Stellar Dynamics and Structure of Galaxies

Circular and nearly circular orbits

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Outline I

1. Circular and Nearly Circular Orbits
    - Precession
    - Epicyclic approximation
        - Example: pseudo black hole potential
    - More general potentials
        - Circular orbits in the $z = 0$ plane

Circular and Nearly Circular Orbits

Rotation in a disk galaxy is the obvious example of such orbit.
Given a central force f_r due to a fixed potential Φ, we have the familiar orbit equation

$$\ddot{r} - r\dot{\phi}^2 = f_r = -\frac{d\Phi}{dr} \tag{3.1}$$

$$r^2\dot{\phi} = h = \text{constant} \tag{3.2}$$

In this section of the course, we will consider typical orbits of particles, stars or gas, in the disk of a spiral galaxy, a galaxy like our own, the Milky Way.

The force $\mathbf{f}_r$ acting on the particle is radial, meaning it depends only on the distance r from the galactic center. This radial dependence implies that we are approximating the galaxy's mass distribution as effectively spherically symmetric (or axisymmetric in the case of a disk). While a disk is not truly spherically symmetric, this assumption is often made for simplicity in analyzing orbital mechanics, especially in the mid-plane of the disk where forces are approximately radial.

By setting up the central force field f_r as a radial derivative of Φ (i.e., $f_r = -\frac{d\Phi}{dr}$), we are neglecting non-radial forces. This implies an idealization of the galactic disk as dynamically cold, where particles move in approximately circular orbits without significant perturbations from random motions, spiral arms, or bars. Further on, in this set of lectures, we will consider a more realistic setup in which the orbits near the disk plane are allowed to deviate from a perfectly circular shape.

Circular and Nearly Circular Orbits

$\boxed{\ddot{r} - r\dot{\phi}^2 = f_r = -\frac{d\Phi}{dr} \quad 3.1}$ $\boxed{r^2\dot{\phi} = h = \text{constant} \quad 3.2}$

For a circular orbit $r = R =$constant and $\dot{\phi} = \Omega =$constant (**angular frequency**)
Then (3.2) is satisfied trivially, and (3.1)$\Rightarrow$

$$R\Omega^2 = -f_r = \left.\frac{d\Phi}{dr}\right|_{r=R} \tag{3.3}$$

For example, if $\Phi = -\frac{GM}{r}$, then

$$R\Omega^2 = \frac{GM}{R^2} \Rightarrow \Omega = \left(\frac{GM}{R^3}\right)^{\frac{1}{2}}$$

and the period

$$T = \frac{2\pi}{\Omega} = 2\pi\sqrt{\frac{R^3}{GM}}$$

From the earlier Keplerian orbit discussion, $R = a =$ the radius of the orbit, or the separation between the two stars for a binary system with circular orbits.

We are stating the obvious, but nonetheless, for a circular orbit with a constant angular momentum and fixed radius, the angular velocity $\dot{\phi}$ (denotes the rate of change of the azimuthal angle ϕ with respect to time) is constant.

For a circular orbit, $\dot{\phi}$ becomes a constant value Ω, known as the **orbital frequency** or **angular frequency** of the orbit

For example, for a Keplerian potential, the derivative of the potential is $\frac{GM}{R^2}$ and we recover a familiar value for the orbital period for an orbit with a semi-major axis $R = a$

Another way to look at the angular frequency and Equation 3.3 is that, in the axi-symmetric and spherical case, Ω is the property of the potential.

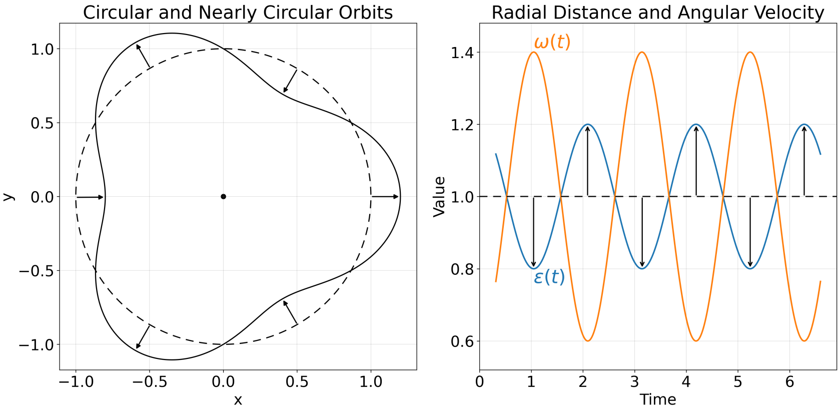
Circular and Nearly Circular Orbits

Now consider an orbit which is nearly circular, so we take

$$r = R + \varepsilon(t) \quad \text{with } \varepsilon << R$$

and

$$\dot{\phi} = \Omega + \omega(t) \quad \text{with } \omega << \Omega$$



near_circular_orbit.ipynb

Let’s imagine instead that our orbit deviates slightly from the circular one.

It remains in the plane but makes a small excursion radially with an amplitude $\varepsilon(t)$. This is illustrated in the left panel where the original circular orbit is shown with a dashed line and the perturbed one with a solid line.

Now, the particle’s angular speed will not be constant anymore like in the case of a circular orbit, it will rise and fall, speeding up at closest approach and slowing down at the furthest. Let’s designate the angular speed variations $\omega(t)$

Changes in radial distance $\varepsilon(t)$ and angular velocity $\omega(t)$ with time are shown in the right panel as blue and orange curve correspondingly.

Circular and Nearly Circular Orbits

If we choose to characterize orbits by their angular momentum, we keep the angular momentum unchanged, and the (3.2)⇒

$$\begin{aligned} h = R^2\Omega &= (R+\varepsilon)^2(\Omega+\omega) \\ &= (R^2+2R\varepsilon)(\Omega+\omega) \\ &= R^2\Omega + 2R\varepsilon\Omega + R^2\omega \end{aligned} \tag{3.4}$$

if we retain only terms to first order. Therefore

$$R\omega = -2\varepsilon\Omega \tag{3.5}$$

So we have induced a small-amplitude radial variation in what previously was a circular orbit. As we slightly "kick" the particle radially, the angular momentum remains unchanged.

Let's express the AM in terms of the variable we have just introduced, $\varepsilon(t)$ and $\omega(t)$.

For the radial component in the first bracket, let's drop the square of ε

Now, as we open the brackets, let's drop the term where we have a product of two small quantities, namely $2R\varepsilon\omega$

$R\Omega^2$ on either side cancel out and we end up with a nice relation between ω and ε

This condition has a simple meaning: a small inward or outward shift in radius ε requires the opposite change in angular velocity ω to maintain the angular momentum unchanged to first order.

What controls $\varepsilon(t)$ and through this equation $\omega(t)$? The properties of the potential. We can find out exactly how by writing down the orbit equation.

Circular and Nearly Circular Orbits

$$\boxed{\ddot{r} - r\dot{\phi}^2 = f_r = -\frac{d\Phi}{dr} \quad 3.1}$$

$$\boxed{R\Omega^2 = -f_r = \left.\frac{d\Phi}{dr}\right|_{r=R} \quad 3.3}$$

Now, using (3.1) and retaining only terms to first order, the perturbation's behaviour is described by:

$$\ddot{\varepsilon} - (R + \varepsilon)(\Omega^2 + 2\Omega\omega) = f(R + \varepsilon) \tag{3.6}$$

$$\ddot{\varepsilon} - R\Omega^2 - \varepsilon\Omega^2 - 2R\Omega\omega = f(R) + \varepsilon f'(R) \tag{3.7}$$

$R\Omega^2 = -f(R)$ from (3.3), and using (3.5) $-2R\Omega\omega = 4\varepsilon\Omega^2$, so we have

$$\ddot{\varepsilon} + 3\varepsilon\Omega^2 = \varepsilon f'(R) \tag{3.8}$$

$$\text{or} \quad \ddot{\varepsilon} + \left(3\Omega^2 - f'(R)\right)\varepsilon = 0 \tag{3.9}$$

Let's take the orbit equation and plug in expression for the radial coordinate which we have written down as a small deviation around constant R.

Because R is constant, in the first term (the second derivative with respect to time) we will only have $\ddot{\varepsilon}$

In the left hand side, for the $r\dot{\phi}^2$ term we apply a similar trick we have just used for $R^2\Omega$ - let's only keep first order terms. This means dropping ω^2 from $\dot{\phi}^2$ and also dropping $2\Omega\omega\varepsilon$

In the right hand side, we Taylor expand the force

In Equation 3.7, on the left and the right we have expression for the angular frequency Ω and the force at the circular orbit that are linked, so these terms cancel out.

Moving a term with $\varepsilon f'$ from right to left, we get a linear second-order homogeneous ordinary differential equation for radial perturbation.

Circular and Nearly Circular Orbits

$$R\Omega^2 = -f_r = \left.\frac{d\Phi}{dr}\right|_{r=R} \quad 3.3$$

$$\ddot{\varepsilon} + \left(3\Omega^2 - f'(R)\right)\varepsilon = 0 \tag{3.9}$$

This is stable simple harmonic motion if $\Omega_R^2 = 3\Omega^2 - f'(R) > 0$ so, using (3.3), if

$$f'(R) + 3\frac{f(R)}{R} < 0 \Leftrightarrow \frac{d}{dR}(R^3 f) < 0$$

. Here Ω_R is the radial frequency.

e.g. $f(R) \propto -R^{-n}$ is stable only if $n < 3$ i.e. unstable if potential is steep.

This equation resembles the standard form of a simple harmonic oscillator if the term in front of ε is positive, which implies that ε will undergo oscillations around R

Using the definition of the angular frequency 3.3 and writing $\Omega = -\frac{f(R)}{R}$ we obtain the condition on the force to keep the oscillations radially stable

To see why this expression relates to $\frac{d}{dR}(R^3 f)$, let's differentiate $R^3 f$ with respect to R:

$$\frac{d}{dR}(R^3 f) = \frac{d}{dR}(R^3) \cdot f + R^3 \cdot f'(R) = 3R^2 f + R^3 f'(R) = \frac{d}{dR}(R^3 f) = R^2 \left(3\frac{f}{R} + f'(R)\right)$$

Because R^2 is positive we can divide by it without changing the inequality.

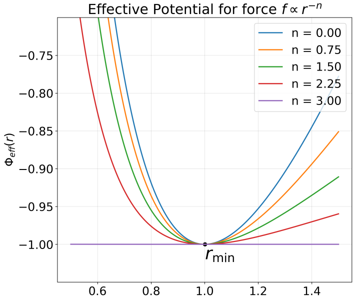
If we approximated the force with a power-law, the stability of the radial perturbation would put a bound on the steepness in terms of the power-law index.

Considering a stable radial oscillation, what does actually provide the restoring force? When the particle moves outwards in radius, it is clearly the gravity. But what about when it moves inwards?

The centrifugal force. It is instructive to look at the corresponding effective potential to understand the stability around the circular orbit.

Circular and Nearly Circular Orbits

Effective potential and stability



`near_circular_orbit_phieff.ipynb`

Given the power-law force: $f(r) = -k\,r^{-n}$

The corresponding potential:

$$\Phi(r) = \begin{cases} \int k\,r^{-n}\,dr = \dfrac{k}{1-n}\,r^{1-n} + C, & \text{if } n \neq 1 \\ \int k\,r^{-1}\,dr = k\ln r + C, & \text{if } n = 1 \end{cases}$$

The effective potential $\Phi_{\mathrm{eff}}(r)$: $\Phi_{\mathrm{eff}}(r) = \Phi(r) + \dfrac{L^2}{2mr^2}$

$$\Phi'_{\mathrm{eff}}(r) = \frac{d\Phi}{dr} - \frac{L^2}{mr^3} = k\,r^{-n} - \frac{L^2}{mr^3} \quad \rightarrow \quad \Phi''_{\mathrm{eff}}(r) = -nkr^{-n-1} + \frac{3L^2}{mr^4}$$

$$r_{\mathrm{min}} = \left(\frac{mk}{L^2}\right)^{1/(n-3)} \rightarrow \Phi''_{\mathrm{eff}}(r_{\mathrm{min}}) = (3-n)\left(\frac{L^2}{mr_{\mathrm{min}}^4}\right)$$

Thinking back to the concept of the effective potential, the circular orbit corresponds to the minimum of the effective potential. This is where the centrifugal and the gravitational force balance and where the two extrema of the orbit (apo and peri) merge and disappear.

For the near-circular orbit to be stable the shape of the effective potential needs to be convex near minimum.

Let's consider a power-law force. Because force is $\frac{d\Phi}{dr}$, the corresponding potential is obtained by integrating the force expression. For cases when the power-law index is not 1, this gives a potential of a power-law form with the index raised by 1. For $n = 1$, the potential is $\ln(r)$

For stability, we need the convex effective potential, i.e. its second derivative near the minimum must be positive. Let's compute the first derivative and set it to zero to get $r_{\min}$. Then compute the second derivative and evaluate it at $r_{\min}$

Given that the expression with L is non-negative, the expression in brackets is positive for $n < 3$

Precession

To a first approximation, a particle circles the origin with a period $T = 2\pi/\Omega$.
It executes radial motion with a period $T_r = 2\pi/\Omega_R$ where $\Omega_R^2 = 3\Omega^2 - f'(R)$.
In general, the two frequencies are not commensurate, $\Omega_R \neq \Omega$, so the orbit is not guaranteed to be closed.
The orbit is like an ellipse which rotates (or precesses) with a period $2\pi/\Omega_p$ where $\Omega_p = \Omega - \Omega_R$

Note: Often Ω_R^2 is written K^2, and K called the **epicyclic frequency**.

In general for galaxies precession is retrograde (i.e. opposite to the rotation direction of the stars) since T_r is usually less than T_ϕ. We have already discussed this and we'll see again later, but the basic results are for a harmonic (uniform density) model $\Delta\phi = \pi$ in one radial period, and for Keplerian orbits $\Delta\phi = 2\pi$ in one radial period, and real galaxies fall between these extremes

For Keplerian potential $f(R) = -\frac{GM}{R^2}$, $\Omega^2 = \frac{GM}{R^3}$ and $f'(R) = \frac{2GM}{R^3}$, so $\Omega_R^2 = 3\Omega^2 - f'(R) = \frac{GM}{R^3} = \Omega^2$, so the orbits are closed.

The two frequencies, angular and radial, are not commensurate, meaning the radial and azimuthal motions don't repeat in sync. This causes the radial oscillation to "wrap" around the azimuthal motion.

Let's look closely at the motion along the nearly circular orbit as the particle appears to executes two separate excursions: one along the circular orbit with a fixed radius and a fixed angular frequency and another one along the so-called epicyclic ellipse.

Epicyclic approximation

Move to a frame in which the unperturbed particle is at rest, with the coordinates in the direction of rotation and in the radial direction. This is necessarily a rotating frame.

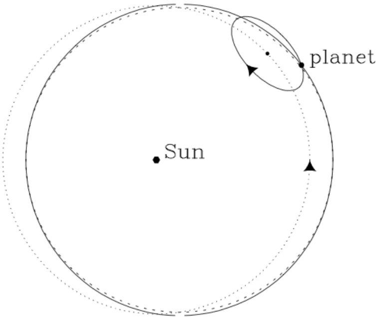


Figure 3.7 An elliptical Kepler orbit (dashed curve) is well approximated by the superposition of motion at angular frequency κ around a small ellipse with axis ratio $\frac{1}{2}$, and motion of the ellipse's center in the opposite sense at angular frequency Ω around a circle (dotted curve).

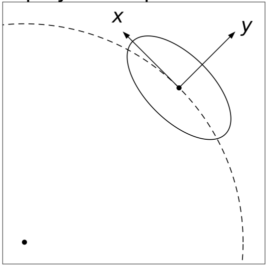
In this setup the centre of the ellipse is rotating on the circle. To describe the ellipse we are translating into a coordinate frame attached to the centre of the ellipse.

This nomenclature goes back to the 2nd century CE (AD). Epicycles were introduced in ancient astronomy, particularly by the Greek astronomer Claudius Ptolemy, as part of his geocentric model described in the Almagest. In Ptolemaic astronomy, epicycles (small circles) were used to explain the apparent retrograde motion of planets, with each planet moving along an epicycle that itself orbited Earth. While we have long abandoned this Geocentric view, the idea of epicycles is still useful to describe small perturbations around a circular orbit.

In this figure, the planet's orbit around the Sun (shown in solid black) is slightly perturbed radially, giving it a bit of eccentricity compared to the unperturbed circular orbit (dotted curve).

Epicyclic approximation

Epicyclic Ellipse Frame



In this rotating frame, y is aligned with r and x is aligned with ϕ

$$r = R + y$$
$$R\dot{\phi} = R\Omega + \dot{x}$$

so

$$y = \varepsilon$$
$$\dot{x} = R\omega = -2\varepsilon\Omega$$

The second equality from the conservation of angular momentum $R\omega = -2\varepsilon\Omega$. So can use relation $\ddot{\varepsilon} + \left(3\Omega^2 - f'(R)\right)\varepsilon = 0$, which becomes

$$\ddot{y} + K^2 y = 0$$

Epicyclic approximation

so if we take $y = -b\cos(Kt)$, then $\dot{x} = 2\Omega b\cos(Kt)$, so

$$x = \frac{2\Omega b}{K}\sin(Kt) = a\sin(Kt)$$

defines a, and then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \sin^2(Kt) + \cos^2(Kt) = 1$$

$\Rightarrow$ motion is an ellipse along which the particle moves retrograde (opposite to its rotation on the circle) at frequency K and is such that $b = \frac{K}{2\Omega}a$

In different gravitational potentials, the value of K changes, affecting the shape of the epicycle.

For Keplerian potential $K = \Omega$ so $b = a/2$
For harmonic potential (to come) $K = 2\Omega$ so $b = a$ (circle)

In general (cases in between Keplerian and Harmonic) epicycle is elongated along tangential direction.

Radial Displacement$y(t) = -b\cos(Kt)$

- b is the amplitude of the radial oscillation.
- The negative sign indicates the choice of phase, starting the oscillation at $y = -b$ when $t = 0$.

Tangential Velocity Perturbation. Using the relation $\dot{x} = -2\Omega y$:

$$\dot{x} = -2\Omega(-b\cos(Kt)) = 2\Omega b\cos(Kt)$$

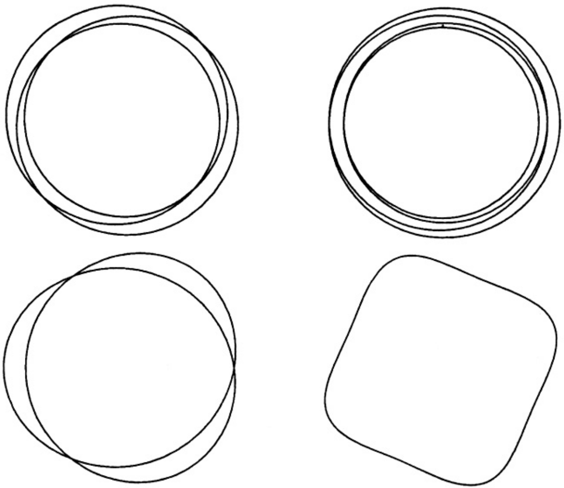
Tangential Displacement. Integrate $\dot{x}$ with respect to time to find $x(t)$:

$$x(t) = \int \dot{x}\,dt = 2\Omega b \int \cos(Kt)\,dt = \frac{2\Omega b}{K}\sin(Kt) = a\sin(Kt)$$

Squaring and adding the equations for x and y we obtain the standard equation of an ellipse. The particle's small deviations from a circular orbit can be approximated by an elliptical "epicycle" superimposed on the circular motion.

Harmonic potential $\Phi \propto r^2$ corresponds to a constant density (think homogeneous sphere) so represents an extreme case.

Epicyclic approximation



Quasi-circular orbits when the ratio of angular to radial frequency is rational (3/2, upper left; 2/3 lower left; 4, upper right; 1/4, lower right).

Here are examples of possible nearly circular orbits with different ratios of angular to radial frequency.

What controls these shapes?

The ratio of radial and angular frequencies is controlled by the properties of the potential. Specifically,

$$K^2 = \Omega_R^2 = 3\Omega^2 - f'(R)$$

and

$$\Omega^2 = -\frac{f(r)}{R}$$

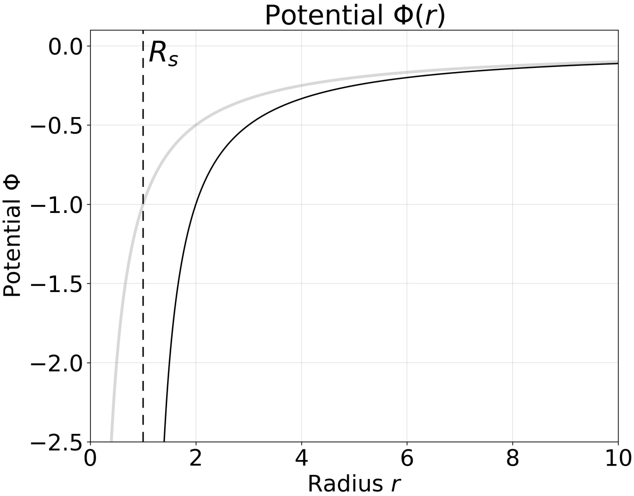
So, angular frequency is determined by the radial gradient of the potential and the radial frequency is controlled by that and the second derivative of the potential.

In this figure, the top row shows orbits where, unusually, the angular frequency is higher than the radial one. This gives a prograde motion on the epicyclic ellipse.

Before we consider more interesting perturbations of the circular orbit, let's consider a curious case of stability of circular orbits near a black hole

Epicyclic approximation

Example: pseudo black hole potential



`pseudo_black_hole_potential.ipynb`

$$\Phi(r) = -\frac{GM}{r - R_s}$$

Black is the modified potential, light grey is the point-mass potential

Let's modify the point-mass potential only slightly.

We introduce a boundary beyond which the particle can not travel, let's call it R_s

This is meant to mimic key relativistic effects around the black hole, specifically the event horizon, but in Newtonian gravity.

At R_s now we have a singularity, a barrier beyond which the particle can not travel.

Epicyclic approximation

Example: pseudo black hole potential

$$\Phi(r) = -\frac{GM}{r - R_s}$$

$$f(r) = -\frac{d\Phi}{dr} = -\frac{GM}{(r - R_s)^2}$$

For a circular orbit $\Omega^2 = -\frac{f(R)}{R}$ so

$$\Omega^2 = \frac{GM}{R\,(R - R_s)^2} \qquad \textit{compare to} \quad \Omega^2_{\mathrm{point\ mass}} = \frac{GM}{R^3}$$

Also

$$f'(R) = \frac{2GM}{(R - R_s)^3}$$

so

$$K^2 = 3\Omega^2 - f'(R) = \frac{3GM}{R\,(R - R_s)^2} - \frac{2GM}{(R - R_s)^3}$$

Given this modified potential, let's compute the force by taking the derivative

Writing down the familiar force balance for a particle on a circular orbit, we can arrive at the expression for the angular frequency $\Omega(R)$

To get to the stability condition, we need the derivative of the force.

Now we can write down the expression for the square of the radial frequency K^2

For the orbit to be stable, $K^2 > 0$

Epicyclic approximation

Example: pseudo black hole potential

We have just derived that the square of the radial frequency:

$$K^2 = \frac{3GM}{R\left(R - R_s\right)^2} - \frac{2GM}{\left(R - R_s\right)^3}$$

Stable circular orbits are those for which $K^2 > 0$, so require

$$3\left(R - R_s\right)^3 > 2R\left(R - R_s\right)^2$$

so for $R \neq R_s$

$$3\left(R - R_s\right) > 2R$$

or

$$R > 3R_s$$

This is reminiscent of a Schwarzschild black hole: $R_s = \frac{2GM}{c^2}$.

Let's consider the case when $K^2 = 0$ just. Then, move the two terms on either side of the equation and get rid of GM.

Opening the brackets and simplifying, we now have an expression describing for stability of circular orbits in this modified gravitational potential.

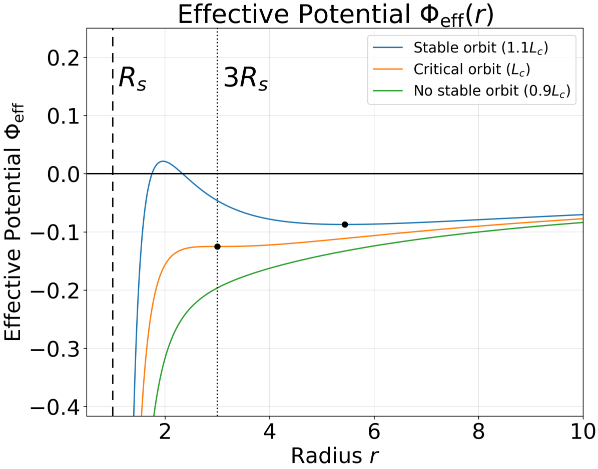
Stable orbits can only exist outside of $3R_s$. This is known as the radius of the Innermost Stable Circular Orbit or ISCO.

How come? All we did is shift the origin, the location of the point mass. Why does this have such a profound effect? After all, in the original, unshifted, point-mass potential, stable orbits exist all the way to $r = 0$.

To understand what is going on here, let's look at th eeffective potential

Epicyclic approximation

Example: pseudo black hole potential



`pseudo_black_hole_potential.ipynb`

To find the extrema of the effective potential, we start with the expression for $\Phi_{\text{eff}}(r)$:

$$\Phi_{\text{eff}}(r) = -\frac{GM}{r - R_s} + \frac{L^2}{2mr^2}$$

The extrema of $\Phi_{\text{eff}}(r)$ occur where its derivative with respect to r is zero:

$$\frac{d\Phi_{\text{eff}}}{dr} = \frac{GM}{(r - R_s)^2} - \frac{L^2}{mr^3} = 0$$

The equation for the extrema of $\Phi_{\rm eff}$ is a cubic.

This means that together with a region where stable orbits exist, there is a portion of the parameter space where the shape of $\Phi_{\rm eff}$ switches from convex to concave and stability disappears

The critical angular momentum L_c is the value of L at which the effective potential has a single extremum point at $r = 3R_s$

The critical AM is $L_c = \sqrt{\frac{27}{4} GMmR_s}$

For angular momentum $L > L_c$. The cubic equation has three real roots, of which two positive real roots correspond to physically meaningful radii greater than R_s. These roots represent the inner (unstable) and outer (stable) circular orbits.

For $L = L_c$. The cubic equation has a repeated real root at $r = 3R_s$. This corresponds to the critical angular momentum where the maximum and minimum of $\Phi_{\rm eff}$ merge into an inflection point.

For $L < L_c$. The cubic equation has one real root corresponding to a radius greater than R_s, but this root does not yield a stable circular orbit. No stable circular orbits exist in this case.

Going back to the earlier question, why simply offsetting the potential by fixed distance has such a profound effect on the orbit stability? It is because the centrifugal force has not been offset. This means that at locations closer to R_s, the gravitational potential is relatively stronger compared to the centrifugal force. The centrifugal force is now relatively weaker and can not balance the gravity.

More general potentials

Axisymmetric Potentials

In most of the things we are interested in, the density distribution is not always (or even often) spherically symmetric, but it may be approximately axisymmetric.

In such cases we use cylindrical polar coordintes (R, ϕ, z).
If $\rho = \rho(R, z)$, then $\Phi(\mathbf{r}) = \Phi(R, z)$.

Often also have plane symmetry, where $\rho(R, z) = \rho(R, -z)$ (with choice of origin in the plane of symmetry of course).

To build a galaxy, to get the full potential, we combine spherically symmetric and axi-symmetric components together, e.g. central bulge in a spiral plus its thin disk and so on

Other examples of flattened mass distributions include

- fast rotating planet (Jupiter, Saturn) has equatorial bulge
- the time averaged potential of the moon (for the study of long timescale effects)

More general potentials

Axisymmetric Potentials

So we have to consider orbits in axisymmetric potentials, where there is no ϕ-dependence so $\Phi(R, \phi, z) = \Phi(R, z)$.

The force

$$\mathbf{F} = \left(-\frac{\partial \Phi}{\partial R}, 0, -\frac{\partial \Phi}{\partial z}\right)$$

Since there is no force in the ϕ direction, the angular momentum about the z-axis L_z is constant, so the equation of motion becomes

$$\ddot{R} - R\dot{\phi}^2 = -\frac{\partial \Phi}{\partial R} \tag{3.10}$$

$$R^2\dot{\phi} = L_z \tag{3.11}$$

$$\ddot{z} = -\frac{\partial \Phi}{\partial z} \tag{3.12}$$

More general potentials

Axisymmetric Potentials

We can remove the $\dot{\phi}$ term from the first two to obtain

$$\ddot{R} = -\frac{\partial \Phi}{\partial R} + \frac{L_z^2}{R^3} = -\frac{\partial \Phi_{\rm eff}}{\partial R} \tag{3.13}$$

where

$$\Phi_{\rm eff} = \Phi + \frac{L_z^2}{2R^2}$$

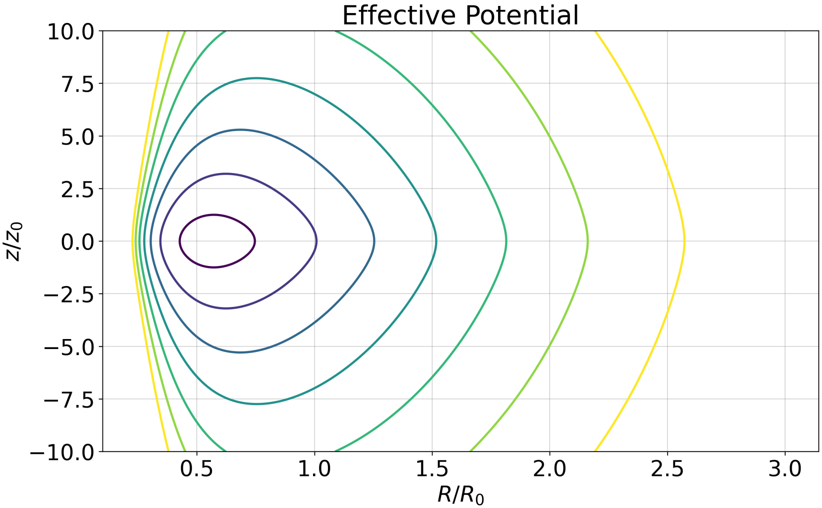
and since $\frac{L_z}{2R^2}$ is independent of z,

$$\ddot{z} = -\frac{\partial \Phi_{\rm eff}}{\partial z} \tag{3.14}$$

So we have reduced a 3D problem to a 2D one.
In astronomical situations we also have plane symmetry, so $\Phi(R, z) = \Phi(R, -z)$.
General orbits are complicated, and beyond the scope of this course (but see Part III).
We will deal with circular and nearly circular orbits close to the $z = 0$ plane.

More general potentials

Axisymmetric Potentials



`axisymmetric_phieff.ipynb`

These are contours of equal effective potential in a composite multi-component model of the Milky Way.

The X and Y axes are shown in scaled coordinates, i.e. in units of the scale radius and scale height. But the plot shows the correct aspect ratio of the contours because the z scale is approximately 10 times smaller then the R scale

The dominant component at the locations shown is the thin disk of our Galaxy. But there are other components in this model too, including the Dark Matter halo

The colours represent the depth of the effective potential, with darker violet colours showing the lower depths, i.e. larger negative values.

In essence, this is a 2D version of 1D effective potential plots we saw before

The radial shape, i.e. tighter packing of the contours at small R and more sparse spacing of the contours at larger R is due to the interplay between the gravitational potential and the centrifugal potential

But there is a vertical shape to the contours too

Using our previous intuition, a near-circular orbit close to the mid-plane of the disk will have to remain inside one of these contours

More general potentials

Circular orbits in the $z = 0$ plane

Look for solution $z = 0$, $R = R_c$ =constant, $\dot{\phi} = \Omega$ =constant. Equation (3.14) is satisfied because $\frac{\partial \Phi}{\partial z} = 0$ at $z = 0$, from the plane symmetry condition. Equation (3.13)⇒

$$\frac{L_z^2}{R^3} = \frac{\partial \Phi}{\partial R}$$

Since $R_c^2 \Omega_c = L_z$, then

$$\Omega_c^2 = \frac{1}{R} \left. \frac{\partial \Phi}{\partial R} \right|_{R=R_c}$$

as before.

More general potentials

Nearly circular orbits close to the $z = 0$ plane

Stars on orbits in the plane in a flattened potential have no way of perceiving that the potential they are moving in is not spherically symmetric. Therefore our deductions apply: star oscillates between two extrema in the radial coordinate.

What happens to stars whose orbits carry them out of the plane?

The star in on an orbit close to circular $R = R_c + x$, and $z = z$, where $x, z << R_c$.

At $z = x = 0$, from symmetry, we have:

$$\left.\frac{\partial \Phi_{\rm eff}}{\partial z}\right|_{z=0} = 0$$

since $\ddot{R} = 0 = \frac{\partial \Phi_{\rm eff}}{\partial R}$

$$\left.\frac{\partial \Phi_{\rm eff}}{\partial R}\right|_{x=0} = 0$$

More general potentials

Nearly circular orbits close to the $z = 0$ plane

We can expand the function $\Phi_{\rm eff}$ about $z = x = 0$ to obtain

$$\begin{aligned}\Phi_{\rm eff}(R_c + x, z) &= \Phi_{\rm eff}(R_c, 0) + x\,\frac{\partial\Phi_{\rm eff}}{\partial R}\bigg|_{(R_c,0)} + z\,\frac{\partial\Phi_{\rm eff}}{\partial z}\bigg|_{(R_c,0)} \\ &\quad + \frac{x^2}{2!}\,\frac{\partial^2\Phi_{\rm eff}}{\partial R^2}\bigg|_{(R_c,0)} + \frac{2xz}{2!}\,\frac{\partial^2\Phi_{\rm eff}}{\partial R\partial z}\bigg|_{(R_c,0)} \\ &\quad + \frac{z^2}{2!}\,\frac{\partial^2\Phi_{\rm eff}}{\partial z^2}\bigg|_{(R_c,0)} \qquad (3.15)\end{aligned}$$

The linear terms are zero from the considerations above, and the cross term (xz) coefficient is also zero from the plane symmetry.

More general potentials

Nearly circular orbits close to the $z = 0$ plane

Thus, from (3.13) ($\ddot{R} = -\frac{\partial \Phi_{\text{eff}}}{\partial R}$)

$$\ddot{x} = -\frac{\partial \Phi_{\text{eff}}}{\partial x} = -x \left. \frac{\partial^2 \Phi_{\text{eff}}}{\partial R^2} \right|_{(R_c,0)}$$

and from(3.14)

$$\ddot{z} = -\frac{\partial \Phi_{\text{eff}}}{\partial z} = -z \left. \frac{\partial^2 \Phi_{\text{eff}}}{\partial z^2} \right|_{(R_c,0)}$$

To obtain these orbit equations we need to obtain forces by taking the derivative of the effective potential. This essentially means differentiating the non-zero (seciond-order) terms in the Taylor expansion in Equation 3.15

More general potentials

Nearly circular orbits close to the $z = 0$ plane

Therefore the equations become

$$\ddot{x} = -K^2 x$$

- the epcyclic frequency, and

$$\ddot{z} = -\mathcal{V}^2 z$$

- the vertical frequency.

Here

$$\mathcal{V}^2 = \left.\frac{\partial^2 \Phi}{\partial z^2}\right|_{(R_c,0)}$$

and

$$K^2 = \left.\frac{\partial^2 \Phi}{\partial R^2}\right|_{(R_c,0)} + \frac{3L_z^2}{R_c^4}$$

These frequencies are the second derivatives of the effective potential. Because only these second-order terms survive in the Taylor expansion.

More general potentials

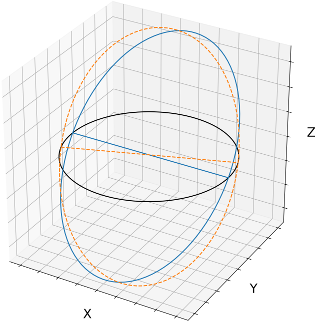
Nearly circular orbits close to the $z = 0$ plane

But

$$\Omega_c^2(R) = \frac{1}{R}\frac{\partial\Phi}{\partial R} = \frac{L_z^2}{R_c^4}$$

$\Rightarrow$

$$K^2 = \left(R\frac{\partial\Omega^2}{\partial R} + 4\Omega^2\right)\Bigg|_{(R_c,0)}$$



`line_of_nodes.ipynb`

Thus there are two types of precession - radial precession (or rotation of pericentre, as before) $\Omega_p = \Omega - K$, and vertical or <u>nodal</u> precession $\Omega_z = \Omega - \mathcal{V}$.

The orbit is in a tilted plane which rotates at rate Ω_z.

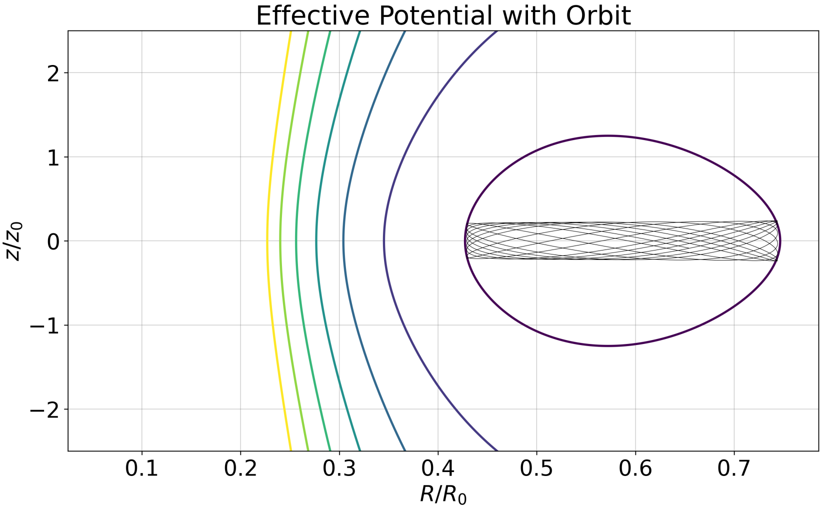
A **node** is the place where the orbit crosses the $z = 0$ plane upwards (by convention, also called the ascending node).

Because the angular frequency Ω squared is both the derivative of the potential with respect to R divided by R and the ratio L_z and R_c squared, we can express K^2 solely in terms of Ω and its derivative.

OK, let's look at the orbits in a flattened potential!

More general potentials

Axisymmetric Potentials



`axisymmetric_phieff.ipynb`

Here is a zoom-in onto the region of interest in the effective potential that we saw before (something very close to what the Milky Way's potential should be if it actually were axi-symmetric)

$\Phi_{\rm eff}$ includes the contributions of the potential and the azimuthal velocity squared, but total energy has additional contributions from V_R^2 and V_Z^2, in other words:

$$E = \Phi_{\rm eff} + \frac{V_R^2 + V_z^2}{2}$$

Energy is conserved, therefore an orbit with energy E will be confined to a region $E \geq \Phi_{\rm eff}$. On the contour where $E = \Phi_{\rm eff}$, the additional contributions from the radial and vertical velocity components have to be zero. These are the turn-around points.

Let's produce an orbit corresponding to the lowest energy level plotted

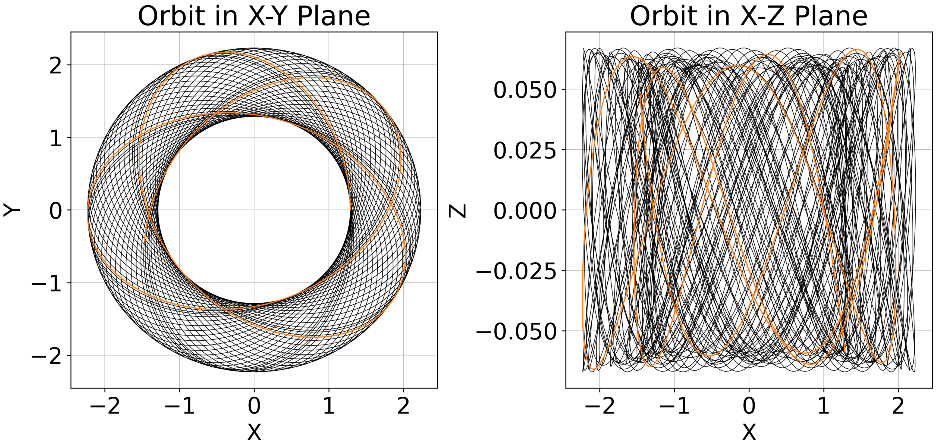
Here is one, shown in black

As expected, in accordance with our intuition based on 1D motion in spherical potentials, the orbit stays inside the energy contour. In other words, its extrema, the points of largest excursion in the R and z directions lie on the energy contour.

In 3D, what does this orbit look like?

More general potentials

Axisymmetric Potentials



`axisymmetric_phieff.ipynb`

It is a doughnut! Or a torus.

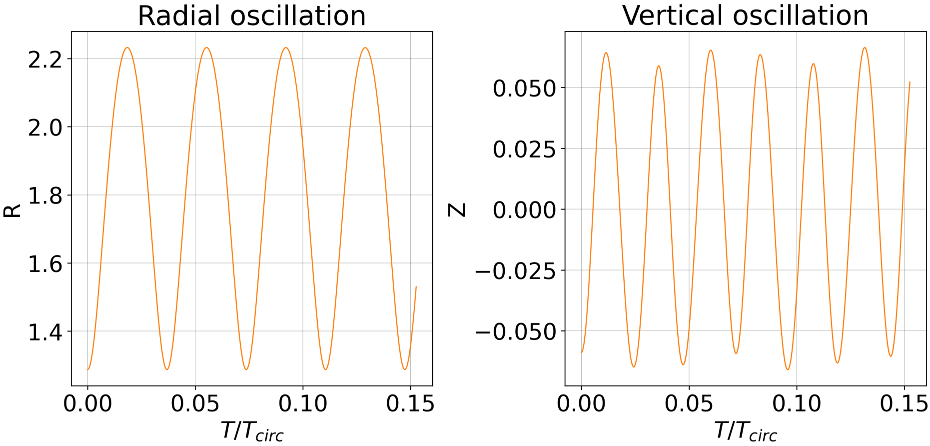
Indeed, you can see that if we integrate the orbit for sufficiently long time (black line, compare to the orange line which is integrated for a few periods only), it will fill a torus in the X, Y, Z space while oscillating around the circular orbit in the radial direction and oscillating around the disk plane, i.e. $z = 0$ plane in the vertical direction

Note that the vertical motion is much smaller in amplitude compared to the radial motion

The orbit's behaviour can also be described by its angular momentum, let's see what the individual components of the AM are doing as a function of time.

More general potentials

Axisymmetric Potentials



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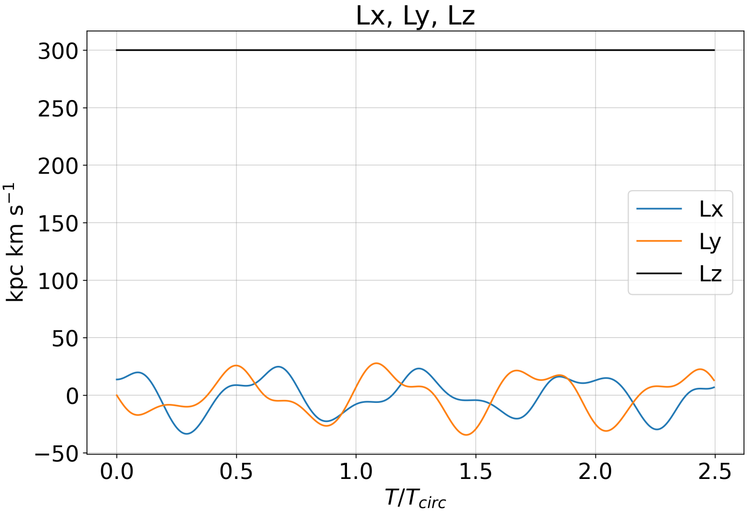
At the same time, radial and vertical motions are quite regular and quasi-periodic

Because the orbital motion and quasi-periodic, the orbit does not explore the entire region of $E \geq \Phi_{rmeff}$

How does the AM vector evolve with time?

More general potentials

Axisymmetric Potentials



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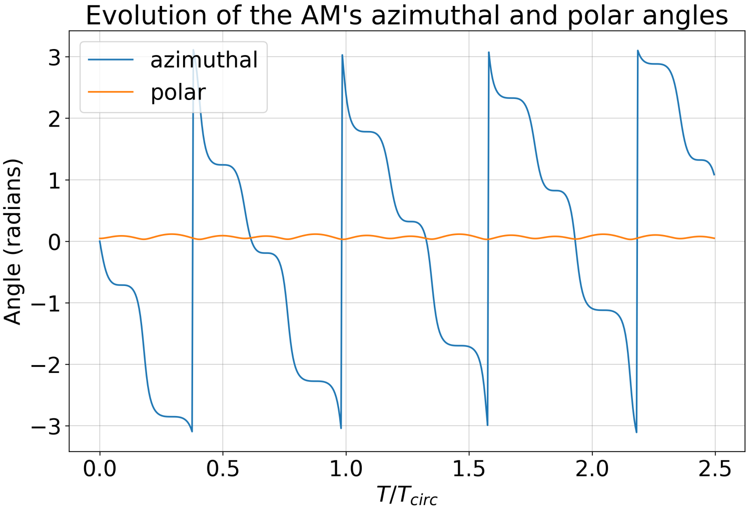
As predicted, the vertical component of the AM stays constant (black line)

The other two components are not conserved, and constantly change with time.

It is perhaps more instructive, to convert the Cartesian components into spherical polars and look at the evolution of the azimuthal (along the equator, in our case, that's where the disk's plane of symmetry crosses the sphere) and polar (the pole is where the vector perpendicular to the plane of the disk pierces the sphere) angles

More general potentials

Axisymmetric Potentials



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This plot shows how the direction of th AM vector changes with time.

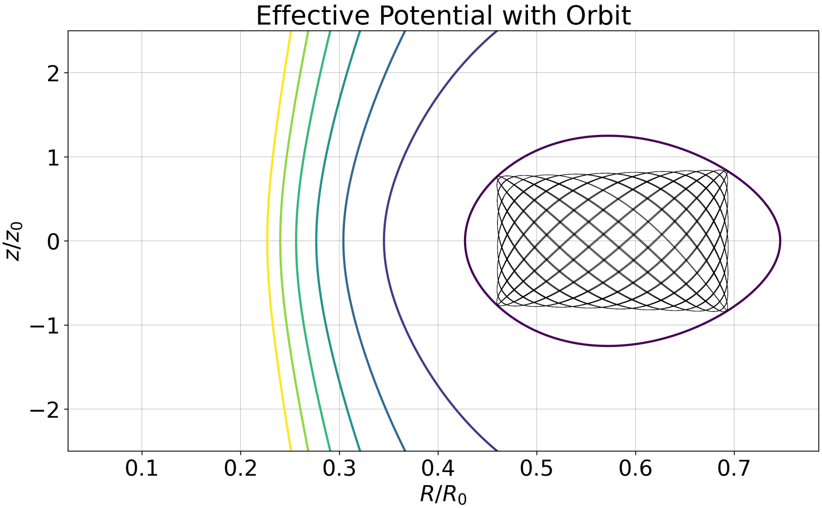
Blue curve shows the evolution of the azimuthal angle ϕ of the angular momentum vector with time

The orbital plane constantly rotates around the vertical axes, or in other words, as the orbit **precesses** around the direction perpendicular to the disk, where the polar angle $\theta = 0$ the azimuthal angle ϕ constant evolves (increases) with time (blue curve)

The precession (rotation around the vertical axes) does not happen at constant polar angle. The polar angle changes slightly with time. This effect is known as **nutation**, when the AM pole nodes up and down slightly while precessing

More general potentials

Axisymmetric Potentials



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Other different orbits are possible too with the same E and L_z

While these different orbits are bounded by a particular $\Phi_{\rm eff}$ contour, they are not filling all of the available space inside it