and this is only way to achieve that)

$$\therefore\ \nu\Sigma = \frac{\dot{m}}{3\pi}\left[1 - (R_*/R)^{1/2}\right]$$

More realistically $\Omega \neq \Omega_K$ near star, but reaches a peak at $R = R_* + \epsilon$ of $\Omega_K(R_*) + \delta$ then decreases to Ω_* at star

But using (3), and integrating from $R_* + \epsilon$ to R, gives same result.

Consider rate of energy lost per unit area from an annulus due to viscous dissipⁿ

$$F_{diss} = \int \sigma_{ij}'\,\partial_j u_i\, dV \,/\, 2\pi R\, dR$$

$$= \tfrac{1}{2}\int \eta\left(\partial_j u_i + \partial_i u_j\right)^2 dz$$

$$= \nu\Sigma\left(R\, d\Omega/dR\right)^2$$

(NB this was derived assuming fluid was incompressible, turns out to be ok as other terms are much smaller for thin disk)

Using Kepler's law and above $\nu\Sigma$ equⁿ:

$$F_{diss} = \frac{3GM\dot{m}}{4\pi R^3}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right]$$

If disk is optically thick, that energy is radiated away as a BB at temperature T_{eff} s.t.

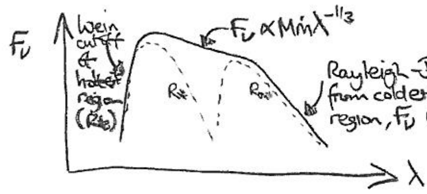
as radiates from top and bottom of disk — $2\sigma_{SB}\, T_{eff}(R)^4 = F_{diss}$ (Stefan Boltzmann) (ie energy balance)

$\therefore$ for $R \gg R_*$, $T_{eff} \propto R^{-3/4}$

And emission spectrum is

$$F_\nu = \int_{R_*}^{R_{out}} B_\nu\left(T_{eff}(R)\right) 2\pi R\, dR$$

where $B_\nu = \frac{2h}{c^2}\nu^3 \Big/ \left(e^{\frac{h\nu}{kT}} - 1\right)$ = Planck function (h = Planck's const)



NB this is independent of form of η and depends only on $\dot{m}$, R_*, R_{out}, M. So can't learn η from steady disks. (rather look at non steady disks, as changes to L occur on t_{visc} timescales)

Disk luminosity $L = \int_{R_*}^{R_{out}} F_{diss}\, 2\pi R\, dR$

$= GM\dot{m}/2R_*$

This energy comes from grav. pot en. of infalling material, and since it loses GM/R_* by time it reaches the star $\Rightarrow$ $GM\dot{m}/R_*$ is rate of loss due to inflow

$\Rightarrow$ half of this energy is radiated from disk (L)

$\Rightarrow$ other half is in KE of material as it reaches $*$ (lost in boundary layer?)

Note that locally (ie. in an annulus) it is not true that grav. pot en is converted into heat, since the energy equⁿ shows that this may be transferred to other annuli through viscous stresses (eg).

Magnetohydrodynamics (MHD)

Astrophysical fluids usually highly ionised ($\therefore$ highly conducting) and permeated by magnetic fields

MHD is an approximation that considers interaction betw fluid and mag fields

→ particles moving through mag. field experience Lorentz force
(as this is $\propto \underline{u} \wedge \underline{B}$ particles move freely along mag field lines but are constrained $\perp$)

→ particle motion also changes mag. field

Start with Maxwell's eqns

(M1) $\nabla . \underline{B} = 0$ (no source or sink of mag. flux)

(M2) $\nabla . \underline{E} = q / \varepsilon_0$ ($\underline{E}$ = electric field, q = charge density, ε_0 = (vacuum) electrical permittivity)

(M3) $\frac{1}{\mu_0} \nabla \wedge \underline{B} = \underline{j} + \varepsilon_0 \partial \underline{E} / \partial t$ (μ_0 = magnetic permeability, $\underline{j}$ = current density, 2nd term is displacement current)

(M4) $\partial \underline{B} / \partial t = - \nabla \wedge \underline{E}$ (electromagnetic induction)

Use Ohm's law to relate $\underline{j}$ to $\underline{E}$ and $\underline{B}$.
In the frame moving with fluid (dashes)

(O) $\underline{j}' = \sigma \underline{E}'$ (σ = electrical conductivity)

Use Lorentz transformation (see relativity course) for frame ' wrt inertial frame (nodash)

$$\underline{E}' = (1-\gamma)\left(\frac{\underline{u}.\underline{E}}{u^2}\right)\underline{u} + \gamma\left(\underline{E} + \underline{u} \wedge \underline{B}\right)$$

$$\underline{B}' = (1-\gamma)\left(\frac{\underline{u}.\underline{B}}{u^2}\right)\underline{u} + \gamma\left(\underline{B} - \frac{1}{c^2}\underline{u} \wedge \underline{E}\right)$$

where $\gamma = \left(1 - u^2/c^2\right)^{-1/2}$

Approximation ① $u^2 \ll c^2$ (non relativistic)

→ Simplifies transformation to

$$\underline{E}' = \underline{E} + \underline{u} \wedge \underline{B}$$
$$\underline{B}' = \underline{B}$$

→ Can neglect displacement current st. (M3) → $\underline{j} = \frac{1}{\mu_0} \nabla \wedge \underline{B}$ (Ampère's law)

Why? Consider the flow has typical lengthscale L and timescale T, ie $u \simeq L/T$

fractional contribution of d.c. $X_{dc} = \mu_0 \varepsilon_0 \partial \underline{E} / \partial t \,/\, \nabla \wedge \underline{B} \simeq \mu_0 \varepsilon_0 \left(\frac{E}{B}\right)\left(\frac{L}{T}\right)$

But $\mu_0 \varepsilon_0 = \frac{1}{c^2}$ and (M4) → $E/B \simeq L/T$

$\therefore X_{dc} \simeq (u/c)^2$

→ This in turn → $\underline{j}' = \underline{j}$ (from the new (M3) & $\underline{B} = \underline{B}'$)

→ Ohm's law (O) → $\underline{j} = \sigma\left(\underline{E} + \underbrace{\underline{u} \wedge \underline{B}}_{\text{current induced by Lorentz forces}}\right)$

Magnetic flux

Equate (O) with (M3) $\Rightarrow \nabla\wedge\underline{B} = \mu_0\sigma(\underline{E}+\underline{u}\wedge\underline{B})$

Curl of this $\Rightarrow \nabla\wedge(\nabla\wedge\underline{B}) = \mu_0\sigma[\nabla\wedge\underline{E} + \nabla\wedge(\underline{u}\wedge\underline{B})]$

Vector ID and (M4) $\Rightarrow -\nabla^2\underline{B} + \nabla(\cancel{\nabla\cdot\underline{B}}) = \mu_0\sigma[-\partial\underline{B}/\partial t + \nabla\wedge(\underline{u}\wedge\underline{B})]$

Use (M1) $\therefore \underbrace{\partial\underline{B}/\partial t}_{\text{rate of change of mag field}} - \underbrace{\nabla\wedge(\underline{u}\wedge\underline{B})}_{\text{convection of field by fluid}} = \underbrace{\frac{1}{\mu_0\sigma}\nabla^2\underline{B}}_{\text{diffusion through conductive term}}$

Remember $\frac{D}{Dt}\int\underline{\omega}.d\underline{S} = \int_S d\underline{S}.[\partial\underline{\omega}/\partial t - \nabla\wedge(\underline{u}\wedge\underline{\omega})]$ applies to ANY quantity $\underline{\omega}$

So if $\Phi = \int\underline{B}.d\underline{S}$ = magnetic flux through a surface

$D\Phi/Dt = \int_S d\underline{S}.[\frac{1}{\mu_0\sigma}\nabla^2\underline{B}]$ (ie Φ only changes through diffusion term)

Approximation (2) $\sigma\to\infty$ (infinite conductivity)

$\rightarrow D\Phi/Dt = 0$

$\therefore$ Mag. field moves with fluid

A mag. field line consists of same particles at all times

The mag. field is "frozen in" to a perfectly conducting fluid

$\rightarrow$ Ohm's law (O) $\Rightarrow \underline{E} + \underline{u}\wedge\underline{B} = 0$

$\rightarrow$ Taking $\underline{B}$. this $\Rightarrow \underline{E}.\underline{B} = 0$ and so $\underline{E}$ is $\perp$ to $\underline{B}$

$\rightarrow$ (M4) becomes $\underline{\partial\underline{B}/\partial t = \nabla\wedge(\underline{u}\wedge\underline{B})}$ (induction eq.)

Lorentz force

Need to include the electromagnetic force acting on fluid in momentum eq. s.t.

(mom) $\rho(\partial\underline{u}/\partial t + \underline{u}.\nabla\underline{u}) = -\nabla p + \underline{f_L}$

where $\underline{f_L} = q\underline{E} + \underline{j}\wedge\underline{B}$

$= q\underline{E} + \frac{1}{\mu_0}(\nabla\wedge\underline{B})\wedge\underline{B}$ (from (M3))

Approximation (3) $q\to 0$ (charge neutrality)

Consider fractional contribution of $\underline{E}$ field to $\underline{f_L}$

$X_E = q\underline{E} / \frac{1}{\mu_0}(\nabla\wedge\underline{B})\wedge\underline{B}$

$= \mu_0\varepsilon_0(\nabla.\underline{E})\underline{E} / (\nabla\wedge\underline{B})\wedge\underline{B}$ (from (M2))

$\simeq (u/c)^2$ (ie. comes directly from non-relativistic assumption)

$\rightarrow$ Set $q=0$ s.t. (M2) $\Rightarrow \nabla.\underline{E} = 0$

and in (mom) $\underline{f_L} = \frac{1}{\mu_0}(\nabla\wedge\underline{B})\wedge\underline{B}$

Magnetic pressure

$= \frac{1}{\mu_0}[\underbrace{-\nabla(\frac{B^2}{2})}_{\text{behaves like hydrostatic pressure of magnitude } p_{mag} = B^2/2\mu_0} + (\underline{B}.\nabla)\underline{B}]$

$\therefore$ relative importance of ram vs thermal vs magnetic pressure dep on ratio of $\frac{1}{2}\rho u^2$ to ρc_s^2 to $B^2/2\mu_0$

and equating KE and mag. energy densities defines a velocity

$V_A = [B^2/\rho\mu_0]^{1/2}$ = Alfvén velocity

Charge Neutrality

Consider a static fluid composed of protons (+) and electrons (−)
Proton density is uniform $n^+ = n_0$
But there's small charge imbalance s.t. $n^- = n_0 + n_1(\underline{r}, t)$ where $n_1 \ll n_0$
Resulting charge density $q = n_0 e - (n_0 + n_1)e$ (where e = electronic charge)
$= -n_1 e$

So (M2) $\Rightarrow \nabla . \underline{E} = -n_1 e / \varepsilon_0$

This results in a force on electrons causing a pert^n to their velocity $\underline{u}_1(\underline{r}, t)$
$\Rightarrow m_e \partial \underline{u}_1 / \partial t = -e\underline{E}$

Use continuity eq^n
$\rightarrow \partial n^- / \partial t + \nabla . (n^- \underline{u}_1) = 0$
$\therefore \partial n_1 / \partial t + n_0 \nabla . \underline{u}_1 = 0$ (to 1st order)

Differentiate $\Rightarrow \partial^2 n_1 / \partial t^2 + n_0 \nabla . \partial \underline{u}_1 / \partial t = 0$
$\therefore \partial^2 n_1 / \partial t^2 - n_0 \left(\frac{e}{m_e}\right) \nabla . \underline{E} = 0$ (sub in for $\partial u_1 / \partial t$)
$\therefore \underline{\partial^2 n_1 / \partial t^2 + (n_0 e^2 / m_e \varepsilon_0) n_1 = 0}$ (sub in for $\nabla . \underline{E}$)

So charge imbalance oscillates at plasma frequency $\omega_p = \sqrt{n_0 e^2 / m_e \varepsilon_0}$
$\rightarrow \nu_p = \omega_p / 2\pi \simeq 9.0\sqrt{n_0}$ Hz
eg for $n_0 \sim 10^{12}\ m^{-3}$ in Earth's ionosphere $\nu_p \sim 10^7$ Hz

Thus charge imbalances are rapidly neutralised
and more subtlely → EM rad^n in $\nu < \nu_p$ is reflected from ionosphere
as electrons respond to incoming wave rather than allow it to be transmitted

If the electron velocity is the thermal velocity, ie $u_1 \sim \sqrt{\frac{KT_e}{m_e}}$
then the lengthscale associated with oscillation is
$\lambda_D = u_1 / \omega_p$
$= (\epsilon_0 K T_e / n_0 e^2)^{1/2}$ = Debye length $= 70\sqrt{T_e / n_0}$ m

which is an "effective shielding length"
→ thermal motions iron out plasma oscillations on that scale
eg. in ionosphere where $T_e \sim 10^3$ K, $\lambda_D \sim 1$ mm

Waves in plasmas

ⓐ Equations: $\partial\rho/\partial t + \nabla.(\rho\underline{u}) = 0$

$$\partial\underline{u}/\partial t + \underline{u}.\nabla\underline{u} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\frac{1}{\mu_0}(\nabla\wedge\underline{B})\wedge\underline{B}$$

$$\partial\underline{B}/\partial t = \nabla\wedge(\underline{u}\wedge\underline{B})$$

$$p = K\rho^\gamma$$

ⓑ Unperturbed: $\rho=\rho_0$, $\underline{u}=\underline{u}_0=0$, $p=p_0$, $\underline{B}=\underline{B}_0$ ie uniform

ⓒ Consider Lagrangian perturbation, called subscript 1

Since uniform medium $\delta X = X_1$

∴ substitute $\rho=\rho_0+\rho_1$, $\underline{u}=\underline{u}_1$, $p=p_0+p_1$, $\underline{B}=\underline{B}_0+\underline{B}_1$

ⓓ First order perturbed equ^ns

- $\partial\rho_1/\partial t + \rho_0\nabla.\underline{u}_1 = 0$
- $\rho_0\partial\underline{u}_1/\partial t = -\cancel{\nabla p_1}^{\,c_s^2\nabla\rho_1} + \frac{1}{\mu_0}[\nabla\wedge(\cancel{\underline{B}_0}^{\text{as uniform}}+\underline{B}_1)]\wedge(\underline{B}_0+\cancel{\underline{B}_1}^{\text{as 2nd order}})$

$$\therefore\ \underline{\rho_0\partial\underline{u}_1/\partial t + c_s^2\nabla\rho_1 + \frac{1}{\mu_0}\underline{B}_0\wedge(\nabla\wedge\underline{B}_1) = 0}$$

- $\underline{\partial\underline{B}_1/\partial t = \nabla\wedge(\underline{u}_1\wedge\underline{B}_0)}$

ⓔ Wave equ^n: $\partial/\partial t$ of mom. eq.

$$\therefore\ \rho_0\partial^2\underline{u}_1/\partial t^2 + c_s^2\nabla\partial\rho_1/\partial t + \frac{1}{\mu_0}\underline{B}_0\wedge(\nabla\wedge\partial\underline{B}_1/\partial t) = 0$$

$$\therefore\ \underline{\partial^2\underline{u}_1/\partial t^2 - c_s^2\nabla(\nabla.\underline{u}_1) + \underline{V}_A\wedge(\nabla\wedge[\nabla\wedge(\underline{u}_1\wedge\underline{V}_A)]) = 0}$$

where $\underline{V}_A = \underline{B}_0/\sqrt{\mu_0\rho_0}$ = Alfvén velocity (vectorial)

ⓕ Solution: ~~let~~ $\underline{u}_1 = \underline{\tilde{u}}_1 e^{i(\underline{k}.\underline{x}-\omega t)}$

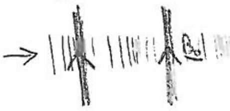
$$\therefore\ -\omega^2\underline{u}_1 + c_s^2(\underline{k}.\underline{u}_1)\underline{k} + \underline{V}_A\wedge(\nabla\wedge[\nabla\wedge(\underline{u}_1\wedge\underline{V}_A)]) = 0$$

see handout: $\therefore\ \underline{-\omega^2\underline{u}_1 + (c_s^2+V_A^2)(\underline{k}.\underline{u}_1)\underline{k} + (\underline{V}_A.\underline{k})[(\underline{V}_A.\underline{k})\underline{u}_1 - (\underline{V}_A.\underline{u}_1)\underline{k} - (\underline{k}.\underline{u}_1)\underline{V}_A] = 0}$

⊥ to lines of const phase; ||'l to unperturbed field

If $\underline{k} \perp \underline{V}_A$ last term vanishes → longitudinal magnetosonic wave

phase velocity $= \sqrt{c_s^2+V_A^2}$; ie. depends on sum of hydrostatic and magnetic pressure ;



Sim to ordinary sound wave, but as field is frozen into fluid, mag field lines are bunched together in compressions → extra magnetic pressure resisting compression

If $\underline{k}$ ||'l V_A $\underline{k} = \underline{V}_A\frac{k}{V_A}$, so $(k^2V_A^2-\omega^2)\underline{u}_1 + \left(\frac{c_s^2}{V_A^2}-1\right)k^2(\underline{V}_A.\underline{u}_1)\underline{V}_A = 0$

∴ Two types of motion

ⓘ Longitudinal wave ie $\underline{u}_1$ ||'l to $\underline{V}_A$ → $\omega/k = c_s$; (as $(\underline{V}_A.\underline{u}_1)\underline{V}_A = V_A^2\underline{u}$) ;

; NB field is undisturbed by compressions, so no hydromagnetic forces ;

Alfvén wave

ⓘⓘ Transverse wave ie $\underline{V}_A.\underline{u}_1 = 0$ travelling at phase velocity $\omega/k = V_A$; ie first term = 0 ;

NB motion of fluid elements is transverse to dir^n of propagation

This is purely magnetohydrodynamic wave that dep on tension in magnetic field lines and inertia of material which moves ie field ; since field is frozen in ;

EG Sun photosphere $\rho_0 \simeq 10^{-4}$ kg/m³ → $V_A \simeq 10^5 B$ m/s whereas T = 5770K so $c_s \simeq 10^4$ m/s

at surface $B \simeq 10^{-4}$ T ($V_A \ll c_s$), but in sunspots $B \simeq 0.3$T ($V_A > c_s$)

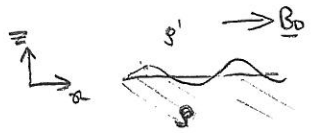
EG GMCs KE and mag E densities comparable and ≫ thermal energies

∴ shocks softened if fluid elements collide at supersonic but sub Alfvenic speeds as Alfvén wave can carry news of impending collision to soften blow

→ structure of dense ISM req. understanding of magnetised shocks.

Rayleigh Taylor Instability $\bar{w}$ Magnetic fields

Can mag. fields stabilise fluids that are R-T unstable?



Consider 2 incompressible inviscid fluids in uniform (vertically down) grav. field $\bar{w}$ interface at $z = 0$

i.e. as before but w/o flow in x dir^n.

and apply uniform mag field ‖'l to interface

Equations:

$$\partial\rho/\partial t + \nabla.(\rho \underline{u}) = 0 \qquad \text{cont}$$

$$\rho(\partial \underline{u}/\partial t + \underline{u}.\nabla \underline{u}) = -\nabla p + \rho \underline{g} + \frac{1}{\mu_0}(\nabla \wedge \underline{B}) \wedge \underline{B} \qquad \text{mom}$$

$$\partial \underline{B}/\partial t = \nabla \wedge (\underline{u} \wedge \underline{B}) \qquad \text{M4}$$

where $\underline{g} = -g\hat{e}_z$

Equilibrium: $\underline{u}_0 = 0,\ \underline{B}_0 = B_0\hat{e}_x,\ \rho = \rho_0,\ dp_0/dz = -\rho g$

Perturb (cont): $\underline{\xi} = \underline{\xi}(z)e^{i(kx-\omega t)}$ where $\underline{\xi}(z) = [\xi_x(z),\ 0\ ,\ \xi_z(z)]$ (NB not looking at pert^{ns} in y dir^n here but could do a more general analysis)

$\underline{u} = \underline{u}_1 = \partial\underline{\xi}/\partial t = -i\omega\underline{\xi}$

continuity with incompressibility $\rightarrow \nabla.\underline{u} = 0$

$\therefore ik\xi_x + d\xi_z/dz = 0 \qquad *$

Perturb (M4): $\underline{B} = \underline{B}_0 + \underline{B}_1(z)e^{i(kx-\omega t)}$ (NB this is Eulerian pert^n, note that $\underline{B}$ is frozen into fluid) (to 1st order)

So (M4) $\rightarrow -i\omega\underline{B}_1 = \nabla\wedge(-i\omega\underline{\xi}\wedge\underline{B}_0)$

$\therefore \underline{B}_1 = \nabla\wedge(\underline{\xi}\wedge\underline{B}_0)$

$= \nabla\wedge[0, B_0\xi_z e^{i(kx-\omega t)}, 0]$

$= [-d\xi_z/dz,\ 0\ ,\ ik\xi_z]B_0 e^{i(kx-\omega t)}$

So mag force term in perturbed mom. eq is (as $\nabla\wedge\underline{B}_0 = 0$)

$$\underline{f}_{mag} = \frac{1}{\mu_0}(\nabla\wedge\underline{B}_1)\wedge\underline{B}_0 = \frac{B_0^2}{\mu_0}[0, 0\ , d^2\xi_z/dz^2 - k^2\xi_z]e^{i(kx-\omega t)}$$

Perturb (mom): incompressible $\rightarrow \Delta\rho = 0$

but Eulerian pert^n is $\rho_1 = \cancel{\Delta\rho}^{\,0} - \underline{\xi}.\nabla\rho_0 = -\xi_z e^{i(kx-\omega t)}d\rho_0/dz$

also define Eulerian pert^n p_1 st. $p = p_0 + p_1$ (to 1st order)

So (mom) $\rightarrow (\rho_0 - \cancel{\xi_z d\rho_0/dz})(-\omega^2\underline{\xi} + \cancel{2^{nd}\text{ order}}) = -\cancel{\nabla p_0} - \nabla p_1 - (\cancel{\rho_0} - \xi_z d\rho_0/dz)g\hat{e}_z + \frac{B_0^2}{\mu_0}\left(\frac{d^2\xi_z}{dz^2} - k^2\xi_z\right)e^{i(kx-\omega t)}\hat{e}_z$ (to 1st order) (equilibrium)

$\boxed{\hat{e}_x} \rightarrow -\omega^2\rho_0\xi_x e^{i(kx-\omega t)} = -dp_1/dx$

$\therefore p_1 = \frac{\omega^2}{ik}\rho_0\xi_x e^{i(kx-\omega t)}$ (integrate)

$= -\frac{\omega^2}{(ik)^2}\rho_0\frac{d\xi_z}{dz}e^{i(kx-\omega t)}$ (using $*$)

$\therefore dp_1/dz = (\omega^2/k^2)d[\rho_0 d\xi_z/dz]/dz\ e^{i(kx-\omega t)}$

$$\boxed{\hat{e}_z} \rightarrow -\omega^2\rho_0\xi_z = -(\omega^2/k^2)d[\rho_0 d\xi_z/dz]/dz + \xi_z(d\rho_0/dz)g + \left(\frac{B_0^2}{\mu_0}\right)\left(\frac{d^2\xi_z}{dz^2} - k^2\xi_z\right)$$

Integrate this equ^n w.r.t. z across interface apply b.c. that $\xi_z(z\rightarrow\pm\infty)\rightarrow 0$ and that ξ_z is continuous across the interface at $z=0$

$$\therefore \omega^2 = kg\left(\frac{\rho - \rho'}{\rho + \rho'}\right) + \frac{2}{\mu_0}\left(\frac{k^2B_0^2}{\rho' + \rho}\right)$$

this is $(\underline{k}.\underline{B}_0)^2$ in more general case that oscill^n is not necessarily in same dir^n as $\underline{B}_0$

So, same result as before if $B_0 = 0$, (ie $\omega^2 < 0$ $\therefore$ unstable if $\rho' > \rho$)

But for $B_0 \neq 0$ mag fields have stabilising effect (as 2nd term is +ve) as work is expended in bending the field lines, $\bar{w}$ stronger effect for shorter wavelengths (as 2nd term $\propto k^2$)

For perts ‖'l to $\underline{B}_0$ there is critical k above which stable $k > \frac{g\mu_0}{2B_0^2}(\rho' - \rho)$

But ... ⊥ to B_0 there is no stabilising effect.