P.O.A EXAMPLES I

1) i) For source at distance R

$$m = -2.5 \log_{10}\left(\frac{L}{R^2}\right) + \text{const.}$$

∴ magnitude limited survey sample objects from a volume $V(L)$ s.t. $\frac{L}{R(L)^2}$ = constant.

i.e. $V(L) \propto L^{3/2}$.

Hence (assuming that number density doesn't vary with distance from observer), the number of objects in a bin of given L is boosted by a factor $\propto L^{3/2}$. To correct for this

$$n(L) \propto N(L)\, L^{-3/2}$$

ii) Observers should always be suspicious of results which indicate 'special' properties in the vicinity of the Sun. It's likely that the student was simply counting objects down to a limiting magnitude, i.e. down to a L which scales as R^2. This means - given that white dwarfs have a range of luminosities as they cool - one can detect a higher fraction close to the Sun.

2) PDMF takes account of fact that after time T, only a fraction $t_{ms}(M)/T$ (if < 1) of stars originally formed are still on the main sequence (assuming constant rate of creation).

So PDMF $\propto f(m)\, t_{ms}(M)$

in case that $t_{ms}(m_{min}) < T$ so this fraction is < 1 for all stars

This distribution does not change with time as T increases since will always be in the regime $t_{ms}(m_{min}) < T$.

$L \sim \frac{M}{t_{ms}(M)}$ because t_{ms} is set by roughly constant mass fraction of star being 'burnt'!

PDLF $= L(m) \cdot$ PDMF

$= L(m)\, t_{ms}(M)\, f(m)$

$\sim m\, f(m)$.

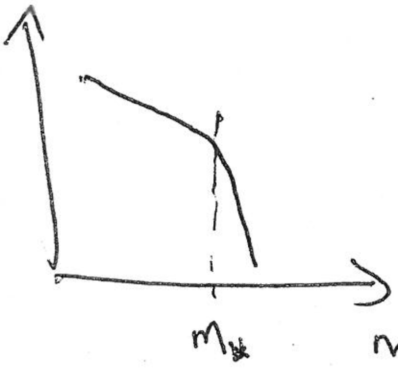
↑ this is the fraction of stars by number in the orig. IMF

This is the fraction of stellar mass in the original IMF

Define m_* s.t. $t_{ms}(m_*) = T$

$$PDMF = f(m) \qquad m < M_*$$

$$= f(m)\frac{t_{ms}(m)}{T} \qquad m \geq M_*$$



$T = 10^8$ years $\Rightarrow m_* \simeq 6\,M_\odot$

PDLF : $m > m_* \rightarrow m f(m)$

so if Salpeter mass dominated by lower end (ie m_*).

$$m < m_* \rightarrow m \frac{T}{t_{ms}(m)} f(m)$$

$$\frac{m f(m)}{t_{ms}(m)} \propto m^{-1.35} \cdot m^{2.5}$$

dominated by upper end of range (ie m_*).

So PDLF dominated by stars of mass $\sim 6\,M_\odot$.

3) Assume all KBOs are same distance from Earth (approx. ok – 40 AU cf 1 AU). Also assume relative orientation of Sun-earth-KBO doesn't matter (ignore phase effects).

$$L_{KBO} = \left(\frac{\pi R^2}{4\pi a_{KBO}^2}\right) \cdot L_\odot \propto R^2.$$

↗ fraction subtended by KBO

$$m = -2.5 \log_{10} L_{KBO} - \text{const.}$$

$$= -5 \log_{10} R - \text{const}' \quad (*)$$

$$dN \propto R^{-q}\, dR \Rightarrow \text{fraction} > R'$$

$\propto R'^{(1-q)}$ assuming $q > 1$ and min. size is R_{min} (magnitude of smallest KBO).

$$\text{Fraction} > R' = \left(\frac{R'}{R_{min}}\right)^{1-q} = 10^{\frac{(-m' + m_{min})}{5}(1-q)}$$

from (*)

$$\equiv F(m')$$ fraction of population with magnitude < m' (i.e. brighter than m').

$$\log_{10} F(m') = \left(\frac{1-q}{5}\right)(m' + m_{min})$$

$-0.3 \Rightarrow 1 - q = -1.5$

$q = 2.5$

Total mass $R \to R + dR \propto R^{-2.5} R^3 \, dR$

(assuming constant density) $\propto R^{0.5} \, d\ln R$

$0.5 > 0$ so mass dominated by upper end

This means that going deep does not make much difference to the total mass budget. An observer would be better advised to conduct a large area survey in order to maximise the probability of detecting the largest objects that dominate the mass. [Note that the total mass diverges unless there is an upper limit on the size distribution so this has to be determined observationally.]

$$\frac{F(28)}{F(25)} = 10^{0.3(28-25)}$$

$$= 8$$

So since 5 objects $m < 25$, expect 40 objects $m < 28$.

Physics of Astrophysics I: Solutions

4) Relative velocity is ~ sound speed in gas
[equipartition of energy ⇒ $v_{grain} << v_{gas}$]
[~ $\sqrt{\frac{R_* T_{gas}}{\mu}}$ ~ 650 ms^{-1}.

Gas molecules arrive with K.E. distribution characteristic of gas temperature & leave with K.E. characteristic of grain temperature. For $T_{gas} >> T_{grain}$ [reasonable because grain is also radiatively cooling as an (efficient) black body] we have a net grain heating associated with gas collisions.

$$\Gamma_{\substack{heat \\ per\ grain}} = \underbrace{2 n_{gas} \pi a_{grain}^2 c_s}_{\substack{\text{no. collisions on} \\ \text{a grain with gas} \\ \text{molecules per unit time}}} \cdot \underbrace{k(T_{gas} - T_{grain})}_{\substack{\text{energy change} \\ \text{per collision}}}$$

factors of order unity are unimportant here.

$$= 6 \times 6.5 \times 10^2 \times 10^{13} \times 10^{-14} \times 1.3 \times 10^{-21}\ W$$
$$= 5 \times 10^{-19}\ W$$

In equilibrium $\Gamma_{heat} = \sigma T_{grain}^4 \, 4\pi a_{grain}^2$
$5 \times 10^{-19} = 6 \times 10^{-8} \times 12 \times 10^{-14} \times T_{grain}^4$

⇒ $T_{grain} = 3$ K (indeed << T_{gas})

* See end of question for thermal timescale calculation

Dust is 1% by mass ⇒ 1 kg gas ⇒ 10^{-2} kg dust. Mass of grain = $\frac{4\pi}{3} \rho_{grain} a_{grain}^3 = 1.2 \times 10^{...}$
(ρ_{grain} ~ 3000 kg/m³)

so corresponds to $N \simeq 10^{15}$ grains due to gas-grain collisions

Heating rate on dust, (= cooling of gas per kg ~~of dust~~ of gas by grain collisions) = $N \Gamma_{\text{heat per grain}}$

$= 10^{15} \times 5 \times 10^{-19}$ W

$= 5 \times 10^{-4}$ W

Compare with 0.02 W kg^{-1} gas given in question — much less so negligible contribution to thermal equilib. of gas

If $a_{gr} \rightarrow f a_{gr}$ ($f > 1$) then

$N \rightarrow f^{-3} N$, $\Gamma_{\text{heat per grain}} \rightarrow f^{2} \Gamma_{\text{heat per grain}}$

So net cooling is $\times f^{-1}$ i.e decreased
[i.e as expect because just depends on total surface area for given mass].

* $$\tau_{therm} = \frac{C_V \frac{4\pi}{3} \rho a^3 T_{gr}}{\underbrace{\text{collision rate} \times kT_{gas}}_{\text{heating rate}}}$$ ← thermal content of grain.

collision rate $= \frac{1}{\text{collision time}}$

So $$\frac{\tau_{therm}}{\text{collision time}} = \frac{C_V \frac{4\pi}{3} \rho a^3 T_{gr}}{kT_{gas}}$$

* thermal energy of grain / thermal energy of gas

This is time because need this * number of collisions to change grain T

$= \frac{840 \times 4 \times 3000 \times 10^{-21} \times 3}{10^{-21}}$ (840 → 840×10^{3})

$\sim 3 \times 10^{7}$.

3) Approx. expression for speed of star.

Can e.g. invoke centrifugal balance for star on circular orbit at radius R with enclosed mass Nm [or invoke virial equilibrium].

$$\Rightarrow \quad v \sim \sqrt{\frac{GNm}{R}} \quad *$$

Expression given in lectures $= \dfrac{v^3}{G^2 m^2 n}$

subst. (*) and $t_{dyn} = \dfrac{R}{v}$

so $t_{deflect} \sim N \dfrac{R}{v}$ — we have ignored constants of order unity in this expression

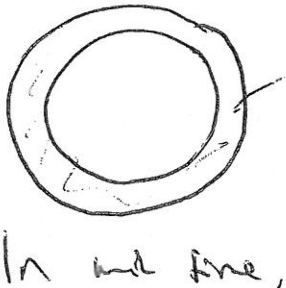
apply to GCs ~~or open clusters~~: $v = \sqrt{\dfrac{10^5}{6\times10^5}} \times 30\ \mathrm{km\,s^{-1}}$
[normalised to earth speed round Sun] $\sim 10\ \mathrm{km\,s^{-1}}$

$t_{dyn} \sim \dfrac{9\times10^{16}}{10^4}\ \mathrm{s} \sim 3\times10^5$ years.

(discuss consequences in supervision) → $t_{deflect} = 0.1 \times 10^5 \times 3\times10^5$ years $= 3$ Gyr $\ll$ age

Apply to ONC: scale N and R to above: $t_{def} = 3\ \mathrm{Gyr} \times \left(\dfrac{10^3}{10^5}\right)^{1/2} \times \left(\dfrac{1}{3}\right)^{3/2} \sim 6\times10^7$ yrs $\gg$ age.

Restrict to annulus $b \to b+db$



$2\pi b\, db$

$$\Delta V_{\perp}^{(b)} = \frac{Gm}{bv}$$

In unit time, no. encounters $= n \cdot 2\pi b\, db\, v$ (don't worry about factors of order unity in relative velocity)

Mean square deflection over that time

$$= n \, 2\pi b \, db \, v \left(\frac{Gm}{b v} \right)^2$$

To get total, simply integrate db since mean square deflections are additive.

$$\frac{d}{dt} \left[\Delta v_{\perp}^2 \right] = \frac{(Gm)^2 \, 2\pi n}{v} \underbrace{\int \frac{db}{b}}$$

so equal contributions per logarithmic bin of b ⇒ dominated by neither end of the distribution.

Answer depends only logarithmically on b_{max}, b_{min} which is fortunate because not well defined ... maximum size of cluster and stellar radius would be reasonable guesses ....

b). Criterion is that don't want encounter within f 20 A.U

(f: constant of order unity)

within 4.5×10^9 years

To decide if gravitational focusing is important compare v (f 20 AU) ~ 7 kms^{-1} with $\hat{v} = 20$ km/s – can ignore

$$\tau_{coll} \sim \frac{1}{n_* f^2 (20 \times 1.5 \times 10^{11})^2 \, 2 \times 10^4}$$

$$\sim 4.5 \times 10^9 \times 3 \times 10^7$$

$$\Rightarrow \quad n_* = \frac{f^{-2} 10^{-25} \times 3 \times 10^{-45}}{4.5 \times 3 \times 10^{16}} = f^{-2} \times 3 \times 10^{-47} \; m^{-3} (!!)$$

$$= f^{-2} \times 3 \times 10^{-47} \times \underbrace{30 \times 10^{48}}_{1 pc^3} \; pc^{-3}$$

$$= f^{-2} \times 10^{4} \; pc^{-3}$$

[adopt 10^4 say].

For grazing encounter, grav focusing is important. To collide at r_* need $b^2 \lesssim \frac{Gm}{v^2} \times r_* = 2$ A.U $\times r_*$.

Collision rate is that calculated above (once every 4.5×10^9 years) $\times \frac{2\left(\frac{6 \times 10^8}{1.5 \times 10^{11}}\right) A}{f^2 \times (20)^2}$

f: 10

A: r_* in AU.

$$\text{Time} = \frac{f^2 \times 200 \times 4.5 \times 10^9}{4 \times 10^{-3}} \text{ years}$$

$\sim 2 \times 10^{15}$ years.

$$\text{Probability} = \frac{4 \times 10^{-5}}{f^2 \times 200} = 2 \times 10^{-6}$$

If the field density is 0.1 pc^{-3}, probability is lower by 10^5 $\rightarrow 2 \times 10^{-11}$.

So for 10^{11} stars in Galaxy expect ~ 1 solar system!

7) Energy from tidal power.

Derived in lecture

$$h \sim \frac{G M_m R_E^2}{g\, r_{EM}^3}$$

$h \sim 1m$

Associated g.p.e. $\sim \rho h R_E^2 g$

$\sim 1000 \times 1 \times (6\times10^6)^2 \times 10$

$E_{tide} \sim 3 \times 10^{17}$ J.

Thought experiment: container attached to spinning earth gathers water in tidal bulge as they coincide, transports it by 90° and lets water fall to local sea level [equivalent to tidal "dam"], extracting g.p.e. released. Earth's spin is decelerated in the process.

Available energy $\sim M_E r_E^2 \Omega_{Earth}^2$

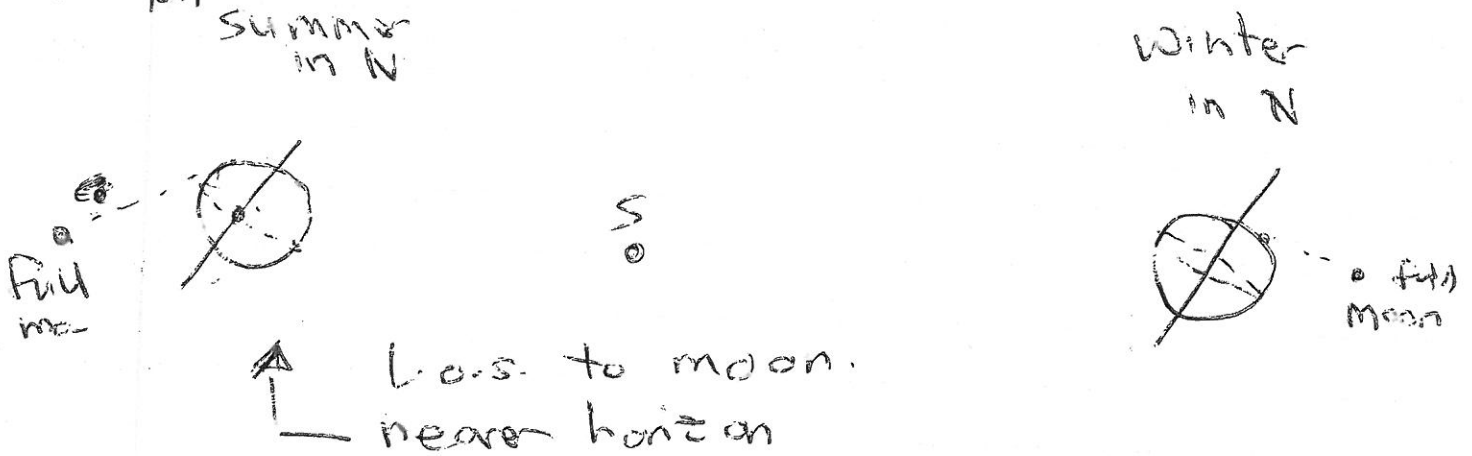
$\sim 6\times10^{24} \times (6\times10^6)^2 \cdot \left(\frac{2\pi}{8\times1\ldots}\right)^2$

$\sim 10^{30}$ J

$\sim 3\times10^{12} \times E_{tide}$

Time to exhaust $\sim 10^{10}$ years.

8(i)a)



ii b) Tidal effects from sun and moon are additive when sun, moon, earth are co-linear [ie 'spring' tide at full moon or new moon]. Tidal bulge is ~ in plane of ecliptic. At equinoxes, the equator intersects the ecliptic along the earth sun (full/new moon) line. Since tidal variation results from the earth's spin relative to the tidal bulge this is maximised at equinoctial spring tides.

iii c) Solar eclipse require that moon crosses ecliptic at a point along the earth-sun line. The moon's orbit is inclined at ~ 5° to the ecliptic and processes (because of torque on earth-moon orbit due to the Sun). Hence incidence of eclipse depends on the relative phasing of crossing the ecliptic and earth-sun line.

iv a) Rising/setting moon appearing red — same as sun at sunrise/set: scattering in earth's atmosphere (blue light preferentially scattered out of l.o.s.).

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$$J_{EM} \sin\Theta \, \Omega_p \sim G$$

$$\frac{M_m}{1\,yr^2} r_{EM}^2 \sin\Theta \sim M_m r_{EM}^2 \Omega_{EM} \sin\Theta \, \Omega_p$$

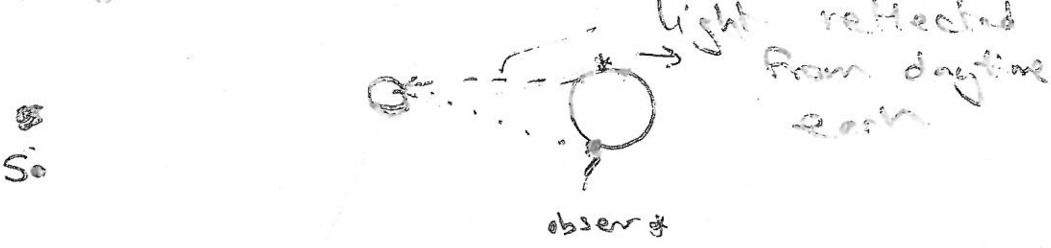
$$\Rightarrow \Omega_p \sim \frac{1 \text{ month}}{(1 \text{ year})^2}$$

$\Rightarrow$ Precession period ~ 12 years.

[We've omitted all factors like π because this is an order of magnitude derivation but this shows that the suggestion is plausible].

vi) Optical illusion (proximity of reference line)

vii) Old moon in new moon's arms is due to earthshine



viii) Tidal bulges follow the moon on period of one lunar month — prograde wrt earth's spin

Each day earth has to rotate through $> 2\pi$ to line up with bulge — hence high tide time is later.

Torque provided by Sun

$$G \sim \frac{G M_\odot M_m r_{EM}^2}{(1\,AU)^3} \sin\theta \sim \frac{M_m r_{EM}^2}{(1\,yr)^2} s$$

$$J_{EM} \sim M_m r_{EM}^2 \Omega_{EM}$$