Stellar Dynamics and Structure of Galaxies

Potentials due to axisymmetric distributions of matter

Vasily Belokurov
vasily@ast.cam.ac.uk

Institute of Astronomy

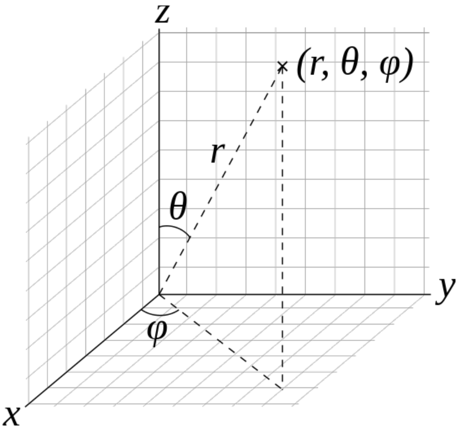
Lent Term 2016

Outline I

Potential from axisymmetric matter distribution

Outside matter, $\rho = 0$, so $\nabla^2 \Phi = 0$.
Axisymmetric, so independent of spherical polar ϕ,

$$\Phi = \Phi(r, \theta)\,\textbf{only}$$



Potential from axisymmetric matter distribution

Outside matter, $\rho = 0$, so $\nabla^2\Phi = 0$.
Axisymmetric, so independent of spherical polar ϕ,

$$\Phi = \Phi(r, \theta)\,\textbf{only}$$

Then $\nabla^2\Phi = 0$ becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) = 0\,.$$

If we **assume** the function $\Phi(r, \theta)$ is separable, ie

$$\Phi(r, \theta) = \mathcal{R}(r) \times \Theta(\theta)$$

then, as substituting and dividing by $\mathcal{R}\Theta$,

$$\frac{1}{\mathcal{R}}\frac{d}{dr}\left(r^2\frac{d\mathcal{R}}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = 0$$

i.e. each term has to be constant (and one = - the other)

Potential from axisymmetric matter distribution

Outside matter, $\rho = 0$, so $\nabla^2\Phi = 0$.
Axisymmetric, so independent of spherical polar ϕ,

$$\Phi = \Phi(r,\theta)\,\textbf{only}$$

Then $\nabla^2\Phi = 0$ becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) = 0\,.$$

If we **assume** the function $\Phi(r,\theta)$ is separable, ie

$$\Phi(r,\theta) = \mathcal{R}(r)\times\Theta(\theta)$$

then, as substituting and dividing by $\mathcal{R}\Theta$,

$$\frac{1}{\mathcal{R}}\frac{d}{dr}\left(r^2\frac{d\mathcal{R}}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = 0$$

i.e. each term has to be constant (and one = - the other)

Potential from axisymmetric matter distribution

Let

$$\frac{1}{\mathcal{R}}\frac{d}{dr}\left(r^2\frac{d\mathcal{R}}{dr}\right) = n(n+1) \quad \text{(say)} \tag{4.1}$$

$$\text{then} \quad \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -n(n+1)\Theta(\theta) \tag{4.2}$$

Set $\mu = \cos\theta \quad (-1 \leq \mu \leq 1)$

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{d\Theta}{d\mu}\right\} + n(n+1)\Theta = 0 \tag{4.3}$$

since $d\mu = -\sin\theta d\theta$

– Legendres equation.
Want $\Theta(\theta)$ to be finite at $\theta = 0, \pi$ $(\mu = \pm 1)$. And the only way this can happen is if $n = 0, 1, 2 \ldots$- a non-negative integer

Potential from axisymmetric matter distribution

Let

$$\frac{1}{\mathcal{R}}\frac{d}{dr}\left(r^2\frac{d\mathcal{R}}{dr}\right) = n(n+1) \quad \text{(say)} \tag{4.1}$$

$$\text{then} \quad \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = -n(n+1)\Theta(\theta) \tag{4.2}$$

Set $\mu = \cos\theta \quad (-1 \leq \mu \leq 1)$

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{d\Theta}{d\mu}\right\} + n(n+1)\Theta = 0 \tag{4.3}$$

since $d\mu = -\sin\theta d\theta$

– Legendres equation.

Want $\Theta(\theta)$ to be finite at $\theta = 0, \pi\,(\mu = \pm 1)$. And the only way this can happen is if $n = 0, 1, 2 \ldots$- a non-negative integer

Potential from axisymmetric matter distribution

Legendré Polynomials

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{d\Theta}{d\mu}\right\} + n(n+1)\Theta = 0 \quad 4.3$$

Solution of (4.1) is

$$\mathcal{R}(r) = Ar^n + \frac{B}{r^{n+1}}$$

And (4.3) is the defining differential equation for Legendré polynomials, so we have

$$\text{and} \quad \Theta(\theta) = CP_n(\cos\theta) \quad \text{for a given } n$$

Presumably you've seen Legendré polynomials before:

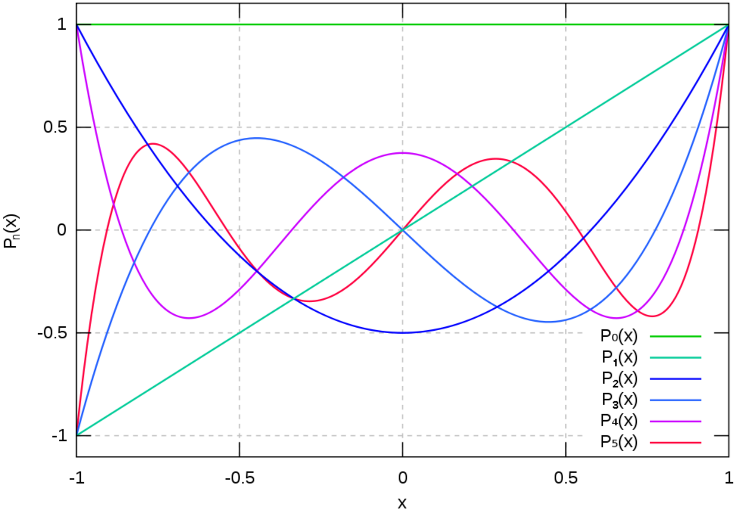
$$P_0(x) = 1 \qquad P_n(1) = 1\forall n$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

Potential from axisymmetric matter distribution

Legendré Polynomialsmials



Potential from axisymmetric matter distribution

Legendré Polynomials

Recurrence relation:

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)\ .$$

Orthogonal polynomials on $[-1,1]$ i.e.

$$\int_{-1}^{1} P_n(x)P_m(x) = 0 \ \text{ if } \ n \neq m$$

$$\left(= \frac{2}{2n+1} \ \text{ if } \ n = m \right)$$

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n.$$

$P_n(x)$ is oscillatory: $P_n(x)$ has n zeros in [-1,1] (so higher $n \Rightarrow$ more oscillations)
$\Rightarrow$ Any function on [-1,1] can be reproduced as a sum of these (cf Fourier series).

Legendré and Fourier

The only known portrait of Legendré, recently unearthed, is found in the 1820 book *Album de 73 portraits-charge aquarelés des membres de l'Institut*, a book of caricatures of seventy-three famous mathematicians by the French artist Julien-Leopold Boilly

Potential from axisymmetric matter distribution

Legendré Polynomials

Note then for general solution outside axisymmetric body

$$\mathcal{R}(r) = \frac{B}{r^{n+1}}$$

(want Φ finite as $r \to \infty$).
And if the body is symmetric about $\theta = \pi/2$ plane, then only need even n, since these are symmetric about $\mu = 0$ and the odd n ones are antisymmetric.

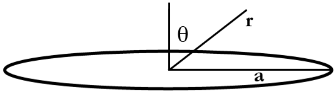
$$\Phi(r,\theta) = \sum_{n=0}^{\infty} \frac{B_n P_n(\cos\theta)}{r^{n+1}} = \sum_{k=0}^{\infty} \frac{B_{2k} P_{2k}(\cos\theta)}{r^{2k+1}}$$

$$= -\frac{GM}{r} + \frac{J_2}{r^3}\frac{1}{2}\left(3\cos^2\theta - 1\right) + \frac{J_4}{r^5}\left(\frac{35}{8}\cos^4\theta - \frac{15}{4}\cos^2\theta + \frac{3}{8}\right) \ldots\ldots$$

If the body is nearly spherical, treat J terms as perturbations.

Potential from axisymmetric matter distribution

Example



Potential due to a ring of matter
Mass M , radius a (in full generality)

Axisymmetric $\Rightarrow$ no ϕ dependence $\Rightarrow$ $\nabla^2\Phi = 0$ has solution

$$\Phi = \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos\theta)$$

So if we need to work out Φ somewhere useful to get the coefficients A_n and B_n.
Trick: let's try the easiest place - on the z axis.
At height z on the z-axis

$$\Phi = -G \int \frac{\rho d^3\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} = -\frac{GM}{(a^2 + z^2)^{\frac{1}{2}}}$$

$\uparrow$ usual sum

Potential from axisymmetric matter distribution

Example

For small z,

$$\begin{aligned}\Phi &= -\frac{GM}{a}\left(1+\frac{z^2}{a^2}\right)^{-\frac{1}{2}} \\ &= -\frac{GM}{a}\left\{1-\frac{z^2}{2a^2}+\frac{3}{8}\frac{z^4}{a^4}-\cdots\right\}\end{aligned}$$

But on axis also $\theta = 0, \cos\theta = 1, r = z$ so

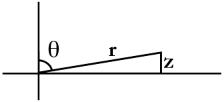
$$\Phi(r) = \sum_{n=0}^{\infty}\left(A_n z^n + \frac{B_n}{z^{n+1}}\right) \text{ there}$$

Match terms, and so for small $r < a$

$$\Phi = -\frac{GM}{a}\left\{1-\frac{r^2}{2a^2}P_2(\cos\theta)+\frac{3}{8a^4}r^4P_4(\cos\theta)\cdots\right\}$$

Potential from axisymmetric matter distribution

Example



$$\Phi = -\frac{GM}{a}\left\{1 - \frac{r^2}{2a^2}P_2(\cos\theta) + \frac{3}{8a^4}r^4 P_4(\cos\theta)\cdots\right\}$$

$$\begin{aligned} P_0(\cos\theta) &= 1 \\ P_2(\cos\theta) &= \frac{1}{2}(3\cos^2\theta - 1) \\ P_4(\cos\theta) &= \frac{1}{8}(35\cos^4\theta - 30\cos^2\theta + 3) \end{aligned}$$

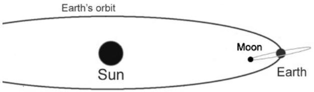
So if you want the potential close to the plane:

$$\theta \sim \pi/2 \qquad r \longrightarrow R, \qquad \text{and } r\cos\theta = z \qquad \text{or} \cos\theta = \frac{z}{R}$$

$$\Rightarrow \Phi(R, Z) \simeq -\frac{GM}{a}\left[1 - \frac{1}{4a^2}\left\{3z^2 - R^2\right\} + \dots\right]$$

Potential from axisymmetric matter distribution

Motion of the Moon



Can apply this to the motion of the Moon due to the time average of the Sun (axisymmetric part of the potential)

$$\Phi = -\frac{GM_\oplus}{R} - \frac{GM_\odot}{a}\left\{1 - \frac{1}{4a^2}\left(3z^2 - R^2\right)\right\}$$

$$R = R_{\oplus M} = 4 \times 10^8 \text{ m} \qquad a = R_{\odot\oplus} = 1.5 \times 10^{11} \text{ m}$$

$$M_\oplus = 6 \times 10^{24} \text{ kg} \qquad M_\odot = 2 \times 10^{30} \text{ kg}$$

Mass of moon is about 1/81th that of the Earth, so it is the 'test particle'.

Ratio of Earth-Moon and Moon-Sun extra part is

$$\frac{GM_\oplus}{R} \Big/ \frac{GM_\odot R^2}{a^3} = M_\oplus a^3 / M_\odot R^3 \sim 200$$

so can treat the Moon-Sun force as a perturbation.

Potential from axisymmetric matter distribution

Eclipse cycle

$R_{\oplus} = 6360\,\mathrm{km}$ – need accurate alignment.
Moon orbit inclination to ecliptic 5^{o}, eccentricity 0.055.
Moon's period:

- Siderial (relative to fixed star)$=27^{\mathrm{d}}.322 = 2\pi/\Omega$
- Synodical (new moon - new moon)$=29^{\mathrm{d}}.530 = 2\pi/(\Omega - \Omega_{\oplus})$
- Anomalistic month (perigee to perigee)$=27^{\mathrm{d}}.554 = 2\pi/\Omega_r = 2\pi/K$
- Nodical month (node to node)$=27^{\mathrm{d}}.212 = 2\pi/\mathcal{V}$

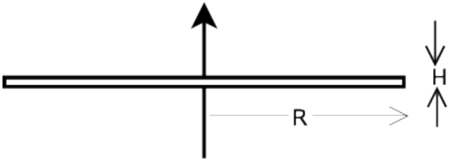
To get a solar ecplise we must have

- Moon at "new moon" - once every synodical month
- Moon passing through a node - once every nodical month
- For the best eclipse, moon at perigee - once every anomalistic month

The "saros" or eclipse cycle comes about because
223 synodic months = 6585.32 days
242 nodical months = 6585.36 days
239 anomalistic months = 6585.54 days

... so every 6585 days, or 18 years get a similar pattern of eclipses.

Potential due to thin disk



$H << R$

Let's use cylindrical polar, (R, ϕ, z), coordinates

Expect $\Phi \equiv \Phi(R, z)$ and $\Phi(R, z) = \Phi(R, -z)$ by symmetry.

Outside disk $\nabla^2\Phi = 0$.

$$\Rightarrow \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\Phi}{\partial R}\right) + \frac{\partial^2\Phi}{\partial z^2} = 0$$

We solve this by separation of variables, letting

$$\Phi(R, z) = J(R)Z(z)$$

Potential due to thin disk

Bessel Functions

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \Phi}{\partial R}\right) + \frac{\partial^2 \Phi}{\partial z^2} = 0 \qquad \Phi(R,z) = J(R)Z(z)$$

$$\Rightarrow \quad Z(z)\frac{1}{R}\frac{d}{dR}\left(R\frac{dJ(R)}{dR}\right) + J(R)\frac{d^2}{dz^2}Z(z) = 0$$

$$\Rightarrow \quad \underbrace{\frac{1}{JR}\frac{d}{dR}\left(R\frac{dJ}{dR}\right)}_{\text{function of } R} = - \underbrace{\frac{1}{Z}\frac{d^2 Z}{dz^2}}_{\text{function of } z} = -k^2 \text{ , say}$$

$$\Rightarrow \quad \frac{d^2 Z}{dz^2} - k^2 Z = 0 \tag{4.4}$$

so $Z = A\exp(kz) + B\exp(-kz)$

$$\text{and} \quad \frac{1}{R}\frac{d}{dR}\left(R\frac{dJ}{dR}\right) + k^2 J(R) = 0 \tag{4.5}$$

Potential due to thin disk

Bessel Functions

$$\frac{1}{R}\frac{d}{dR}\left(R\frac{dJ}{dR}\right) + k^2 J(R) = 0 \quad 4.5$$

We would quite like $\Phi(R, \infty)$ and $\Phi(R, -\infty)$ to be zero, so

$$Z(z) = A\exp(-k|z|)$$

is the appropriate solution for $Z(z)$.

The R equation (4.5) is the defining equation for a Bessel function. These are the analogues of sines and cosines now for cylindrical as opposed to linear problems (e.g. drum beats).

$$\text{So while} \quad \frac{d^2y}{dz^2} + k^2 y = 0 \quad \text{has solutions} \quad \sin(kz)\,,\cos(kz) \qquad (4.6)$$

$$\text{similarly} \quad \frac{1}{s}\frac{d}{ds}\left(s\frac{dy}{ds}\right) + k^2 y = 0 \quad \text{has solutions} \quad J_0(ks)\,,\, Y_0(ks) \qquad (4.7)$$

which you can look up in e.g. Abramowitz & Stegun "Handbook of Mathematical Functions"

Potential due to thin disk

Bessel Functions

Note that as $x \to 0$ $J_0(x) \to 1$ and $Y_0(x) \to -\infty$.
More generally the equation

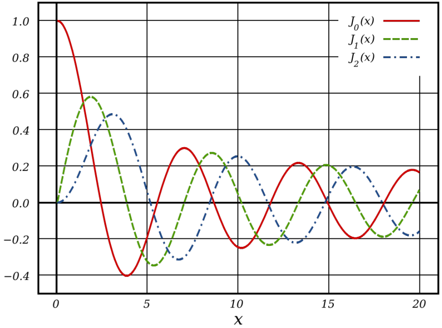
$$\frac{1}{s}\frac{d}{ds}\left(s\frac{dy}{ds}\right) + \left(k^2 - \frac{\nu^2}{s^2}\right) y = 0$$

has solutions $J_\nu(ks)$, $Y_\nu(ks)$, a family of Bessel functions characterized by the index ν.

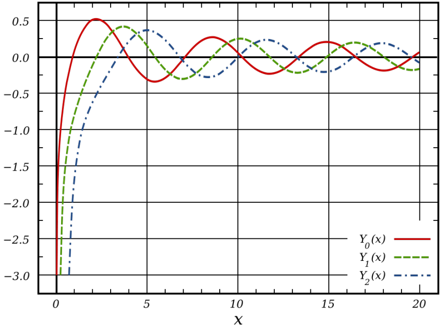
J_ν - Bessel function of the first kind
Y_ν - Bessel function of the second kind

Potential due to thin disk

Bessel Functions



J_ν, Bessel function of the first kind



Y_ν, Bessel function of the second kind

Potential due to thin disk

Bessel Functions

Similarly $$\frac{1}{s}\frac{d}{ds}\left(s\frac{dy}{ds}\right) - k^2 y = 0 \rightarrow \quad I_0(ks), K_0(ks)$$

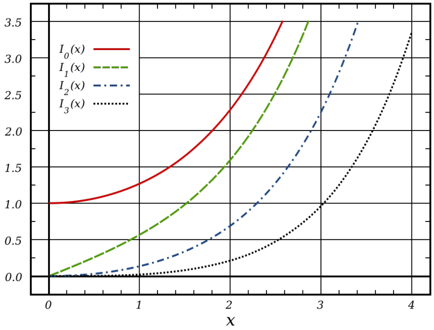
and $$\frac{1}{s}\frac{d}{ds}\left(s\frac{dy}{ds}\right) - \left(k^2 + \frac{\nu^2}{s^2}\right) y = 0 \rightarrow \quad I_\nu(ks), K_\nu(ks)$$

see Abramowitz + Stegun "Handbook of Mathematical functions"

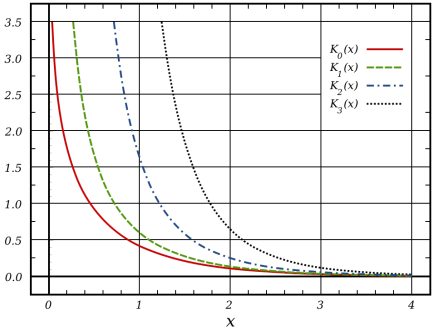
These are **Modified Bessel Functions**.

Potential due to thin disk

Modified Bessel Functions



I_ν, Modified Bessel function of the first kind



K_ν, Modified Bessel function of the second kind

Potential due to thin disk

Bessel Functions

And we can take this even further. By analogy with Fourier transforms where $\sin, \cos \rightarrow$ form the basis, we have $J, Y \rightarrow$ Hankel transforms.
Given a function $g(r)$, then the Hankel transform of g is

$$\tilde{g}(k) = \int_0^\infty g(r) J_\nu(kr) r dr$$

and the inverse transform is:

$$g(r) = \int_0^\infty \tilde{g}(k) J_\nu(kr) k dk$$

look these up in books of Hankel transforms!

Friedrich Bessel

Potential due to thin disk

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0 \quad 4.4$$

$$\frac{1}{R}\frac{d}{dR}\left(R\frac{dJ}{dR}\right) + k^2 J(R) = 0 \quad 4.5$$

Returning to the axisymmetric plane distribution, we have
$(4.4) \Rightarrow Z(z) = \exp(-k|z|)$
$(4.5) \Rightarrow J(R) = J_0(kr)$
choose J to get Φ finite at $R = 0$

Let $k > 0$ then

$$\Rightarrow \quad \Phi_k(R, Z) = \begin{cases} Ce^{-kz} J_0(kR) & z > 0 \\ Ce^{kz} J_0(kR) & z < 0 \end{cases}$$

This is true $\forall\ k > 0$, but a specific k for each Φ_k.

General potential $\rightarrow \sum_k \Phi_k$

Potential due to thin disk

$$\Rightarrow \Phi(R,z) = \int_0^\infty f(k) e^{-k|z|} J_0(kR) dk \tag{4.8}$$

Here $f(k)$ is a weighting function, corresponding to the C values in the sum. So what we need to do for a particular mass distribution is find $f(k)$.

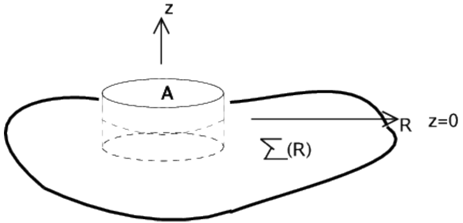
If we are going to relate it to a mass distribution, the next thing we should do is look at the $z=0$ plane, i.e. the region we have neglected so far since we have taken $\nabla^2\Phi = 0$ and so considered regions outside the plane.

Note that Φ_k is continuous across $z=0$ but $\nabla\Phi_k$ **is not**, due to $|z|$ dependence.That is where the mass is, so that is not a surprise.

$\Rightarrow \nabla^2\Phi_k = 0$ except at $z=0$ and $\Phi_k \to 0$ as $z, R \to \infty \Rightarrow$ satisfies conditions for potential from an isolated mass distribution. Still need to link with ρ (or $\Sigma(R)$) in the plane.

Potential due to thin disk

Poisson's equation



Use Gauss' Theorem ($\equiv$ Poisson's equation plus divergence theorem) to determine Σ in the $z = 0$ plane.
Over the cylinder

$$\iiint 4\pi G\rho dV = \iiint \nabla^2\Phi dV = \iiint \nabla \cdot (\nabla\Phi)\, dV$$

$$= \iint \nabla\Phi \cdot \hat{\mathbf{n}} d^2\mathbf{S}$$

Potential due to thin disk

Poisson's equation

$$\iiint 4\pi G\rho dV = \iint \nabla\Phi \cdot \hat{\mathbf{n}} d^2\mathbf{S}$$

Consider the limit in which the cylinder height $\to 0$. Then if A is the area of an end of the cylinder

$$LHS = 4\pi G\Sigma A$$

$$RHS = \left(\left[\frac{\partial\Phi}{\partial z}\right]_{z=0+} - \left[\frac{\partial\Phi}{\partial z}\right]_{z=0-} \right) \times A$$

$$\text{Equating these} \Rightarrow \left[\frac{\partial\Phi}{\partial z}\right]_{0-}^{0+} = 4\pi G\Sigma(R)$$

Potential due to thin disk

Poisson's equation

$$
\begin{aligned}
RHS &= -\int_0^\infty kf(k)e^{-k0+}J_0(kR)dk - \int_0^\infty kf(k)e^{-k0-}J_0(kR)dk \\
&= -\int_0^\infty kf(k)J_0(kR)dk - \int_0^\infty kf(k)J_0(kR)dk \\
&= -2\int_0^\infty kf(k)J_0(kR)dk \\
\Rightarrow \quad \Sigma(R) &= -\frac{1}{2\pi G}\int_0^\infty f(k)J_0(kR)kdk
\end{aligned}
$$

Potential due to thin disk

Poisson's equation

Hence determine $f(k)$ [and hence Φ] from inverse Hankel transform

$$f(k) = -2\pi G \int_0^\infty \Sigma(R) J_0(kR) R dR$$

Thus the process for determining Φ from ρ in this case is $\Sigma \rightarrow f \rightarrow \Phi$.

Note: For determining the circular velocity need $\frac{\partial \Phi}{\partial R}$, whch becomes $\frac{dJ_0(x)}{dx}$, and for Bessel function J_0 have $\frac{d}{dx} J_0(x) = -J_1(x)$ [Example].

Potential due to thin disk

Summary of derivation of Φ for thin axisymmetric disk

This has been a bit longwinded, but the steps are clear. They are:

1. $\nabla^2\Phi = 0$ outside disk. Solve by separation of variables.
2. Solutions of form $\Phi_k(R, z) = Ce^{-k|z|}J_0(kR)\ \forall\ k > 0$
3. $\Phi_k \to 0$ as $R, z \to \infty$ **and** satisfies $\nabla^2\Phi = 0$
   $\Rightarrow$ is potential of an isolated density distribution
4. General Φ can be written as
   $$\Phi(R, z) = \int_0^\infty \Phi_k(R, z) f(k) dk$$
   where $f(k)$ is an appropriate weight function.
5. Use Gauss' theorem to determine
   $$\Sigma(R) = -\frac{1}{2\pi G}\int_0^\infty f(k) J_0(kR) k dk$$
6. Hence
   $$f(k) = -2\pi G \int_0^\infty \Sigma(R) J_0(kR) R dR$$

So given Σ, use item (6) to determine $f(k)$, and then (5) to obtain Φ.

Potential due to thin disk

Circular velocity in the thin disk

$$\Phi(R,z) = \int_0^\infty f(k) e^{-k|z|} J_0(kR) dk \quad 4.8$$

The circular velocity in the plane of a plane distribution of matter is given by

$$\frac{v_C^2(R)}{R} = \left.\frac{\partial \Phi}{\partial R}\right|_{z=0}$$

and we have

$$x = kR \qquad \frac{d}{dR} J_0(kR) = k \frac{d}{dx} J_0 x = -k J_1(kR)$$

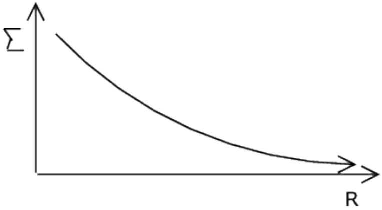
$$\frac{d}{dx}[x^m J_m(x)] = x^m J_{m-1}(x)$$

Then since we have equation (4.8), then

$$\frac{v_C^2(R)}{R} = -\int_0^\infty f(k) J_1(kR) k dk$$

Thin Disk: Examples

Mestel disk



A Mestel disk has the surface density distribution $\Sigma(R) = \frac{\Sigma_0 R_0}{R}$

Thus

$$M(<R) = \int_0^R 2\pi\Sigma(R')R'dR' = 2\pi R_0\Sigma_0 \int_0^R dR'$$

$$= 2\pi\Sigma_0 R_0 R$$

$$\rightarrow \infty \text{ as } R \rightarrow \infty$$

Thin Disk: Examples

Mestel disk

$$f(k) = -2\pi G \int_0^\infty \Sigma(R) J_0(kR) R dR$$

$$\begin{aligned} f(k) &= -2\pi G \Sigma_0 R_0 \int_0^\infty J_0(kR) dR \\ &= -\frac{2\pi G \Sigma_0 R_0}{k} \end{aligned}$$

From Gradshteyn and Ryzkik 6.511.1 $\int_0^\infty J_\nu(bx)dx = \frac{1}{b}$ $\quad {\scriptstyle Re(\nu)>-1,\ b>0}$

$$\Rightarrow \Phi(R,z) = -2\pi G \Sigma_0 R_0 \int_0^\infty e^{-k|z|} \frac{J_0(kR)}{k} dk$$

Thin Disk: Examples

Mestel disk

$$\frac{v_C^2(R)}{R} = -\int_0^\infty f(k)J_1(kR)kdk$$

$$f(k) = -\frac{2\pi G\Sigma_0 R_0}{k}$$

$$\text{and} \quad \frac{v_c^2(R)}{R} = 2\pi G\Sigma_0 R_0 \int_0^\infty J_1(kR)dk$$

$$\Rightarrow v_c^2(R) = 2\pi G\Sigma_0 R_0 = \text{const}$$

$$M(< R) = 2\pi\Sigma_0 R_0 R$$

Note that

$$v_c^2(R) = \frac{GM(R)}{R}$$

exactly in this case even though distribution is a disk, not spherical.
More generally, find $v_c^2 \equiv \frac{GM(R)}{R}$ to within 10% [reasonable accuracy] for most smooth Σ distributions.
Conclude that measurement of $v_c(R)$ is a good measure of mass **inside** R.

Thin Disk: Examples

Exponential Disk

Here

$$\Sigma(R) = \Sigma_0 \;\exp\left[-R/Rd\right]$$

This has finite mass

$$\begin{aligned} M &= \int_0^{\infty} 2\pi\Sigma_0 \;\exp\left[-R/Rd\right] RdR \\ &= 2\pi\Sigma_0 R_d^2 \underbrace{\int_0^{\infty} e^{-x} x dx}_{=1} \\ &= 2\pi\Sigma_0 R_d^2 \end{aligned}$$

Then

$$f(k) = -2\pi G\Sigma_0 \int_0^{\infty} e^{-R/Rd} J_0(kR) RdR$$

Gradshteyn + Ryzhik $\int_0^{\infty} e^{-\alpha x} J_0\left(\beta x\right) x dx = \frac{\alpha}{\left[\beta^2+\alpha^2\right]^{3/2}}$

Thin Disk: Examples

Exponential Disk

[Actually they have something (6.623.2) which requires a little work:

$$\int_0^\infty e^{-\alpha x} J_\nu(\beta x) x^{\nu+1} dx = \frac{2\alpha(2\beta)^\nu \Gamma(\nu + \frac{3}{2})}{\sqrt{\pi}\,(\alpha^2 + \beta^2)^{\nu+\frac{3}{2}}}$$

and you need to put in $\nu = 0$, $\Gamma(3/2) = \sqrt{\pi}/2$.
Then put $\alpha = 1/R_d$ and $\beta = k$.]

$$\int_0^\infty e^{-\alpha x} J_0(\beta x)\, x dx = \frac{\alpha}{[\beta^2+\alpha^2]^{3/2}}$$

OK, so

$$f(k) = -\frac{2\pi G \Sigma_0 R_d^2}{[1 + (kR_d)^2]^{3/2}}$$

Hence

$$\Phi(R, z) = -2\pi G \Sigma_0 R_d^2 \int_0^\infty \frac{J_0(kR) e^{-k|z|}}{\left(1 + (kR_d)^2\right)^{3/2}} dk$$

Thin Disk: Examples

Exponential Disk

Use

$$\int_0^\infty \frac{J\nu(xy)dx}{(x^2+a^2)^{1/2}} = I_{\nu/2}\left(\frac{1}{2}ay\right)K_{\nu/2}\left(\frac{1}{2}ay\right)$$

You can do this with help from Gradshteyn + Ryzhik again, using (6.552.1)

$\frac{d}{da}$ of this gives

$$-a\int_0^\infty \frac{J_\nu(xy)\ dx}{(x^2+a^2)^{3/2}} = \frac{y}{2}I_{\nu/2}(\frac{1}{2}ay)K'_{\nu/2}(\frac{1}{2}ay) + \frac{y}{2}I'_{\nu/2}(\frac{1}{2}ay)K_{\nu/2}(\frac{1}{2}ay)$$

so for $\nu = 0$

$$-a\int_0^\infty \frac{J_0(xy)\ dx}{(x^2+a^2)^{3/2}} = -\frac{y}{2}I_0(\frac{1}{2}ay)K_1(\frac{1}{2}ay) + \frac{y}{2}I_1(\frac{1}{2}ay)K_0(\frac{1}{2}ay)$$

Thin Disk: Examples

Exponential Disk

or

$$\int_0^\infty \frac{J_\nu(xy)\ dx}{(x^2+a^2)^{3/2}} = \frac{y}{2a}\left[I_0(\frac{1}{2}ay)K_1(\frac{1}{2}ay) - I_1(\frac{1}{2}ay)K_0(\frac{1}{2}ay)\right]$$

Then with $x = k$, $y = R$ and $a = 1/R_d$ this becomes

$$\int_0^\infty \frac{J_\nu(kR)\ dk}{(1+(kRd)^2)^{3/2}} = \frac{R}{2R_d^2}\left[I_0(\frac{R}{2R_d})K_1(\frac{R}{2R_d}) - I_1(\frac{R}{2R_d})K_0(\frac{R}{2R_d})\right]$$

or

$$\Phi(R,0) = -\pi G\Sigma_0 R\left[I_0(\frac{R}{2R_d})K_1(\frac{R}{2R_d}) - I_1(\frac{R}{2R_d})K_0(\frac{R}{2R_d})\right]$$

Also you find for the circular velocity (with $y = R/2R_d$)

$$v_C^2 = R\frac{\partial\Phi}{\partial R} = 4\pi\Sigma_0 R_d y^2[I_0K_0 - I_1K_1]$$

which is helpfully left as an example...

Thin Disk: Examples

Exponential disk

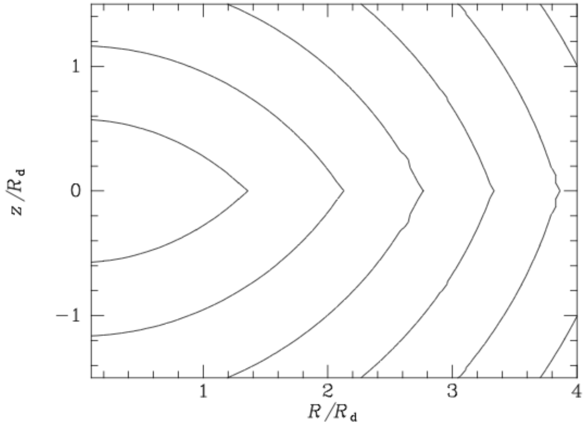


Figure 2.16 Contours in the (R, z) plane of constant potential Φ for a razor-thin exponential disk. The contour levels are GM_d/R_d divided by 1.5, 2, 2.5, . . ., where $M_d = 2\pi\Sigma_0 R_d^2$ is the mass of the disk.

Thin Disk: Examples

Exponential disk

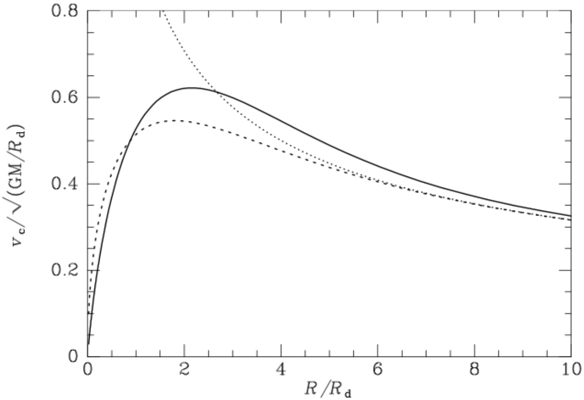


Figure 2.17 The circular-speed curves of: an exponential disk (full curve); a point with the same total mass (dotted curve); the spherical body for which $M(r)$ is given by equation (2.166) (dashed curve).

Thin Disk: Examples

Kuzmin disk

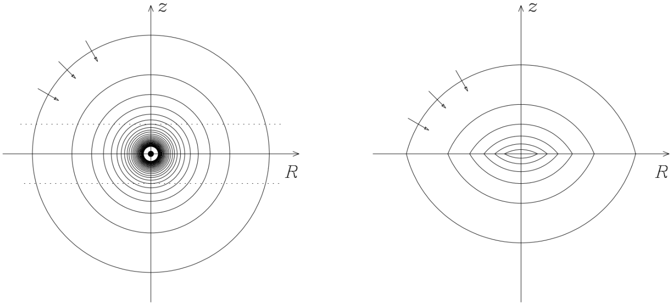
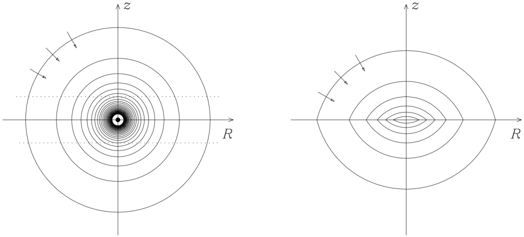


Figure 6.1: Surgery on the potential of a point mass produces the potential of a Kuzmin disk. Left: potential of a point mass. Contours show equal steps of $\Phi \propto 1/r$, while the arrows in the upper left quadrant show the radial force field. Dotted lines show $z = \pm a$. Right: potential of a Kuzmin disk, produced by excising the region $|z| \leq a$ from the field shown on the left. Arrows again indicate the force field; note that these no longer converge on the origin.

Image credit: Joshua Barnes

Thin Disk: Examples

Kuzmin disk

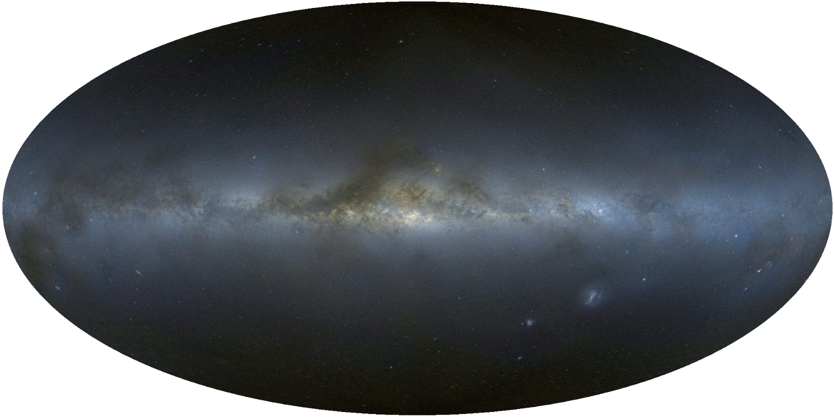


$$\Phi = \frac{-GM}{\sqrt{R^2 + (a + |z|)^2}}$$

Most (over)used disk model in astronomy is that credited to Miyamoto-Nagai (>1000 citations), which is a slight modification of Kuzmin disk.

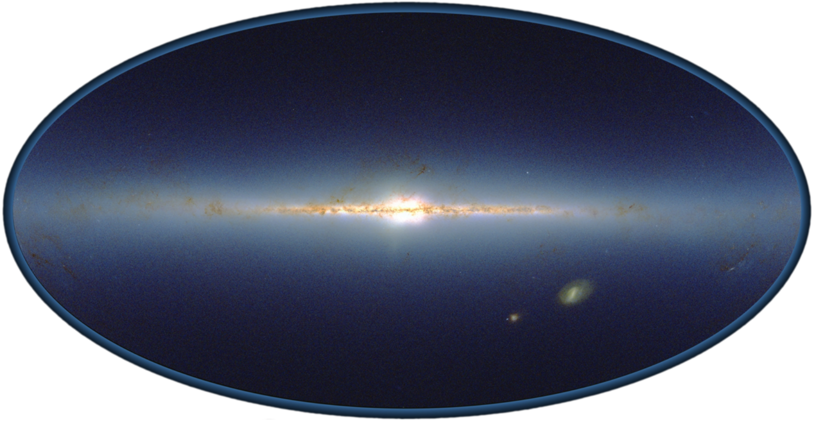
The Oort Constants: the rotation of our Galaxy

The Galaxy in the optical



The Oort Constants: the rotation of our Galaxy

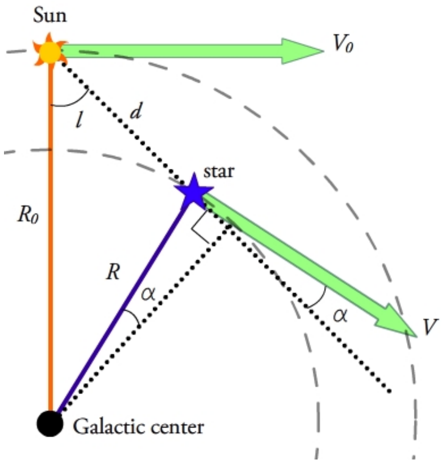
The Galaxy in the infrared



The Galactic coordinate system

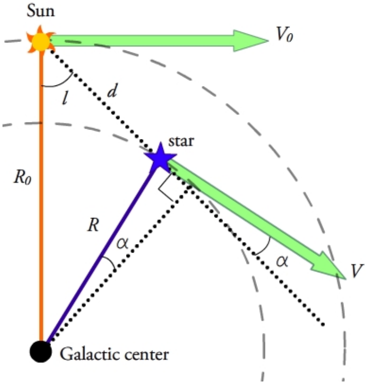
The Oort Constants: the rotation of our Galaxy

Suppose all stars are in circular orbits, with velocities $v(R) = R\Omega(R)$
Observe a star at distance d, galactic longitude ℓ.



The Oort Constants: the rotation of our Galaxy

Line of sight velocity



The radial velocity, as seen from the Earth, is

$$v_R = v \cos\alpha - v_0 \sin\ell$$

But

$$\frac{\sin\ell}{R} = \frac{\sin\left(\frac{\pi}{2}+\alpha\right)}{R_0} = \frac{\cos\alpha}{R_0}$$

and so

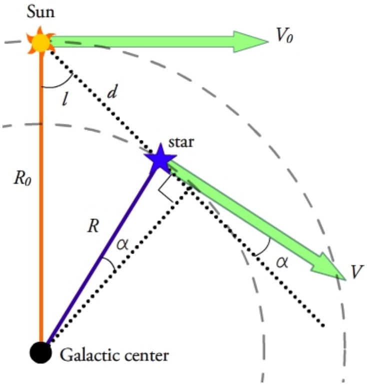
$$v_R = \left(\frac{vR_0}{R} - v_0\right)\sin\ell = \left(\frac{v}{R} - \frac{v_0}{R_0}\right) R_0 \sin\ell$$

Now if we look at only nearby stars ($d << R_0$):

$$R_0 - R \approx d\cos\ell$$

The Oort Constants: the rotation of our Galaxy

Line of sight velocity



$$v_R = \left(\frac{v}{R} - \frac{v_0}{R_0}\right) R_0 \sin\ell$$

$$R_0 - R \approx d\cos\ell$$

If we expand $\frac{v}{R}$ around $R = R_0$:

$$\frac{v}{R} = \frac{v_0}{R_0} + (R - R_0)\frac{d}{dR}\left(\frac{v}{R}\right)\bigg|_{R_0}$$

So the radial velocity

$$v_R \approx -R_0\frac{d}{dR}\left(\frac{v}{R}\right)\bigg|_{R_0} d\sin\ell\cos\ell$$

Since $\sin 2\ell = 2\sin\ell\cos\ell$, we have

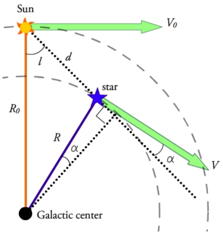
$$v_R = Ad\sin 2\ell$$

$$A = -\frac{R_0}{2}\frac{d}{dR}\left(\frac{v}{R}\right)\bigg|_{R_0} = \frac{1}{2}\left[\frac{v_0}{R_0} - \frac{dv}{dR}\bigg|_{R_0}\right]$$

the Oort constant A

The Oort Constants: the rotation of our Galaxy

Line of sight velocity



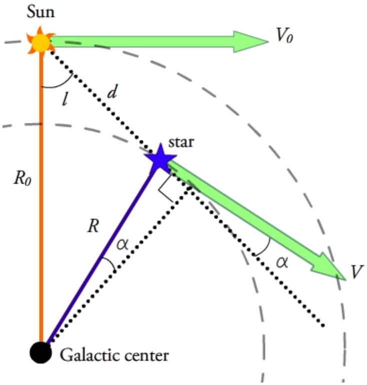
$$v_R = Ad\sin 2\ell$$

$$A = -\frac{R_0}{2}\frac{d}{dR}\left(\frac{v}{R}\right)\bigg|_{R_0} = \frac{1}{2}\left[\frac{v_0}{R_0} - \frac{dv}{dR}\bigg|_{R_0}\right]$$

Then if we measure $\frac{v_R}{d}$ as a function of ℓ, then (in principle) we would know A.

The Oort Constants: the rotation of our Galaxy

Tangential velocity



Similarly we can measure the tangential velocity (from proper motions)

$$v_T = v \sin\alpha - v_0 \cos\ell$$

Note that $R \sin\alpha = R_0 \cos\ell - d$, and so

$$v_T = \frac{v}{R}\left[R_0 \cos\ell - d\right] - v_0 \cos\ell$$

i.e.

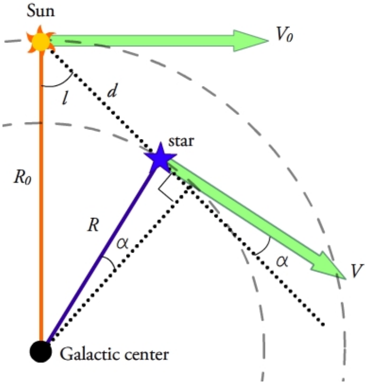
$$v_T = \left(\frac{v}{R} - \frac{v_0}{R_0}\right) R_0 \cos\ell - \frac{v}{R} d$$

For small d we have $R_0 - R \approx d \cos\ell$ and

$$\frac{v}{R} = \frac{v_0}{R_0} + (R - R_0)\frac{d}{dR}\left(\frac{v}{R}\right)\bigg|_{R_0} \qquad (*)$$

The Oort Constants: the rotation of our Galaxy

Tangential velocity



$$v_T = \left(\frac{v}{R} - \frac{v_0}{R_0}\right) R_0 \cos\ell - \frac{v}{R} d$$

$$\frac{v}{R} - \frac{v_0}{R_0} = (R - R_0)\frac{d}{dR}\left(\frac{v}{R}\right)\Big|_{R_0} \quad (*)$$

$$R_0 - R \approx d\cos\ell$$

$$v_T \approx -R_0 \frac{d}{dR}\left(\frac{v}{R}\right)\Big|_{R_0} d\cos^2\ell - \frac{v}{R} d$$

The Oort Constants: the rotation of our Galaxy

Tangential velocity

$$\frac{v}{R} = \frac{v_0}{R_0} + (R - R_0)\frac{d}{dR}\left.\left(\frac{v}{R}\right)\right|_{R_0} \quad (*)$$

We can remove the $\frac{v}{R}$ term by using the expression (*), and $\cos^2\ell = \frac{1}{2}(1 + \cos 2\ell)$ to obtain

$$\begin{aligned} v_T &\approx -R_0\frac{d}{dR}\left.\left(\frac{v}{R}\right)\right|_{R_0} d\cos^2\ell - \frac{v}{R}d \\ &\approx \frac{1}{2}\left(\frac{v_0}{R_0} - \left.\frac{dv}{dR}\right|_{R_0}\right) d\cos 2\ell - \frac{1}{2}\left(\frac{v_0}{R_0} + \left.\frac{dv}{dR}\right|_{R_0}\right) d + O\left(\frac{d^2}{R_0^2}\right) \end{aligned}$$

Now define

$$v_T = d(A\cos 2\ell + B)$$

i.e.

$$B = -\frac{1}{2}\left[\frac{v_0}{R_0} + \left.\frac{dv}{dR}\right|_{R_0}\right] \qquad A = \frac{1}{2}\left[\frac{v_0}{R_0} - \left.\frac{dv}{dR}\right|_{R_0}\right]$$

The Oort Constants: the rotation of our Galaxy

$$A = \frac{1}{2}\left[\frac{v_0}{R_0} - \left.\frac{dv}{dR}\right|_{R_0}\right] = -\frac{R_0}{2}\frac{d}{dR}\left(\frac{v}{R}\right)\Big|_{R_0} = -\frac{1}{2}R_0\left.\frac{d\Omega}{dR}\right|_{R_0}$$

$$B = -\frac{1}{2}\left[\frac{v_0}{R_0} + \left.\frac{dv}{dR}\right|_{R_0}\right] = -\left(\Omega + \frac{1}{2}R_0\frac{d\Omega}{dR}\right)\Big|_{R_0}$$

- A measures the shear in the disk i.e. deviation from solid body rotation. For solid body rotation $\frac{d\Omega}{dR} = 0$, so $A = 0$.
- B measures vorticity i.e. tendency for material in the disk to circulate due to differential rotation.

Then note that

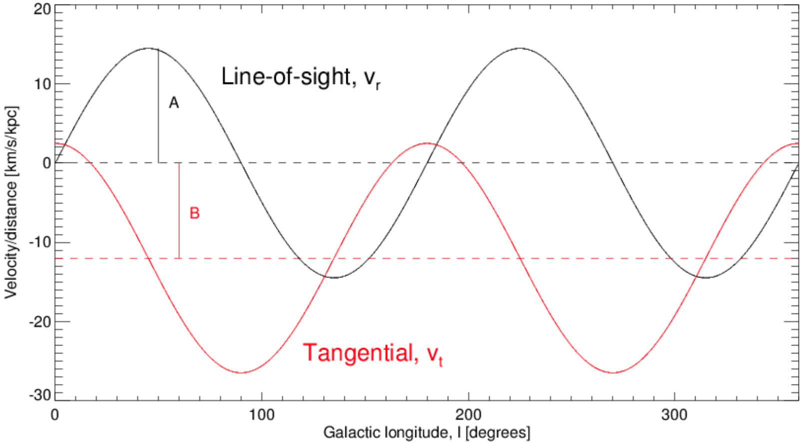
$$\Omega_0 = v_0/R_0 = A - B$$

and

$$\left.\frac{dv}{dR}\right|_{R_0} = -(A + B)$$

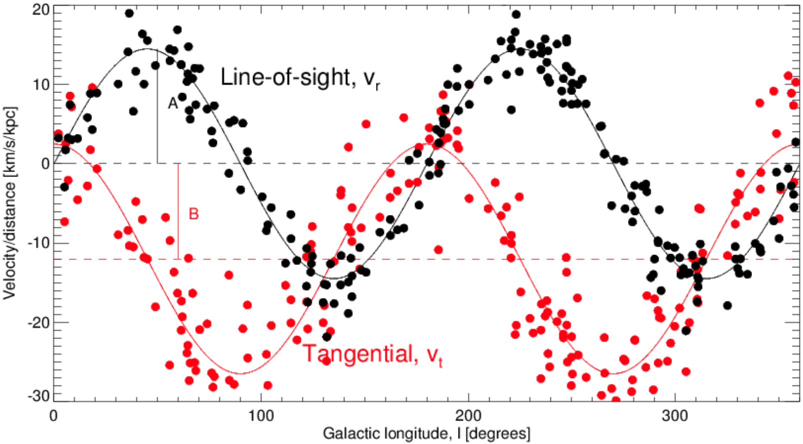
The Oort Constants: the rotation of our Galaxy

Measurement



The Oort Constants: the rotation of our Galaxy

Measurement



Not easy: distances, peculiar velocities, proper motions

The Oort Constants: the rotation of our Galaxy

Best estimates are $A = 14.5$ km s^{-1} kpc^{-1}, $B = -12$ km s^{-1} kpc^{-1}.

$\Rightarrow \Omega_{R_0} = 26$ km s^{-1} kpc^{-1}, so with $R_0 = 8.5$ kpc $v_0 = 220$ km s^{-1}

$\Rightarrow$ Rotation period$= 2.4 \times 10^8$ years.

Epicyclic frequency $K_0 = \sqrt{-4B(A-B)} = 36 \pm 10$ km s^{-1} kpc^{-1}

$K_0/\Omega_0 = 1.3 \pm 0.2$.

$$K^2 = R\frac{d\Omega^2}{dR} + 4\Omega^2$$

See examples sheet 3

The Sun makes 1.3 oscillations in the radial direction in the time it takes to complete the orbit around the Galactic centre

The Oort Constants: the rotation of our Galaxy

$$\nu^2 = \left(\frac{\partial^2 \Phi}{\partial z^2}\right)$$

$$\Omega^2 = \frac{1}{R}\left(\frac{\partial \Phi}{\partial R}\right)$$

Poisson's equation in cylindrical coordinates:

$$\begin{aligned} 4\pi G\rho &= \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \Phi}{\partial R}\right) + \frac{\partial^2 \Phi}{\partial z^2} \\ &\approx \frac{1}{R}\frac{dv_c^2}{dR} + \nu^2 \end{aligned}$$

If we could measure $\mathcal{V}$, the vertical frequency, we could estimate the mass density in the Galactic plane, but we can't.
But, if we assume that the matter distribution was spherical then mass internal to the Sun:

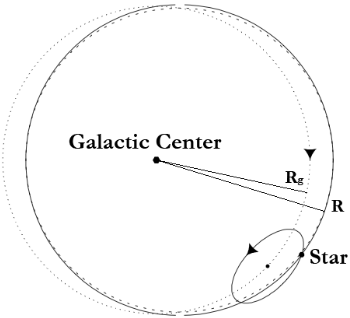
$$\frac{v_0^2}{R_0} \approx \frac{GM}{R_0^2}$$

$$\Rightarrow M \approx \frac{R_0 v_0^2}{G} \approx 10^{11}\ \mathrm{M}_\odot$$

The Oort Constants: the rotation of our Galaxy

Jan Oort, 1930s

The Oort Constants and epicycles



A star on an orbit with the guiding center radius R_g:

$$x \equiv R - R_g$$

The angular speed of the circular orbit $\Omega_g = L_z/R_g^2$

$$\begin{aligned}\dot{\phi} &= \frac{p_\phi}{R^2} = \frac{L_z}{R_g^2}\left(1 + \frac{x}{R_g}\right)^{-2} \\ &\simeq \Omega_g \left(1 - \frac{2x}{R_g}\right)\end{aligned}$$

The Oort Constants and epicycles

$\dot{\phi} \simeq \Omega_g \left(1 - \frac{2x}{R_g}\right)$

If we could measure $v_\phi(R_0) - v_c(R_0)$ and v_R for a group of stars, each with its own guiding center, is there a way to obtain the relevant frequencies?

$$\begin{aligned} v_\phi(R_0) - v_c(R_0) &= R_0(\dot{\phi} - \Omega_0) = R_0(\dot{\phi} - \Omega_g + \Omega_g - \Omega_0) \\ &\simeq R_0 \left[(\dot{\phi} - \Omega_g) - \left(\frac{d\Omega}{dR} \right)_{R_g} x \right] \end{aligned}$$

$$v_\phi(R_0) - v_c(R_0) \simeq -R_0 x \left(\frac{2\Omega}{R} + \frac{d\Omega}{dR} \right)_{R_g}$$

Evaluate at R_0 and dropping terms of order of x^2

$$v_\phi(R_0) - v_c(R_0) \simeq -x \left(2\Omega + R \frac{d\Omega}{dR} \right)_{R_0}$$

The Oort Constants and epicycles

The Oort's constants definition:

$$A(R) \equiv \frac{1}{2}\left(\frac{v_c}{R} - \frac{dv_c}{dR}\right) = -\frac{1}{2}R\frac{d\Omega}{dR}$$

$$B(R) \equiv -\frac{1}{2}\left(\frac{v_c}{R} + \frac{dv_c}{dR}\right) = -\left(\Omega + \frac{1}{2}R\frac{d\Omega}{dR}\right)$$

Therefore,

$$v_\phi(R_0) - v_c(R_0) \simeq -x\left(2\Omega + R\frac{d\Omega}{dR}\right)_{R_0} = 2Bx$$

Epicyclic radial motion solution:

$$x(t) = X\cos(Kt + \alpha)$$

Hence, averaging over the phases of stars near the Sun

$$\overline{[v_\phi - v_c(R_0)]^2} = 2B^2X^2$$

The Oort Constants and epicycles

The Oort's constants definition:

$$A(R) \equiv \frac{1}{2}\left(\frac{v_c}{R} - \frac{dv_c}{dR}\right) = -\frac{1}{2}R\frac{d\Omega}{dR}$$

$$B(R) \equiv -\frac{1}{2}\left(\frac{v_c}{R} + \frac{dv_c}{dR}\right) = -\left(\Omega + \frac{1}{2}R\frac{d\Omega}{dR}\right)$$

Therefore,

$$v_\phi(R_0) - v_c(R_0) \simeq -x\left(2\Omega + R\frac{d\Omega}{dR}\right)_{R_0} = 2Bx$$

Epicyclic radial motion solution:

$$x(t) = X\cos(Kt + \alpha)$$

Hence, averaging over the phases of stars near the Sun

$$\overline{[v_\phi - v_c(R_0)]^2} = 2B^2X^2$$

The Oort Constants and epicycles

$$\overline{[v_\phi - v_c(R_0)]^2} = 2B^2X^2$$

Similarly, for the radial velocity

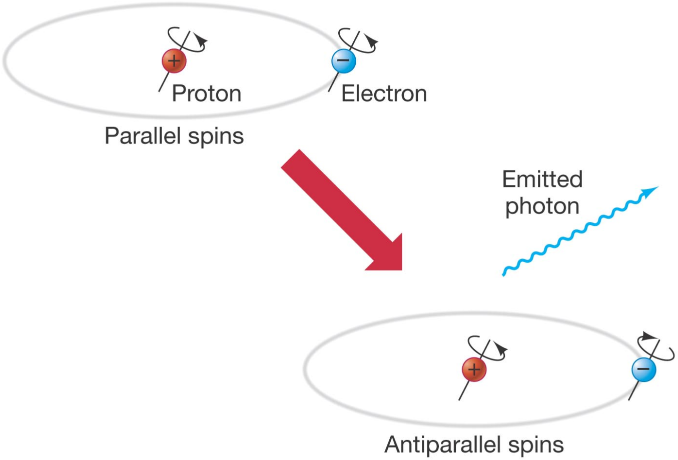
$$\overline{v_R^2} = -2B(A - B)X^2$$

Taking ratio:

$$\frac{\overline{[v_\phi - v_c(R_0)]^2}}{\overline{v_R^2}} \simeq \frac{-B}{A - B} = -\frac{B}{\Omega_0} = \frac{K_0^2}{4\Omega_0^2}$$

The Rotation Curve of our Galaxy

21cm line of neutral hydrogen



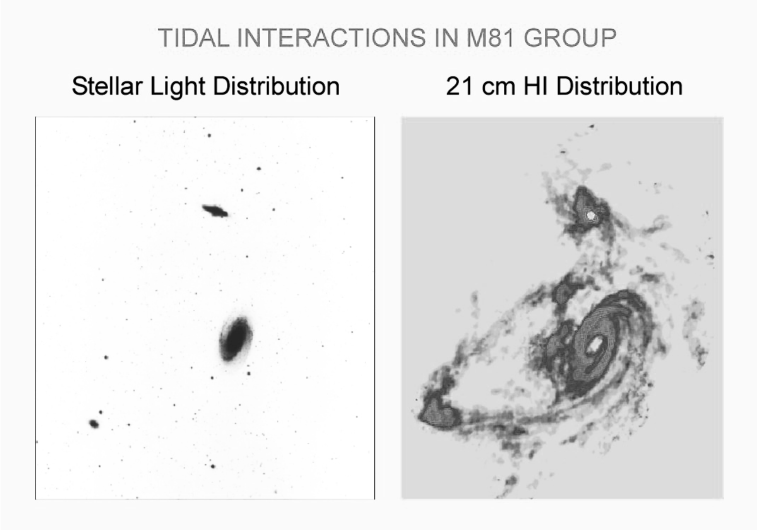
frequency 1420.4058 MHz, wavelength 21 cm, probability **1 in** 10^7 **years**

The Rotation Curve of our Galaxy

21cm line of neutral hydrogen

The Rotation Curve of our Galaxy

21cm line of neutral hydrogen



The Rotation Curve of our Galaxy

Rotation curve from 21cm line

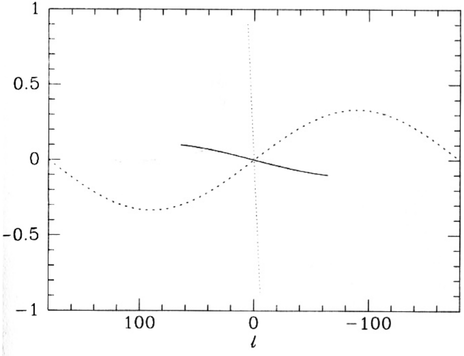


Figure 9.2 The traces in the (l, v) plot of three rings. The radii of the rings are $r = 0.9R_0$ (full curve), $r = 0.1R_0$ (dotted curve) and $r = 1.5R_0$ (dashed curve). The circular speed in the disk is unity at all R.

The Rotation Curve of our Galaxy

Rotation curve from 21cm line

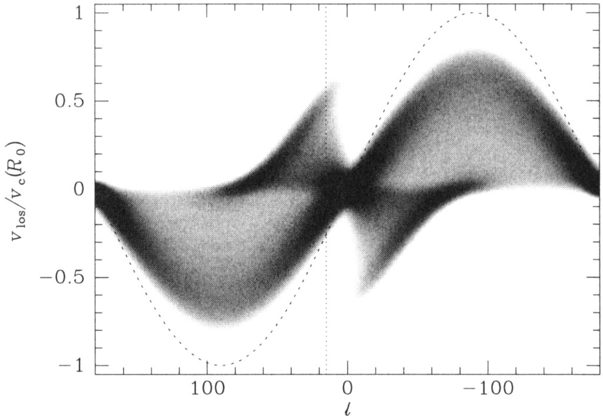


Figure 9.3 Grayscale of optical depth in the (l, v) plane. As for Figure 8.12, the number density of hydrogen atoms has been assumed to equal $2 \times 10^5\,m^{-3}$ at R_0 and to vary with R as $\exp(-R/2R_0)$. The circular speed is given by $v_c(R) = (R/R_0)^{0.1} \times$

The Rotation Curve of our Galaxy

Terminal velocity

Gas moves on circular orbits - use 21cm HI emission, and measure Doppler shift given ℓ. We have

$$v_R = \left(\frac{v}{R} - \frac{v_0}{R_0} \right) R_0 \sin \ell$$

If $\Omega(R) = \frac{v}{R}$ decreases monotonically with R, the we can look at the 21cm line profile and pick the maximum Doppler shift as the one corresponding to the smallest circle along the line of sight at Galactic longitude ℓ. This occurs when $R = R_0 \sin \ell$ Therefore

$$v_R(\mathrm{max}) = v(R) - v_0 \sin \ell$$

and hence

$$v(R) = v_R(\mathrm{max}) + v_0 \sin \ell$$

Note that this works only for $-90^{\mathrm{o}} < \ell < 90^{\mathrm{o}}$, since a line of sight for $90^{\mathrm{o}} < \ell < 270^{\mathrm{o}}$ is never a tangent to a circular orbit.

The Rotation Curve of our Galaxy

Non-circular motion

- Axisymmertic expansion
- Oval distortion (elliptical orbits)
- Spiral structure
- Random motions

The Rotation Curve of our Galaxy

Axisymmetric expansion

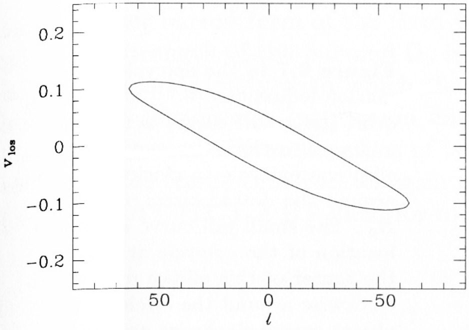
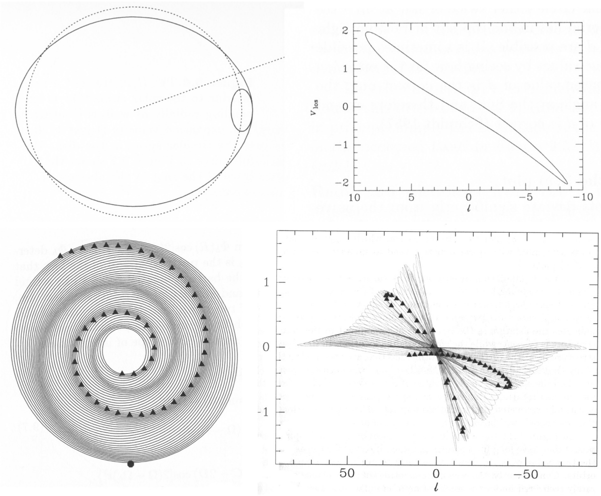


Figure 9.6 The (l, v) trace of a ring interior to the solar circle that is expanding radially as well as rotating. The expansion velocity is just 1/20 of the circular speed in the disk. the full curve in Figure 9.2 shows the trace of the ring in the absence of expansion.

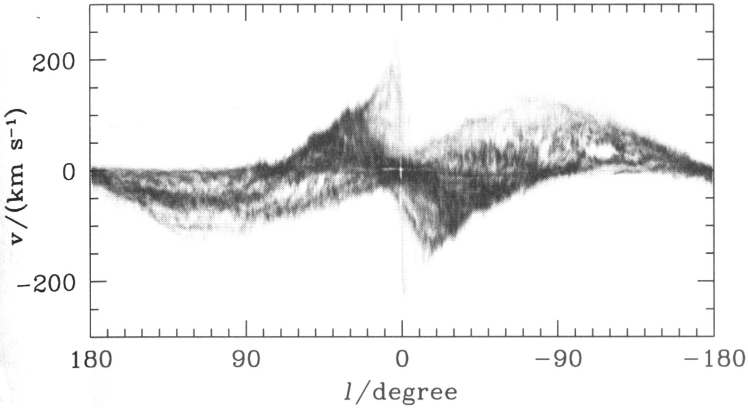
The Rotation Curve of our Galaxy

Oval distortion and spiral structure



The Rotation Curve of our Galaxy

Rotation curve from 21cm line



Spiral Galaxy Rotation Curves

- spirals contain disks of gas. In a symmetric potential settles down to rotate on circular orbits in centrifugal balance
- gas collides with itself and shocks, therefore paths of gas motion rarely intersect, unlike stars
- if the potential is spherically symmetric then the measure of the circular velocity $v_c(R)$ is a good measure of the mass $M(R)$ contained within radius R

$$\frac{v_c^2}{R} = \frac{d\Phi}{dR} \simeq \frac{GM(R)}{R^2}$$

or

$$M(< R) \simeq \frac{v_c^2 R}{G}$$

Spiral Galaxy Rotation Curves

NGC 253

Spiral Galaxy Rotation Curves

We can obtain $v_C(R)$ by observing the Doppler shift of a spectral line. For the gas in the spiral's disk, our options are:

- HI - **radio** - neutral gas, largest range in radius
- H_α - **optical** - warm gas, inner regions
- CO - **mm** - molecular gas, very inner regions

Problems:

- **Instrumental** *beam smearing*: each pointing has data from a range of radii, need to deconvolve
- **Intrinsic** *a) absorption*: especially if galaxy is close to edge on position
  b) finite thickness of gas layer

Spiral arms cause kinks in apparent rotation curve, because

- gas motion is not exactly circular
- arms provide non-axisymmetric potential
- arms contribute to optical extinction

Spiral Galaxy Rotation Curves

We can obtain $v_C(R)$ by observing the Doppler shift of a spectral line. For the gas in the spiral's disk, our options are:

- HI - **radio** - neutral gas, largest range in radius
- H_α - **optical** - warm gas, inner regions
- CO - **mm** - molecular gas, very inner regions

Problems:

- **Instrumental** *beam smearing*: each pointing has data from a range of radii, need to deconvolve
- **Intrinsic** *a) absorption*: especially if galaxy is close to edge on position
  b) finite thickness of gas layer

Spiral arms cause kinks in apparent rotation curve, because

- gas motion is not exactly circular
- arms provide non-axisymmetric potential
- arms contribute to optical extinction

Spiral Galaxy Rotation Curves

We can obtain $v_C(R)$ by observing the Doppler shift of a spectral line. For the gas in the spiral's disk, our options are:

- HI - **radio** - neutral gas, largest range in radius
- H_α - **optical** - warm gas, inner regions
- CO - **mm** - molecular gas, very inner regions

Problems:

- **Instrumental** *beam smearing*: each pointing has data from a range of radii, need to deconvolve
- **Intrinsic** *a) absorption*: especially if galaxy is close to edge on position
  b) finite thickness of gas layer

Spiral arms cause kinks in apparent rotation curve, because

- gas motion is not exactly circular
- arms provide non-axisymmetric potential
- arms contribute to optical extinction

Spiral Galaxy Rotation Curves

NGC 253 rotation curve

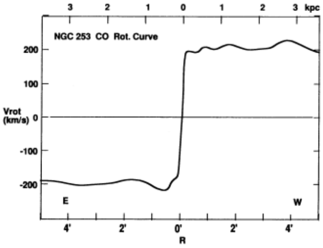


FIG. 2a-1

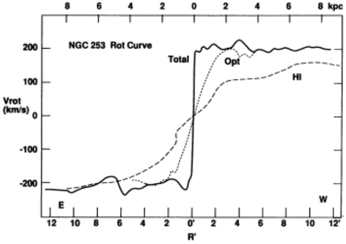


FIG. 2a-2

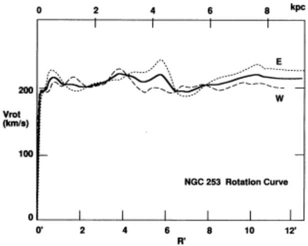


FIG. 2b

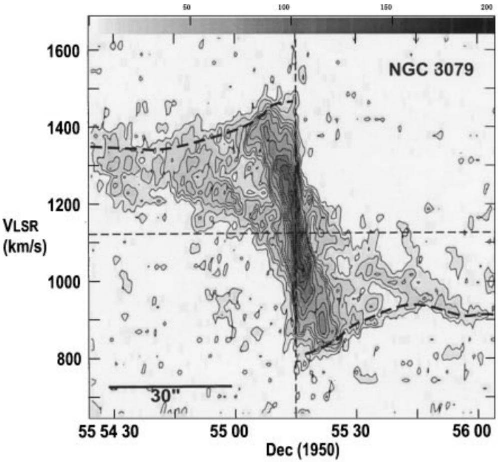
FIG. 2.—(*a*-1) Inner rotation curve of NGC 253 derived by using the CO PV diagram, and (*a*-2) CO + optical (Hα) (= total) rotation curve compared to H I rotation. (*b*) Total rotation curve of NGC 253 obtained by averaging and smoothing the eastern and western halves of the rotation velocities. See Table 1 for observational parameters and references.

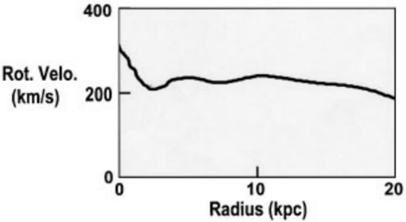
Spiral Galaxy Rotation Curves

NGC 3079

Spiral Galaxy Rotation Curves

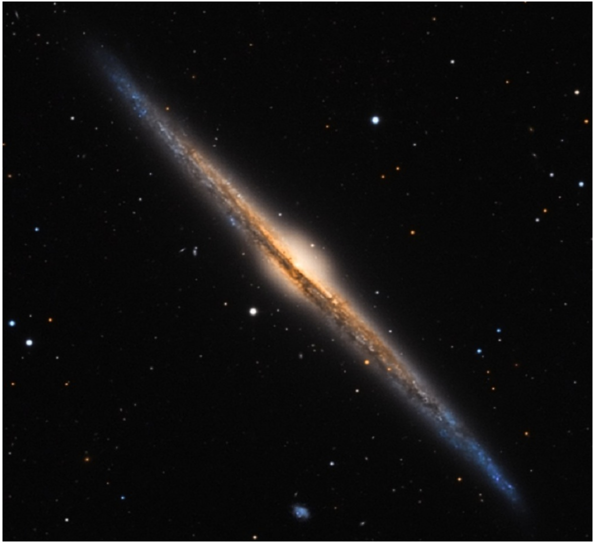
NGC 3079 rotation curve from PV diagram





Spiral Galaxy Rotation Curves

NGC 4565



Spiral Galaxy Rotation Curves

NGC 6946

Spiral Galaxy Rotation Curves

NGC 1808

Spiral Galaxy Rotation Curves

Rotation curves of nearby spirals

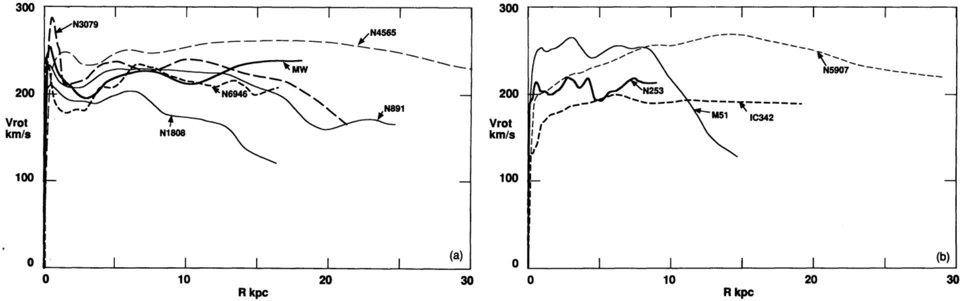


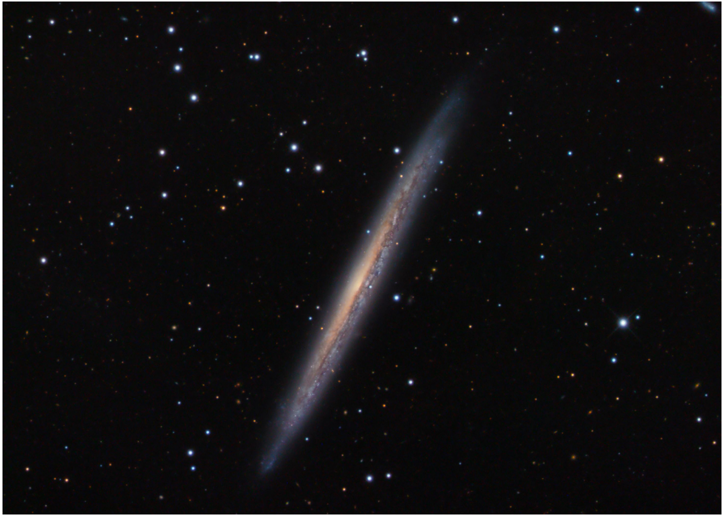
FIG. 12.—Rotation curves of galaxies studied in this paper plotted in the same linear and velocity scales. (*a*) Rotation curves having a central peak similar to that of our Galaxy. (*b*) Rotation curves without significant central peak.

Spiral Galaxy Rotation Curves

IC 342

Spiral Galaxy Rotation Curves

NGC 5907



Spiral Galaxy Rotation Curves

NGC 5907's secret

Spiral Galaxy Rotation Curves

Vera Rubin, 1980

"...This form for the rotation curves implies that the mass is not centrally condensed, but that significant mass is located at large R. The integral mass is increasing at least as fast as R. The mass is not converging to a limiting mass at the edge of the optical image. The conclusion is inescapable that non-luminous matter exists beyond the optical galaxy..."

Spiral Galaxy Rotation Curves

The similarity of the rotation curves

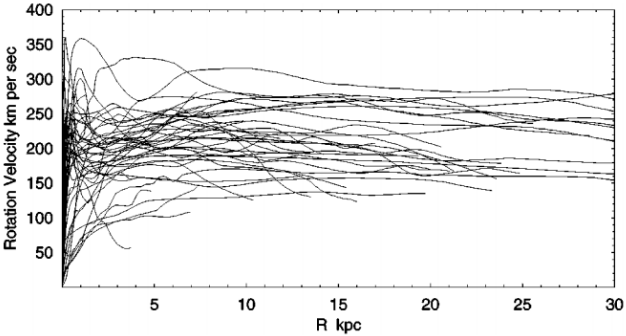


Figure 4 Rotation curves of spiral galaxies obtained by combining CO data for the central regions, optical for disks, and HI for outer disk and halo (Sofue et al. 1999a).

Spiral Galaxy Rotation Curves

The dynamical components

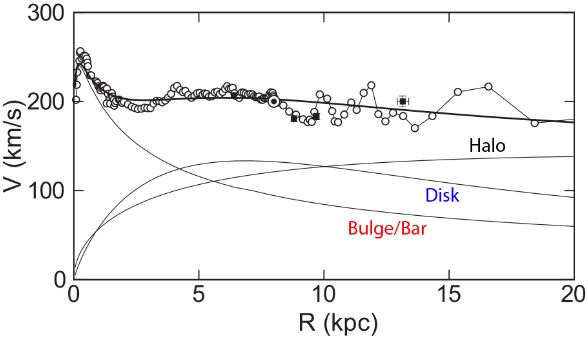


Fig. 4. Least-squares fit by the bulge, disk, and dark halo to the grand rotation curve. The thick line represents the fitted rotation curve, and the thin lines show individual contributions from bulge, disk, and halo. The observed velocities, $V(R_i)$, are shown by open circles, and the most recent accurate results from VERA (Honma et al. 2007; Oh et al. 2010) are shown by squares.

Spiral Galaxy Rotation Curves

Tully-Fisher relation