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PART II ASTROPHYSICS

Stellar Dynamics and Structure of Galaxies

Examples Sheet 1

1.1 In Cartesian coordinates (x, y) a particle obeys the equation of motion

$$\ddot{x} = f_x \; ; \; \ddot{y} = f_y \, ,$$

where $\mathbf{f} = (f_x, f_y)$ is the force per unit mass. Cylindrical polar coordinates (r, ϕ) are defined by

$$x = r\cos\phi \; ; \; y = r\sin\phi \, .$$

Write down the inverse relations $r = r(x, y)$; $\phi = \phi(r, y)$. Show that

$$f_r = f_x \cos\phi + f_y \sin\phi \, ;$$

and find the corresponding expression for f_ϕ. Hence show directly that

$$\ddot{r} - r\dot{\phi}^2 = f_r \; ; \; \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right) = f_\phi \, .$$

1.2 In cylindrical polar coordinates a conic section is given by

$$\frac{l}{r} = 1 + e\cos\phi \, .$$

By *suitable choice* of a Cartesian coordinate system show directly that for $0 \le e < 1$ the equation represents the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $b^2 = a^2(1 - e^2)$ and $l = a(1 - e^2)$.

* Similarly show that $e = 1$ represents a parabola, and $e > 1$ represents (one branch of) a hyperbola.

1.3 In a coordinate system which rotates with constant angular velocity $\mathbf{\Omega}$, the equation of motion of a particle is

$$\ddot{\mathbf{r}} + 2\mathbf{\Omega} \wedge \dot{\mathbf{r}} + \mathbf{\Omega} \wedge (\mathbf{\Omega} \wedge \mathbf{r}) = -\nabla\Phi \, .$$

What is the physical interpretation of the 'extra' terms on the lhs?

In considering the motion of a light particle moving under the influence of the two equal mass stars in a binary system with a circular orbit, we use a Cartesian coordinate system with origin at the centre of mass. Show that with suitable choice of $\mathbf{\Omega}$ the particle obeys an equation of the form

$$\ddot{\mathbf{r}} + 2\mathbf{\Omega} \wedge \dot{\mathbf{r}} = -\nabla\Phi_{tot} \ ,$$

where $\Phi_{tot} = \Phi + \Phi_\Omega$ is independent of time, and

$$\Phi_\Omega = \tfrac{1}{2}\Omega^2 \left(x^2 + y^2\right) \ .$$

Show that along the orbit the quantity

$$J(\mathbf{r}, \dot{\mathbf{r}}) = \tfrac{1}{2}\dot{\mathbf{r}}^2 + \Phi_{tot}(\mathbf{r})$$

is a constant.

* In the (x, y)-plane sketch the curves $\Phi_{tot} =$ const. Deduce that a particle released at rest on the line joining the stars but close to the centre of mass does not escape from the system.

1.4 [00305(ii)] A particle of unit mass is moving in a plane (r, ϕ) subject to the central force

$$f_r = -\frac{GM}{r^2} f(\phi).$$

Write down the equations of motion.

At time $t = 0$, the particle is at $r = a$, $\phi = 0$ and has velocity V, with $\dot{r} = 0$. Show that the orbit of the particle is given by

$$\frac{a}{r} = \cos\phi + \frac{GM}{aV^2} \int_0^\phi \sin(\phi - s)\, f(s)\, ds.$$

If $f(\phi) = \cos\phi$, derive the equation for the orbit $r(\phi)$.

if

$$V^2 = \frac{3\pi GM}{8a} \ ,$$

show that $r \to \infty$ as $\phi \to \pm 3\pi/4$.

[You may assume that

$$2\sin A \cos B = \sin(A + B) + \sin(A - B).]$$

1.5 Two particles, masses M_1 and M_2, approach at relative velocity V_0 and impact parameter b. Show that their paths are each deflected through an angle θ_0, where

$$\tan\left(\frac{\theta_0}{2}\right) = \frac{G\left(M_1 + M_2\right)}{bV_0^2}.$$

What is the distance d of closest approach?

1.6 The simplest, and most energy efficient, way of moving a spacecraft from one circular orbit, radius r_1 to another circular orbit $r_2(> r_1$, say) about a point mass is by use of a Hohman transfer orbit. This orbit is an ellipse with pericentre distance r_1 and apocentre distance r_2. Use of the Hohman transfer orbit requires just two firings of the spacecraft's motors.

(a) You are orbiting the Earth in the Space Shuttle 500 km above the surface. 300 km ahead of you on the same orbit is a malfunctioning satellite. Describe how would you bring your spacecraft alongside the satellite.

(b) To launch Pioneer 10 out of the solar system, it was launched in a Hohman transfer orbit to rendezvous with Jupiter. How long did it take to get there? Suppose that at aphelion it collides elastically with Jupiter in such a way that its velocity reverses direction. Show that it has enough speed now to escape from the solar system.

*(c) Make a more realistic assumption about the interaction with Jupiter and show that escape velocity can still be reached.

1.7 [95305]

(a) A particle of mass m is moving in the gravitational field of a particle of mass M ($\gg m$). Given that the relative velocity of the two particles has magnitude v and the relative distance is r, write down the condition that the two particles are gravitationally bound.

Calculate the orbital velocity of the Earth about the Sun, and the orbital velocity of the Moon about the Earth.

(b) A malevolent deity stops the motion of the Earth, relative to the Sun. Discuss briefly what might happen to the Moon.

If the presence of the Moon can be ignored, show that the Earth falls into the Sun after $1/(4\sqrt{2})$ years.

If the Moon collides with the Earth, give an expression for the ratio of the angular momentum of the Earth's new orbit, compared to the angular momentum of the Earth's original circular orbit. In this case, does the Earth still fall into the Sun?

1.8 [96305(ii)] Derive the circular velocity due to the Plummer potential

$$\Phi(r) = -\frac{GM}{\sqrt{b^2 + r^2}},$$

where b is a constant and sketch the result. Obtain the density profile $\rho(r)$ that gives rise to this potential. Show that M is the mass of the system.

[*For a spherically symmetrical function* F, $\nabla^2 F = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial F}{\partial r}\right)$.]

1.9 [97405(i)] Consider the gravitational potential

$$\Phi(R, z) = -\frac{GM}{\sqrt{R^2 + (z-a)^2}} - \frac{GM}{\sqrt{R^2 + (z+a)^2}},$$

in cylindrical polar coordinates (R, ϕ, z). Show that circular orbits are possible in the $z = 0$ plane, and calculate the corresponding angular velocity $\Omega_0(R)$. Sketch Ω_0 as a function of R.

By considering physically the source of the gravitational potential, comment briefly on the likely stability of the orbits in the regions (a) $R \gg a$ and (b) $R \ll a$?

1.10 Show that the gravitational potential $\Phi(r)$ for a thin spherical shell of mass Δm at radius r' is

$$\Phi = \begin{cases} -G\Delta m/r & r > r' \\ \\ -G\Delta m/r' & r \leq r'. \end{cases}$$

Deduce that the gravitational potential due to a spherically symmetric density distribution $\rho(r')$ is given by

$$\Phi(r) = -\int_0^r \frac{G\rho(r')4\pi}{r} r'^2 dr' - \int_r^\infty G\rho(r')4\pi r' dr' \,.$$

Hence find the potential for the density distribution

$$\rho(r) = \begin{cases} \rho_0 \left(1 - \frac{r^2}{r_0^2}\right) & r < r_0 \\ 0 & r > r_0, \end{cases}$$

where ρ_0 is a constant.