Stellar Dynamics and Structure of Galaxies

Jeans Theorem

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Outline I

Jeans Theorem

If we go back the the **Collisionless Boltzmann Equation** and look for a steady state solution (so $\frac{\partial}{\partial t} = 0$)

$$\mathbf{v} \cdot \nabla f - \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = 0$$

where $f(\mathbf{x}, \mathbf{v}, t)$ is the stellar distribution function in phase space $(\mathbf{x}, \mathbf{v})$.

Recall that each star follows a path in phase space given by $(\mathbf{x}(t), \mathbf{v}(t))$ where

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \mathbf{v} \\ \frac{d\mathbf{v}}{dt} &= -\nabla \Phi \end{aligned} \tag{6.1}$$

Define an integral of the motion as a function of the phase space coordinates $I(\mathbf{x}, \mathbf{v})$ which is constant along the path.

Integrals of Motion

Constants of Motion: any function of the phase-space coordinates and time $C(\mathbf{x}, \mathbf{v}, t)$ that is constant along every orbit where $\mathbf{x}(t)$ and $\mathbf{v}(t)$ are a solution to the equations of motion

$$C[\mathbf{x}(t_1), \mathbf{v}(t_1); t_1] = C[\mathbf{x}(t_2), \mathbf{v}(t_2); t_2] \tag{6.2}$$

for any t_1 and t_2

Any orbit in any force field has six independent constants of motion. For example, the initial phase-space coordinates $(\mathbf{x}_0, \mathbf{v}_0) \equiv [\mathbf{x}(0), \mathbf{v}(0)]$ can always be obtained from the equations of motion and can be regarded as six **constants of motion**.

The above procedure reminds us that physics is invariant to time translations i.e., the time at which we pick our initial conditions does not hold any information regarding the dynamical system.

Integrals of Motion

Integrals of Motion: any function $I(\mathbf{x}, \mathbf{v})$ of the phase-space coordinates alone that is constant along any orbit

$$I[\mathbf{x}(t_1), \mathbf{v}(t_1)] = I[\mathbf{x}(t_2), \mathbf{v}(t_2)] \tag{6.3}$$

Every integral is a constant of motion, but every constant of motion is **not** an integral.

For example, on a circular orbit in a spherical potential, the azimuthal speed

$$\psi = \Omega t + \psi_0$$

Hence, $C(\psi, t) \equiv t - \psi/\Omega$ will be constant of motion, but is not an integral of motion because it depends on time.

Integrals of Motion

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Integrals of Motion

Integrals of motion come in two flavors:

- Isolating Integrals of Motion reduce the dimensionality of the orbit by one, i.e. with energy E or angular momentum $\mathbf{L}$ in hand, the motion is restricted to 5D manifold in 6D dimensional phase-space. These are of great practical and theoretical importance in Dynamics.
- Non-Isolating Integrals of Motion do not affect the phase-space distribution of an orbit, i.e. do not reduce the dimensionality of the motion. These carry no practical value.

And, finally,

Energy is always an isolating integral of motion

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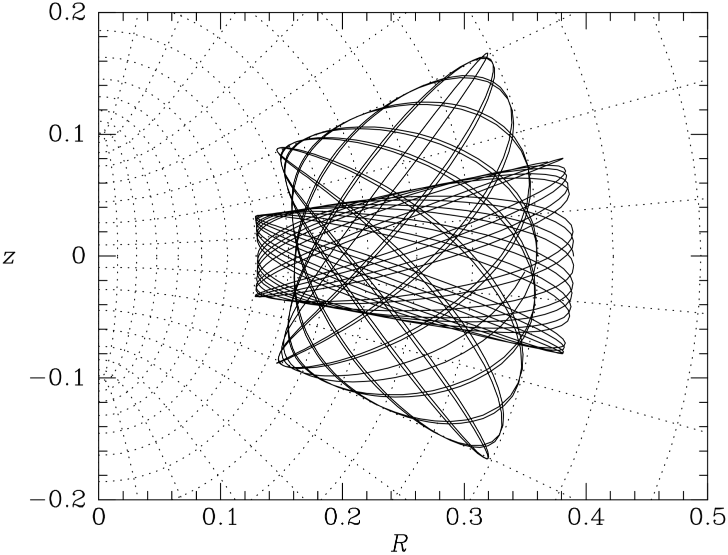
Integrals of Motion

Integrals of Motion



Jeans Theorem

$$\boxed{\frac{d\mathbf{x}}{dt} = \mathbf{v}} \quad \boxed{\frac{d\mathbf{v}}{dt} = -\nabla\Phi}$$

For example, in a static potential $\Phi(\mathbf{x})$, the energy

$$E = \frac{1}{2}\mathbf{v}^2 + \Phi(\mathbf{x}) \tag{6.4}$$

is an integral of the motion because

$$\begin{aligned} \frac{dE}{dt} &= \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} + \nabla\Phi \cdot \frac{d\mathbf{x}}{dt} \\ &= \mathbf{v} \cdot (-\nabla\Phi) + \nabla\Phi \cdot \mathbf{v} \\ &= 0 \end{aligned}$$

Jeans Theorem

Thus, for an integral of the motion I, we require

$$\frac{d}{dt}\left\{I\left[\mathbf{x}(t),\mathbf{v}(t)\right]\right\} = 0 \tag{6.5}$$

$\Rightarrow$

$$\frac{dI}{dt} = \nabla I \cdot \frac{d\mathbf{x}}{dt} + \frac{\partial I}{\partial \mathbf{v}} \cdot \frac{d\mathbf{v}}{dt} = 0$$

i.e.

$$\mathbf{v} \cdot \nabla I - \nabla \Phi \cdot \frac{\partial I}{\partial \mathbf{v}} = 0 \tag{6.6}$$

Recall the steady state collisionless Boltzmann equation

$$\mathbf{v} \cdot \nabla f - \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = 0$$

i.e. f and I obey the same equation.

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Jeans Theorem

Theorem (Jeans Theorem)

i) Any steady state solution of the Collisionless Boltzmann Equation depends on the phase-space coordinates ($\mathbf{x}$,$\mathbf{v}$) *only through integrals of the motion in a static potential, and ii) any function of the integrals yields a steady state solution of the collisionless Boltzmann equation.*

Proof.

Suppose f is a steady state solution of the collisionless Boltzmann equation. Then we have just shown $\frac{df}{dt} = 0$, and so f is an integral of the motion *i.e.* f can depend only on integrals of the motion.

Conversely if there are n integrals of the motion I_1,I_2,...,I_n, and if f is any function of these then

$$\frac{d}{dt}\left[f\left(I_1(\mathbf{x},\mathbf{v}), I_2(\mathbf{x},\mathbf{v}), ..., I_n(\mathbf{x},\mathbf{v})\right)\right] = \sum_{m=1}^{n} \frac{\partial f}{\partial I_m}\frac{dI_m}{dt} = 0$$

and so f satisfies the collisionless Boltzmann equation. □

Jeans Theorem

The value of Jeans theorem is that it gives us a way of closing the loop for solving the Collisionless Boltzmann Equation.

- Taking moments gave us insight about the properties of the solutions but not the actual solutions.
- The Jeans equation approach gave us more models, but no guarantee that they were physical.

Application of Jeans theorem

Obtaining self-consistent models

Given $\Phi(\mathbf{x})$ we know that any function

$$f(E) = f\left(\frac{1}{2}\mathbf{v}^2 + \Phi(\mathbf{x})\right) \tag{6.7}$$

is a solution of the collisionless Boltzmann equation. Now assume that all stars have the same mass m, then

$$\rho(\mathbf{x}) = m \iiint f d^3\mathbf{v} = m\nu(\mathbf{x})$$

or, without loss of generality, redfine f as the mass distribution function (rather than the number). Then

$$\nabla^2\Phi = 4\pi G\rho = 4\pi G \iiint f d^3\mathbf{v} \tag{6.8}$$

If we can find a function $f(E)$ which satisfies both (6.7) and (6.8) then we have a self-consistent solution in which the stars all obey Newton's laws in the potential $\Phi(\mathbf{x})$, and the potential $\Phi(\mathbf{x})$ is due to the stars.

Application of Jeans theorem

Obtaining self-consistent models

Notation: To make things easier we redefine the potential and the energy by adjusting the arbitrary constant and changing the sign.

Let $\Psi = -\Phi + \Phi_0$. This is **relative potential**.
and $\mathcal{E} = -E + \Phi_0 = \Psi - \frac{1}{2}v^2$. This is **relative energy**

Then we choose Φ_0 such that

$$f > 0 \quad \text{for } \mathcal{E} > 0$$

$$f = 0 \quad \text{for } \mathcal{E} \leq 0$$

Then, the relative potential satisfies the Poisson's equation

$$\nabla^2 \Psi = -4\pi G \rho$$

and $\Psi \to \Phi_0$ as $|\mathbf{x}| \to \infty$.

Application of Jeans theorem

If we have **spherical symmetry**, so Φ depends only on r, then

$$\begin{aligned}
\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) &= -4\pi G\rho \\
&= -4\pi G \iiint f d^3\mathbf{v} \\
&= -4\pi G \int_0^{\sqrt{2\Psi}} f(\mathcal{E}) 4\pi v^2 dv, \text{since f depends on v and not on } \mathbf{v} \\
&\quad \text{the upper limit comes from } f \neq 0 \text{ only if } \mathcal{E} = \Psi - \frac{1}{2}v^2 > 0 \\
&= -16\pi^2 G \int_0^{\sqrt{2\Psi}} f(\Psi - \frac{1}{2}v^2) v^2 dv
\end{aligned}$$

Now $d\mathcal{E} = -v dv$, with limits $v = 0$ $\mathcal{E} = \Psi$ and $v = \sqrt{2\Psi}$ $\mathcal{E} = 0$,so

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) = -16\pi^2 G \int_0^{\Psi} f(\mathcal{E})\sqrt{2(\Psi(r) - \mathcal{E})}\, d\mathcal{E}$$

Application of Jeans theorem

Eddington Formula

So, how to get from ρ to f?

Application of Jeans theorem

Eddington Formula

We start by going the other way round:

$$\nu(r) = \nu(\Psi) = \int \mathrm{d}^3\mathbf{v} f = 4\pi \int \mathrm{d}v v^2 f(\Psi - \frac{1}{2}v^2) = 4\pi \int_0^{\Psi} \mathrm{d}\mathcal{E} f(\mathcal{E})\sqrt{2(\Psi - \mathcal{E})} \quad (6.9)$$

Noting that potential Ψ is a monotonic function of r in any spherical system. Differentiating both sides with respect to Ψ

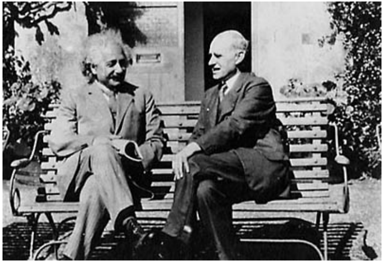
$$\frac{1}{\sqrt{8}\pi}\frac{\mathrm{d}\nu}{\mathrm{d}\Psi} = \int_0^{\Psi} \mathrm{d}\mathcal{E} \frac{f(\mathcal{E})}{\sqrt{\Psi - \mathcal{E}}} \quad (6.10)$$

This is an Abel integral equation having solution:

$$f(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2}\frac{\mathrm{d}}{\mathrm{d}\mathcal{E}}\int_0^{\mathcal{E}} \frac{\mathrm{d}\Psi}{\sqrt{\mathcal{E} - \Psi}}\frac{\mathrm{d}\nu}{\mathrm{d}\Psi} \quad (6.11)$$

$$f(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2}\left[\int_0^{\mathcal{E}} \frac{\mathrm{d}\Psi}{\sqrt{\mathcal{E} - \Psi}}\frac{\mathrm{d}^2\nu}{\mathrm{d}\Psi^2} + \frac{1}{\sqrt{\mathcal{E}}}\left(\frac{\mathrm{d}\nu}{\mathrm{d}\Psi}\right)_{\Psi=0}\right] \quad \textbf{Eddington's formula} \quad (6.12)$$

Sir Arthur Stanley Eddington



Application of Jeans theorem

Eddington Formula

To summarize: Given a spherically symmetric density distribution, which can be written as $\rho = \rho(\Psi)$ (may not always be possible), **Eddington's formula** yields the corresponding distribution function $f = f(\mathcal{E})$

Because we require $f(\mathcal{E}) \geq 0$ everywhere, Eddington's formula

$$\boxed{f(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2}\frac{\mathrm{d}}{\mathrm{d}\mathcal{E}}\int_0^{\mathcal{E}}\frac{\mathrm{d}\Psi}{\sqrt{\mathcal{E}-\Psi}}\frac{\mathrm{d}\nu}{\mathrm{d}\Psi}}$$

demands that the function $\int_0^{\mathcal{E}}\frac{\mathrm{d}\Psi}{\sqrt{\mathcal{E}-\Psi}}\frac{\mathrm{d}\nu}{\mathrm{d}\Psi}$ is an increasing function of $\mathcal{E}$.

Application of Jeans theorem

Harmonic oscillator potential

The problem (for the spherical case) is to find a pair of functions f,Ψ which satisfy this equation.
What does this problem amout to? Instead of looking at the 6-D case, let us illustrate the main ideas by taking a simple example - a 1-D harmonic oscillator potential.
[Part of the motivation for this is that inside a $\rho =$constant sphere

$$\Phi = \frac{2}{3}\pi G\rho_0(r^2 - 3r_0^2) = \frac{1}{2}\omega_0^2(x^2 + y^2 + z^2) + C \tag{6.13}$$

where ω_0 and C are constants, and this is a 3-D harmonic oscillator.]
So we take $E = \frac{1}{2}mv^2 + \frac{1}{2}\omega_0^2 x^2$ (from $\Phi = \frac{1}{2}\omega_0^2 x^2$), and then from Poissons equation

$$\rho(x) = \frac{1}{4\pi G}\frac{d^2\Phi}{dx^2} = \frac{\omega_0^2}{4\pi G} \tag{6.14}$$

Application of Jeans theorem

Harmonic oscillator potential

Phase space (x, v) orbit is an ellipse determined entirely by E, so all orbits with the same E lie on top of each other.

Then $f(E)$ just determines how many orbits there are of a given amplitude.

Note though that the contribution to the density at $x = 0$ is different for each E, since v there increases with E, so ones with higher E spend less time there.

The question is now: can we find $f(E)$ which gives $\rho = \rho_0 =$constant out to some x_0?

Application of Jeans theorem

Harmonic oscillator potential

Let

$$\Psi = -\Phi + \Phi_0 = C - \frac{1}{2}\omega_0^2 x^2$$

$$\mathcal{E} = -E + \Phi_0 = C - \frac{1}{2}\omega_0^2 x^2 - \frac{1}{2}v^2$$

At $x = x_0$ need $v = 0$, so choose $C = \frac{1}{2}\omega_0^2 x_0^2$

$$\mathcal{E} = \frac{1}{2}\omega_0^2 x_0^2 - \frac{1}{2}\omega_0^2 x^2 - \frac{1}{2}v^2 = \Psi - \frac{1}{2}v^2 \tag{6.15}$$

Then

$$f > 0 \quad \text{for } \mathcal{E} > 0$$

$$f = 0 \quad \text{for } \mathcal{E} \leq 0$$

$$\rho(x) = \int_0^\infty f dv = \int_0^{\sqrt{2\Psi(x)}} f dv$$

Application of Jeans theorem

Harmonic oscillator potential

In terms of $\mathcal{E}$ we use $-vdv = d\mathcal{E}$, with limits $v = 0 \leftrightarrow \mathcal{E} = \Psi$ and $v = \sqrt{2\Psi} \leftrightarrow \mathcal{E} = 0$ to obtain

$$\rho(x) = \int_0^{\Psi(x)} \frac{f(\mathcal{E})d\mathcal{E}}{\sqrt{2(\Psi(x) - \mathcal{E})}}$$

where

$$\Psi(x) = \frac{1}{2}\omega_0^2(x_0^2 - x^2).$$

[Note that 1-D differs from 3-D for this]
In fact it is easier to use the v equation, i.e.

$$\rho(x) = \int_0^{\sqrt{\omega_0^2(x_0^2 - x^2)}} f\left(\frac{1}{2}\omega_0^2(x_0^2 - x^2) - \frac{1}{2}v^2\right)\, dv \qquad (6.16)$$

Now need to find a function f which gives us constant ρ. We can do this by trial and error, or inspired guesswork...

Application of Jeans theorem

Harmonic oscillator potential

Try f = constant = f_0. Then

$$\rho(x) = [f_0 v]_0^{\sqrt{\omega_0^2(x_0^2-x^2)}} = f_0\sqrt{\omega_0^2(x_0^2-x^2)}$$

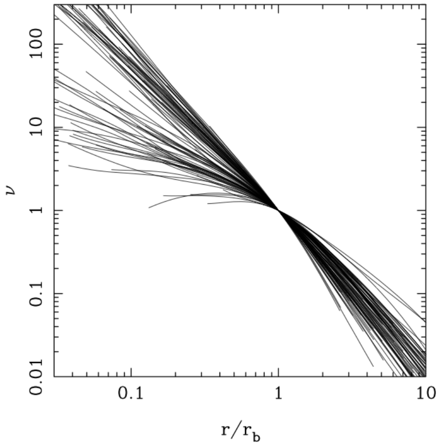
which is not constant, so we have chosen the wrong f.
So try $f = \frac{k}{\sqrt{\mathcal{E}}}$, where k is a constant.

$$\begin{aligned}
\rho(x) &= \int_0^{\sqrt{\omega_0^2(x_0^2-x^2)}} \frac{\sqrt{2}\,k\,dv}{\sqrt{\omega_0^2(x_0^2-x^2)-v^2}} \\
&= \left[\sqrt{2}\,k\sin^{-1}\left(\frac{v}{\sqrt{\omega_0^2(x_0^2-x^2)}}\right)\right]_0^{\sqrt{\omega_0^2(x_0^2-x^2)}} \\
&= \frac{k\pi}{\sqrt{2}} \quad = \text{constant as required}
\end{aligned}$$

Application of Jeans theorem

Surface brightness profiles of elliptical galaxies

Ellipticals either have “cores” or “extra light”

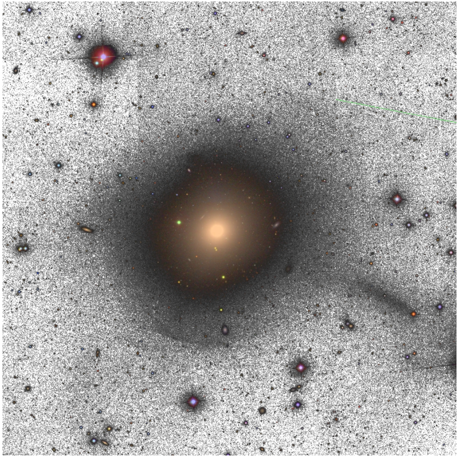


Lauer et al 2007, HST data

Application of Jeans theorem

Surface brightness profiles of elliptical galaxies

“Extra light” = shells of accreted material

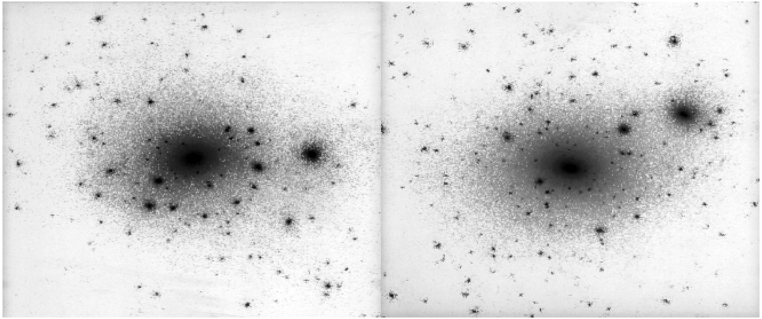


SDSS image manipulation

Application of Jeans theorem

Dark Matter only N-body simulations

Universal DM radial density profile discovered

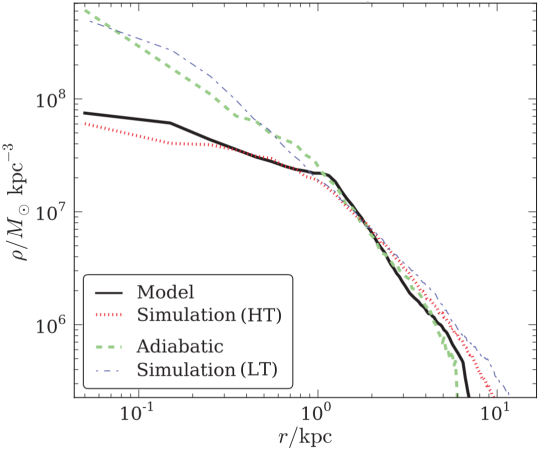


Moore et al, 1999

Application of Jeans theorem

Baryonic physics affects Dark Matter

Cusps are turned into cores with supernova feedback



Pontzen & Governato, 2011

Application of Jeans theorem

Two-Power Law Density Models

The two-power law models motivated by the measurements of the light profile of elliptical galaxies and by the results of dark matter N-body simulations.

$$\rho(r) = \frac{\rho_0}{(r/a)^{\alpha}(1 + r/a)^{\beta-\alpha}} \tag{6.17}$$

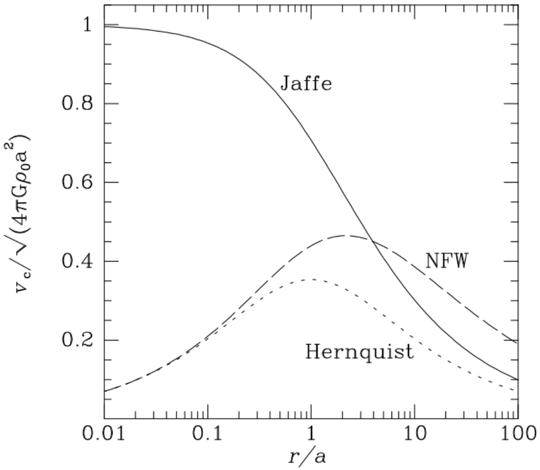
For several α and β there are models with particularly simple analytic properties. For example

- $\beta = 4$ Dehnen (Dehnen 1993)
- $\alpha = 1, \beta = 4$ Hernquist (Hernquist 1990)
- $\alpha = 2, \beta = 4$ Jaffe (Jaffe 1983)
- $\alpha = 1, \beta = 3$ NFW (Navarro, Frenk & White 1993)
- $1 < \alpha < 1.5, \beta \simeq 3$ for dark haloes

Application of Jeans theorem

Two-Power Law Density Models

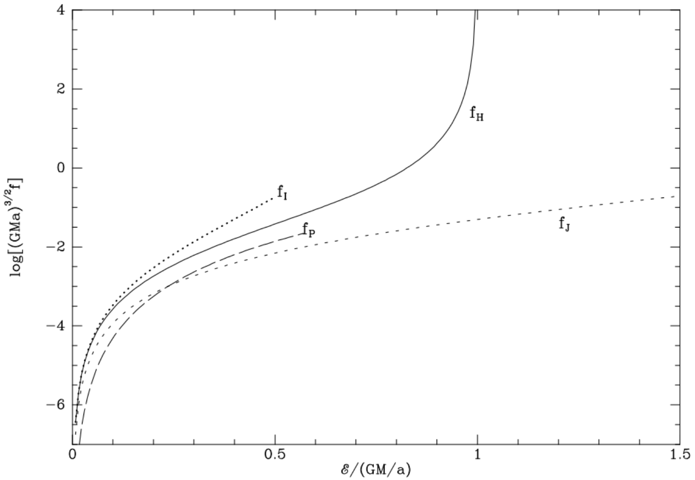
Circular speed versus radius



Application of Jeans theorem

Two-Power Law Density Models

Distribution functions for simple two-power law models



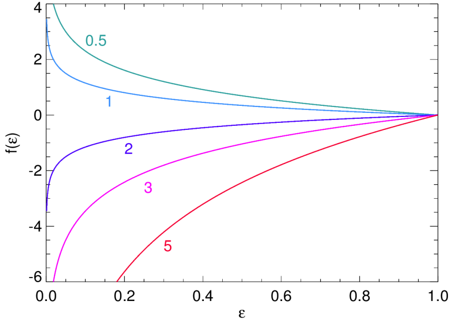
Application of Jeans theorem

Spherically symmetric solutions of the collisionless Boltzmann equation

These still have one spatial coordinate, but note that the orbits are not just radial.
A simple form of the distribution function is

$$f = \begin{cases} F\mathcal{E}^{n-\frac{3}{2}} & \mathcal{E} > 0 \\ 0 & \mathcal{E} \leq 0 \end{cases} \tag{6.18}$$

where F is a constant.



Application of Jeans theorem

Spherically symmetric solutions of the collisionless Boltzmann equation

$$f = \begin{cases} F\mathcal{E}^{n-\frac{3}{2}} & \mathcal{E} > 0 \\ 0 & \mathcal{E} \leq 0 \end{cases}$$

Then

$$\rho(r) = 4\pi \int_0^\infty f(\Psi - \frac{1}{2}v^2)v^2 \, dv$$

with $\Psi = \Psi(r)$. So

$$\rho(r) = 4\pi F \int_0^{\sqrt{2\Psi}} (\Psi - \frac{1}{2}v^2)^{n-\frac{3}{2}} v^2 \, dv \tag{6.19}$$

Application of Jeans theorem

Spherically symmetric solutions of the collisionless Boltzmann equation

$$\boxed{\rho(r) = 4\pi F \int_0^{\sqrt{2\Psi}} (\Psi - \tfrac{1}{2}v^2)^{n-\frac{3}{2}} v^2 \ dv}$$

Let

$$v^2 = 2\Psi \cos^2\theta$$

so

$$v \ dv = -2\Psi \cos\theta \sin\theta \ d\theta$$

$$v^2 \ dv = -(2\Psi)^{\frac{3}{2}} \cos^2\theta \sin\theta \ d\theta$$

Limits are $v = 0 \leftrightarrow \theta = \frac{\pi}{2}$ and $v = \sqrt{2\Psi} \leftrightarrow \theta = 0$

$\Rightarrow$

$$\rho(r) = 4\pi F \int_0^{\frac{\pi}{2}} \Psi^{n-\frac{3}{2}} \sin^{2n-3}\theta (2\Psi)^{\frac{3}{2}} \cos^2\theta \sin\theta \ d\theta \qquad (6.20)$$

Application of Jeans theorem

Spherically symmetric solutions of the collisionless Boltzmann equation

$$\begin{aligned}\rho(r) &= 4\pi F \int_0^{\frac{\pi}{2}} \Psi^{n-\frac{3}{2}} \sin^{2n-3}\theta (2\Psi)^{\frac{3}{2}} \cos^2\theta \sin\theta \; d\theta \\ &= 2^{\frac{7}{2}} \pi F \Psi^n \left[\int_0^{\frac{\pi}{2}} \sin^{2n-2}\theta \; d\theta - \int_0^{\frac{\pi}{2}} \sin^{2n}\theta \; d\theta \right] \\ &= C_n \Psi^n \quad \text{where } \Psi > 0 \quad (\text{otherwise } 0) \end{aligned} \tag{6.21}$$

where

$$C_n = \frac{(2\pi)^{\frac{3}{2}} \left(n - \frac{3}{2}\right)! F}{n!} \tag{6.22}$$

Note that for C_n to be finite we need $n - \frac{3}{2} > -1 \Rightarrow n > \frac{1}{2}$ since $(n - \frac{3}{2})! = \Gamma(n - \frac{1}{2})$, and $\Gamma(x)$ is finite for $x > 0$.

Application of Jeans theorem

Spherically symmetric solutions of the collisionless Boltzmann equation

Gamma function:

$$\Gamma(z+1) = \int_0^\infty t^z e^{-t} dt, \qquad \Gamma(1) = \Gamma(2) = 1$$

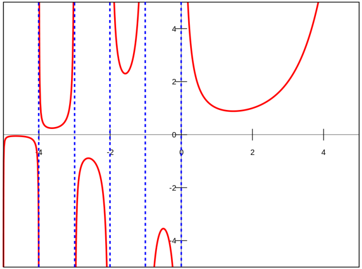
Integration by parts gives $\Gamma(z+1) = z\Gamma(z) \Rightarrow \Gamma(z+1) = z!$ for integer z.
Also have (Euler's reflection formula)

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} = \int_0^\infty \frac{t^{z-1}}{1+t} dt$$

$$\Rightarrow \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$\Gamma(z)$ has simple poles at $z = 0, -1, -2 \ldots$
$\Rightarrow C_n$ finite requires $n > \frac{1}{2}$ for

$$C_n = \frac{(2\pi)^{\frac{3}{2}} \left(n - \frac{3}{2}\right)! F}{n!}$$



Application of Jeans theorem

Spherically symmetric solutions of the collisionless Boltzmann equation

$$\boxed{\rho(r) = C_n \Psi^n \quad \text{where } \Psi > 0 \quad (\text{otherwise } 0)}$$

Now we can substitute the expression for ρ into Poisson's equation, so

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\Psi}{dr}\right) = -4\pi G C_n \Psi^n \tag{6.23}$$

We can rescale this, so $s = r/b$, where

$$b = (4\pi G \Psi_0^{n-1} C_n)^{-\frac{1}{2}} \tag{6.24}$$

$\psi = \Psi/\Psi_0$ with $\Psi_0 = \Psi(0)$ and then

$$\frac{1}{s^2}\frac{d}{ds}\left(s^2 \frac{d\psi}{ds}\right) = \begin{cases} -\psi^n & \psi > 0 \\ 0 & \psi \leq 0 \end{cases} \tag{6.25}$$

$(\Psi \leq 0 \Rightarrow \mathcal{E} \leq 0 \Rightarrow f = 0 \Rightarrow \rho = 0)$

Application of Jeans theorem

Spherically symmetric solutions of the collisionless Boltzmann equation

$$\frac{1}{s^2}\frac{d}{ds}\left(s^2\frac{d\psi}{ds}\right) = \begin{cases} -\psi^n & \psi > 0 \\ 0 & \psi \leq 0 \end{cases}$$

This is the **Lane-Emden equation**, which you are familiar with from the fluids course.

The boundary conditions are: at $s = 0$ $\psi = 1$ by definition, and $\frac{d\psi}{ds} = 0$ because there is no gravitational force at $s = 0$.

The equation for $\psi(r)$ is the same as the equation for $\rho(r)$ for a star with an equation of state $p = K\rho^{1+\frac{1}{n}}$. And we know there are analytic solutions for $n = 0, 1, 5$, and that the one with $n = 5$ has infinite radius. Here we need $n > \frac{1}{2}$.

What we have done here is chosen $f(\mathcal{E})$, and then obtained the differential equation to solve for Ψ and hence ρ.

Application of Jeans theorem

Plummer potential

This is the model with $n = 5$.
Solution is

$$\psi = \frac{1}{\sqrt{1 + \frac{1}{3}s^2}} \tag{6.26}$$

It satisfies the boundary conditions, and you can check it satisfies

$$\frac{1}{s^2}\frac{d}{ds}\left(s^2\frac{d\psi}{ds}\right) = -\psi^5 \tag{6.27}$$

$\Rightarrow$

$$\rho = C_5\Psi^5 = \frac{c_5\Psi_0^5}{(1 + \frac{1}{3}s^2)^{\frac{5}{2}}} \tag{6.28}$$

so the density extends to ∞.

Application of Jeans theorem

Plummer potential

But the mass

$$
\begin{aligned}
M &= \int_0^\infty 4\pi\rho r^2\,dr \\
&= -\int_0^\infty \frac{1}{G}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right)\,dr \\
&= \frac{1}{G}\left[r^2\frac{d\Psi}{dr}\right]_\infty^0 \\
&= \lim_{r\to\infty} -\frac{1}{G}\left(r^2\frac{d\Psi}{dr}\right) \\
&= -\frac{b}{G}\left(s^2\frac{d\Psi}{ds}\right)_{s\to\infty} \\
&= \frac{b\,\Psi_0}{G}\quad\text{which is finite}
\end{aligned}
$$

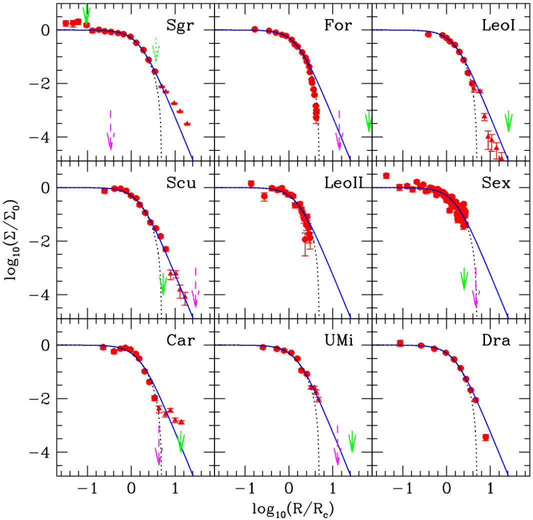
Application of Jeans theorem

Plummer potential

- This is quite a good model of most globular clusters,
- and (for the light profiles) of dwarf spheroidal galaxies.
- But not so good for E0 galaxies because $\rho \sim r^{-5}$ at large radii.

Application of Jeans theorem

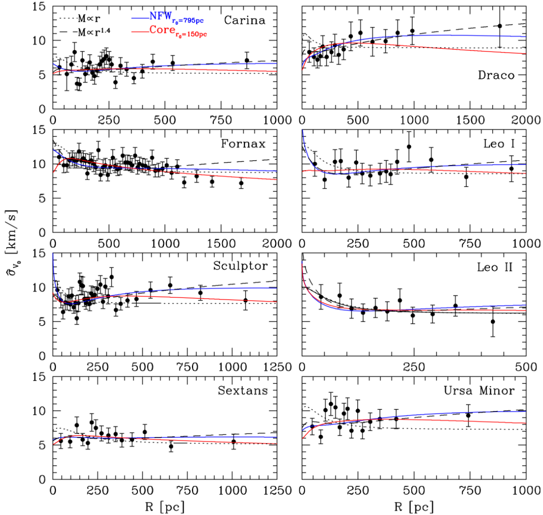
Stellar density in dwarf spheroidals



Peñarrubia et al

Application of Jeans theorem

Constant velocity dispersion in dwarfs



Walker et al

Application of Jeans theorem

Isothermal sphere

i.e. $\sigma^2(r)$ =constant.

This is the limit $n \to \infty$ (as in fluids, where $p = k\rho^{1+\frac{1}{n}}$ with $n \to \infty \Rightarrow p = K\rho$), but it is easier to start again.

Assume that the distribution function is Maxwellian with constant velocity dispersion, so guess

$$\begin{aligned} f(\mathcal{E}) &= \frac{\rho_1}{(2\pi\sigma^2)^{\frac{3}{2}}} \exp\left(\frac{\mathcal{E}}{\sigma^2}\right) \\ &= \frac{\rho_1}{(2\pi\sigma^2)^{\frac{3}{2}}} \exp\left(\frac{\Psi(r) - \frac{1}{2}v^2}{\sigma^2}\right) \end{aligned}$$

where ρ_1 is a constant.

$\Rightarrow$

$$\rho(r) = \int_0^\infty 4\pi v^2 f(v) dv = \rho_1 \exp\left(\frac{\Psi}{\sigma^2}\right) \qquad (6.29)$$

Application of Jeans theorem

Isothermal sphere

$$\boxed{\rho(r) = \int_0^\infty 4\pi v^2 f(v) dv = \rho_1 \exp\left(\frac{\Psi}{\sigma^2}\right)}$$

which means

$$\Psi = \sigma^2(\ln\rho - \ln\rho_1)$$

Poisson's equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) = -4\pi G\rho_1 \exp\left(\frac{\Psi}{\sigma^2}\right)$$

is then

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\ln\rho\right) = -\frac{4\pi G}{\sigma^2}\rho \tag{6.30}$$

<u>One</u> solution to this equation is

$$\rho(r) = \frac{\sigma^2}{2\pi G r^2} \tag{6.31}$$

(which you can easily check).

Application of Jeans theorem

Isothermal sphere

$$\rho(r) = \frac{\sigma^2}{2\pi G r^2}$$

This is called the **Singular Isothermal Sphere**.

- $\rho \to \infty$ as $r \to 0$ (singular)
- $M(r) = \frac{2\sigma^2 r}{G} \to \infty$ as $r \to \infty$ (awkward)
- $\Sigma(R) = \frac{\sigma^2}{2GR}$
- $\Phi(r) = 2\sigma^2 \ln(r) + \mathrm{constant}$

Application of Jeans theorem

Isothermal sphere

We'd prefer a solution which is well behaved at the origin, so $\Psi \rightarrow$constant and $\frac{d\Psi}{dr} \rightarrow 0$ there. It is convenient to rescale the variables first, so

$$\tilde{\rho} = \rho/\rho_0$$

and

$$\tilde{r} = r/r_0$$

where

$$r_0 = \sqrt{\frac{9\sigma^2}{4\pi G\rho_0}}$$

Then in terms of the new variables the Poisson's equation (6.30) becomes

$$\frac{1}{\tilde{r}^2}\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d}{d\tilde{r}}\ln\tilde{\rho}\right) = -9\tilde{\rho} \qquad (6.32)$$

with boundary conditions $\tilde{\rho}(0) = 1$ and $\left.\frac{d\tilde{\rho}}{d\tilde{r}}\right|_{\tilde{r}=0} = 0$.

Application of Jeans theorem

Isothermal sphere

$$\boxed{\frac{1}{\tilde{r}^2}\frac{d}{d\tilde{r}}\left(\tilde{r}^2\frac{d}{d\tilde{r}}\ln\tilde{\rho}\right) = -9\tilde{\rho}}$$

This is a numerical problem (see Fig 4-7 from Binney & Tremaine).
At large radii $r >> r_0$ have $\rho \propto r^{-2}$ and $M(r) \approx \frac{2\sigma^2}{G} r$ so $M \to \infty$ and $v_{\rm escape} = \infty$.
It is of interest to calculate the mean square speed of the stars:

$$\begin{aligned}
\overline{v^2} &= \frac{\int_0^\infty f(\mathcal{E}) v^2 4\pi v^2\, dv}{\int_0^\infty f(\mathcal{E}) 4\pi v^2\, dv} \\
&= \frac{\int_0^\infty \exp\left(\frac{\Psi - \frac{1}{2}v^2}{\sigma^2}\right) v^2 4\pi v^2\, dv}{\int_0^\infty \exp\left(\frac{\Psi - \frac{1}{2}v^2}{\sigma^2}\right) 4\pi v^2\, dv} \\
&\qquad \text{Let } x^2 = v^2/2\sigma^2, \text{ and noting that } \exp\Psi \text{ terms cancel} \\
&= 2\sigma^2 \frac{\int_0^\infty e^{-x^2} x^4\, dx}{\int_0^\infty e^{-x^2} x^2\, dx}
\end{aligned}$$

Application of Jeans theorem

Isothermal sphere

[These are fairly standard:

$$\int_0^\infty e^{-\alpha x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\frac{d}{d\alpha}: \quad -\int_0^\infty x^2 e^{-\alpha x^2} dx = -\frac{\sqrt{\pi}}{4}\alpha^{-\frac{3}{2}}$$

$$\frac{d}{d\alpha}: \quad \int_0^\infty x^4 e^{-\alpha x^2} dx = \frac{\sqrt{\pi}}{4}\frac{3}{2}\alpha^{-\frac{5}{2}}$$

]

Hence

$$\overline{v^2} = 2\sigma^2 \times \frac{3}{2} = 3\sigma^2$$

So σ is the one-dimensional velocity dispersion.

Application of Jeans theorem

Isothermal sphere

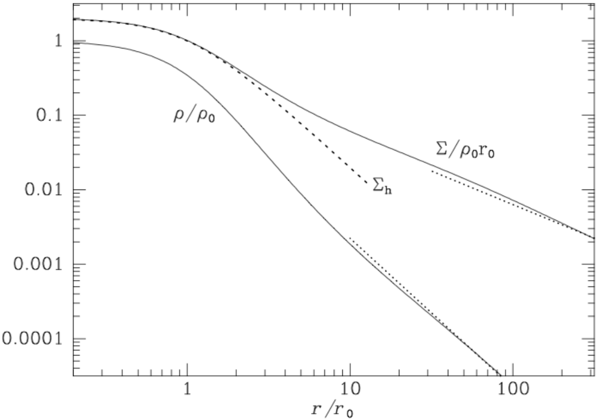


Figure 4.6 Volume (ρ/ρ_0) and projected ($\Sigma/\rho_0 r_0$) mass densities of the isothermal sphere. The dotted lines show the volume- and surface-density profiles of the singular isothermal sphere. The dashed curve shows the surface density of the modified Hubble model (4.109a).