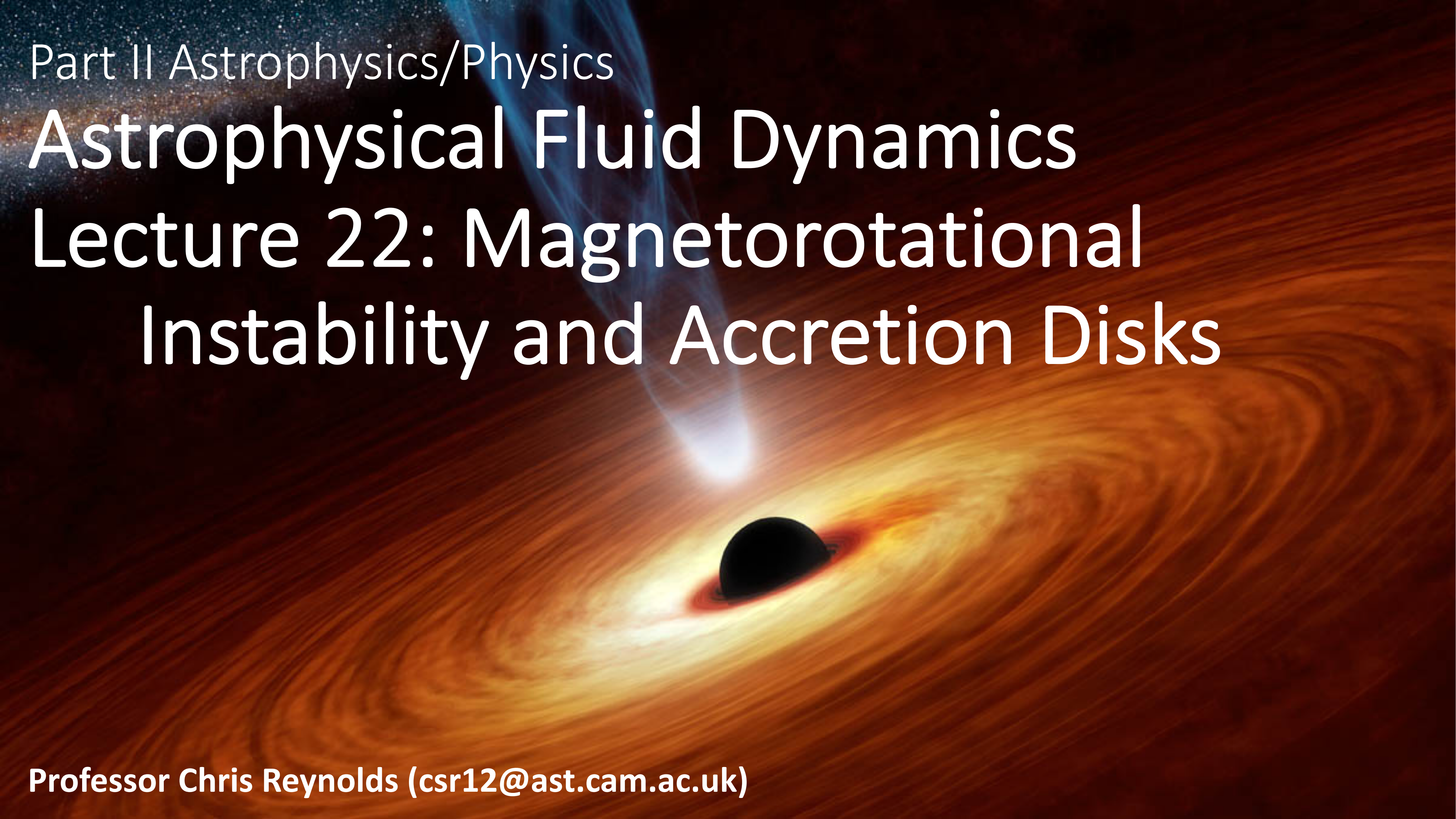
Part II Astrophysics/Physics

# Astrophysical Fluid Dynamics
# Lecture 22: Magnetorotational Instability and Accretion Disks



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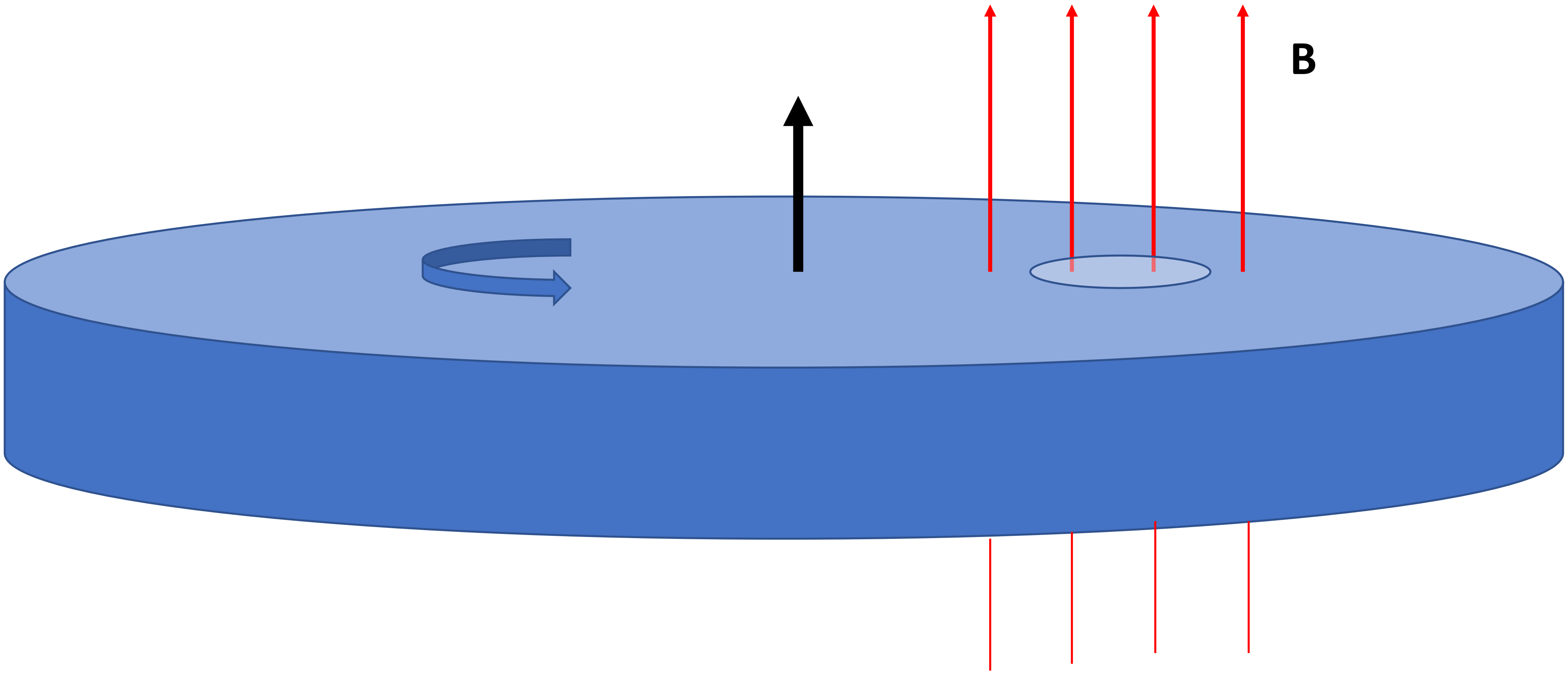
# Recap - last two lectures

- Plasmas (gases of charged particles)
- Equations of ideal, non-relativistic magnetohydrodynamics
  - Assumptions and approximations
  - Flux freezing, magnetic pressure, magnetic tension
- MHD Waves (Alfven waves, fast/slow magnetosonic waves)

# This lecture : Magnetorotational Instability

- Stability of rotating magnetized systems
- Magneto-rotational instability (MRI)
  - Walk through linear theory (in simple case)
  - Instability if angular velocity decreases outwards (true for Keplerian profile!)
  - System most unstable if B-field is weak
- Application of MRI to accretion disks

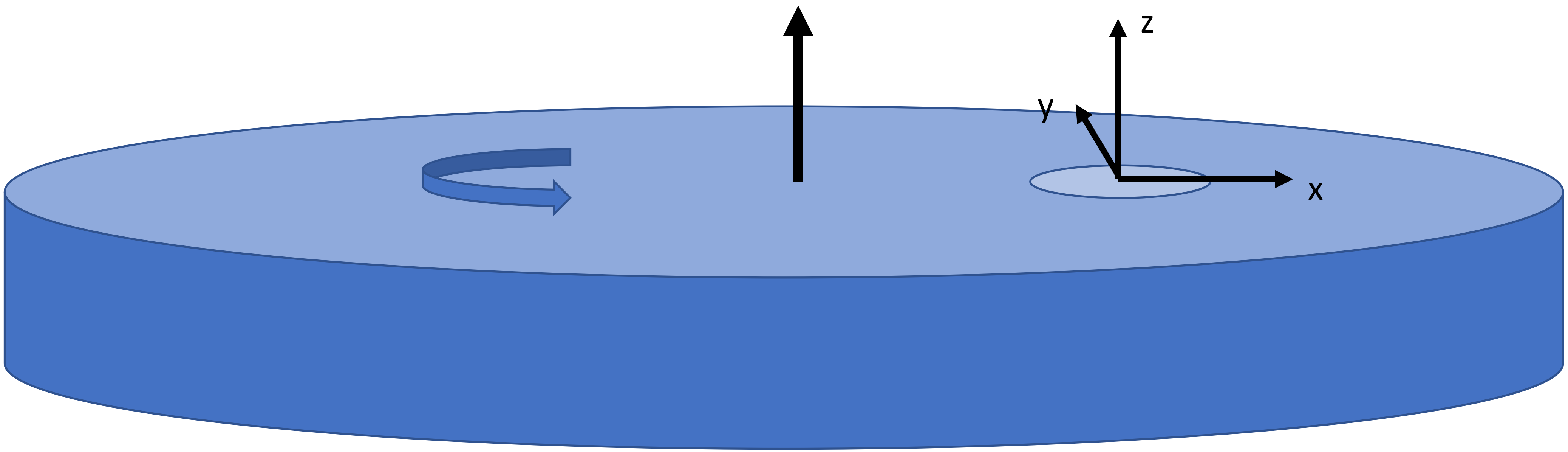
# Magnetorotational Instability



Analyze in a “local reference frame” of some orbiting patch at $R=R_0$.

Let frame rotate with the angular frequency of the center of the patch $\Omega(R_0)$.

Define Cartesian coordinate system in that frame (x-axis in +ve radial direction).



Momentum equation in that frame:

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \underbrace{\frac{1}{\mu_0 \rho}(\nabla \times \mathbf{B}) \times \mathbf{B}}_{\text{Magnetic force}} + \underbrace{2\mathbf{u} \times \boldsymbol{\Omega}}_{\text{Coriolis}} + \underbrace{\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})}_{\text{Centrifugal}} - \underbrace{R\Omega_K(R)^2 \hat{\mathbf{R}}}_{\text{Gravity}}$$

To illustrate the magnetorotational instability while keeping the analysis as simple as possible, we'll make the following restrictions/assumptions (all of which can be relaxed without qualitatively changing the result):

- Equilibrium has uniform magnetic field $\mathbf{B}_0$ aligned with *z*-axis (i.e. direction of Ω)
- Gas is cold, so ignore pressure forces
- Only consider perturbations with $\mathbf{k} \parallel \mathbf{B}_0$ (only $k_z$ is non-zero)

For conceptual convenience, we work in the Lagrangian picture.

Perturb momentum equation

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \frac{1}{\mu_0\rho}(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B} + 2\mathbf{u}\times\boldsymbol{\Omega} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}) - R\Omega_K(R)^2\hat{\mathbf{R}}$$

$$\frac{D\,\Delta\mathbf{u}}{Dt} - 2\Delta\mathbf{u}\times\boldsymbol{\Omega} = \frac{1}{\mu_0\rho}(\mathbf{B}_0\cdot\boldsymbol{\nabla})\Delta\mathbf{B} - \Delta x\, R\frac{d\Omega^2}{dR}\hat{\mathbf{R}}$$

For conceptual convenience, we work in the Lagrangian picture.

Perturb momentum equation

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \frac{1}{\mu_0\rho}(\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B} + 2\mathbf{u}\times\boldsymbol{\Omega} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}) - R\Omega_K(R)^2\hat{\mathbf{R}}$$

$$\frac{D\,\Delta\mathbf{u}}{Dt} - 2\Delta\mathbf{u}\times\boldsymbol{\Omega} = \frac{1}{\mu_0\rho}(\mathbf{B}_0\cdot\boldsymbol{\nabla})\Delta\mathbf{B} - \Delta x\,R\frac{d\Omega^2}{dR}\hat{\mathbf{R}}$$

$$\Rightarrow \qquad -i\omega\Delta\mathbf{u} - 2\Delta\mathbf{u}\times\boldsymbol{\Omega} = \frac{i}{\mu_o\rho}B_0 k\,\Delta\mathbf{B} - \Delta x\,R\frac{d\Omega^2}{dR}\hat{\mathbf{R}}$$

But we also have

$$\frac{D\,\Delta x}{Dt} = \Delta u_x$$

$$\Rightarrow \qquad -i\omega\,\Delta x = \Delta u_x$$

$$\Rightarrow \qquad \Delta x = \frac{i\,\Delta u_x}{\omega}$$

$$\frac{\partial\Delta\mathbf{B}}{\partial t} = \boldsymbol{\nabla}\times(\Delta\mathbf{u}\times\mathbf{B}_0) = (\mathbf{B}_0\cdot\boldsymbol{\nabla})\Delta\mathbf{u}$$

$$\Rightarrow \qquad -i\omega\,\Delta\mathbf{B} = ikB_0\,\Delta\mathbf{u}$$

$$\Rightarrow \qquad \Delta\mathbf{B} = -\frac{kB_0}{\omega}\Delta\mathbf{u}.$$

So,

$= i(kv_A)^2$

$$-i\omega\Delta\mathbf{u} - 2\Delta\mathbf{u}\times\Delta\mathbf{\Omega} = \frac{i}{\mu_o\rho}B_0k\frac{kB_0}{\omega}\Delta\mathbf{u} - \frac{i\,\Delta u_x}{\omega}R\frac{d\Omega^2}{dR}\hat{\mathbf{R}}$$

$$\Rightarrow \quad \omega^2\Delta u_x - 2i\Delta u_y\Omega\omega = (kv_A)^2\Delta u_x + \Delta u_x\frac{d\Omega^2}{d(\ln R)}$$

$$\omega^2\Delta u_y + 2i\Delta u_x\Omega\omega = (kv_A)^2\Delta u_y$$

$$\Rightarrow \quad \begin{pmatrix} \omega^2 - (kv_A)^2 - \frac{d\Omega^2}{d(\ln R)} & -2i\omega\Omega \\ 2i\omega\Omega & \omega^2 - (kv_A)^2 \end{pmatrix}\begin{pmatrix}\Delta u_x \\ \Delta u_y\end{pmatrix} = 0$$

$$\Rightarrow \quad \left[\omega^2 - (kv_A)^2 - \frac{d\Omega^2}{d(\ln R)}\right]\left[\omega^2 - (kv_A)^2\right] - 4\Omega^2\omega^2 = 0$$

$$\Rightarrow \quad \omega^4 - \omega^2\left[4\Omega^2 - \frac{d\Omega^2}{d(\ln R)} + 2(kv_A)^2\right] + (kv_A)^2\left[(kv_A)^2 + \frac{d\Omega^2}{d(\ln R)}\right] = 0$$

$$\omega^4 - \omega^2 \left[4\Omega^2 - \frac{d\Omega^2}{d(\ln R)} + 2(kv_A)^2\right] + (kv_A)^2 \left[(kv_A)^2 + \frac{d\Omega^2}{d(\ln R)}\right] = 0$$

Let's examine consequences of this:

1. If we turn off magnetic physics ($v_A$=0 identically)…

$$\begin{aligned}\omega^2 &= 4\Omega^2 + \frac{d\Omega^2}{d(\ln R)} \\ &= \frac{1}{R^3}\frac{d}{dR}(R^4\Omega^2) \equiv \kappa_R^2\end{aligned}$$

   a) If $\kappa_R^2 > 0$, we get oscillations (radial epicyclic oscillation). For Keplerian case, $\kappa_R^2 = \Omega^2$.
   b) If $\kappa_R^2 < 0$ (specific angular momentum decreasing with radius), flow is unstable. This is the Rayleigh instability.

$$\omega^4 - \omega^2 \left[4\Omega^2 - \frac{d\Omega^2}{d(\ln R)} + 2(kv_A)^2\right] + (kv_A)^2 \left[(kv_A)^2 + \frac{d\Omega^2}{d(\ln R)}\right] = 0$$

2. Including magnetic effects… there will be instability ($\omega^2 < 0$) if final term is negative.

$$(kv_A)^2 + \frac{d\Omega^2}{d(\ln R)} < 0$$ (instability)

   a) For sufficiently weak field or long wavelength, there will always be instability if

$$\frac{d\Omega^2}{dR} < 0$$ (instability; true for Keplerian flow)

   This is the magneto-rotational Instability (MRI).

b) Magnetic tension stabilizes the MRI for $k > k_{\rm crit}$ where

$$(k_{\rm crit} v_A)^2 = -\frac{d\Omega^2}{d(\ln R)} \quad (= 3\Omega^2 \text{ for Keplerian})$$

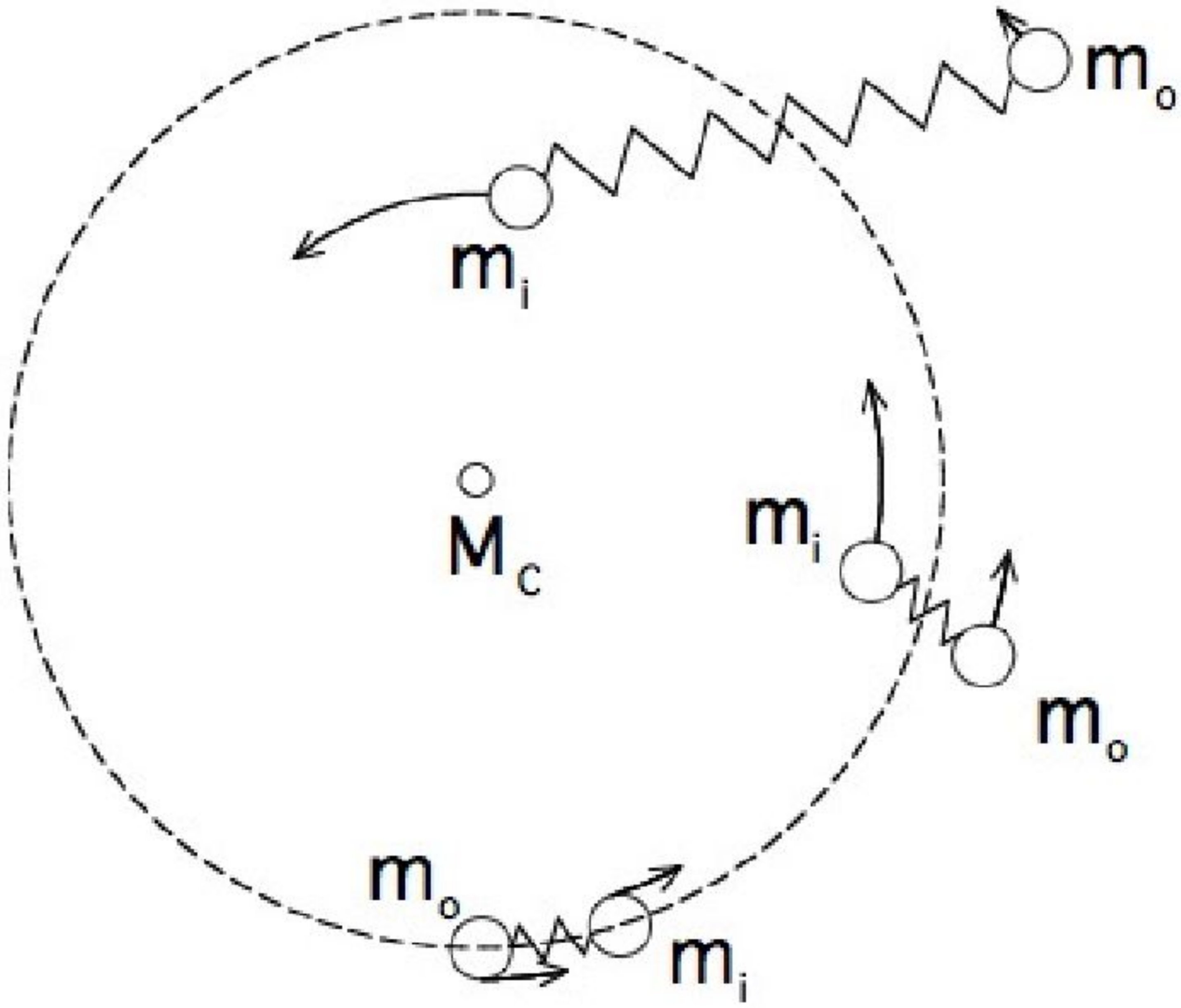
So, the range of unstable modes increases as the magnetic field strength gets weaker
Conversely, range of unstable modes decreases as field gets stronger.

b) Further analysis of dispersion relation (not done here) shows that the fastest growing modes has growth rate

$$|\omega_{\rm max}| = \frac{3}{2}\Omega$$

(formally, do not recover B=0 when we take limit B→0!)
The fastest growing wavenumber

$$k v_A \approx \Omega$$



m_o
m_i
M_C
m_i
m_o
m_o
m_i

# Relevance to accretion disks

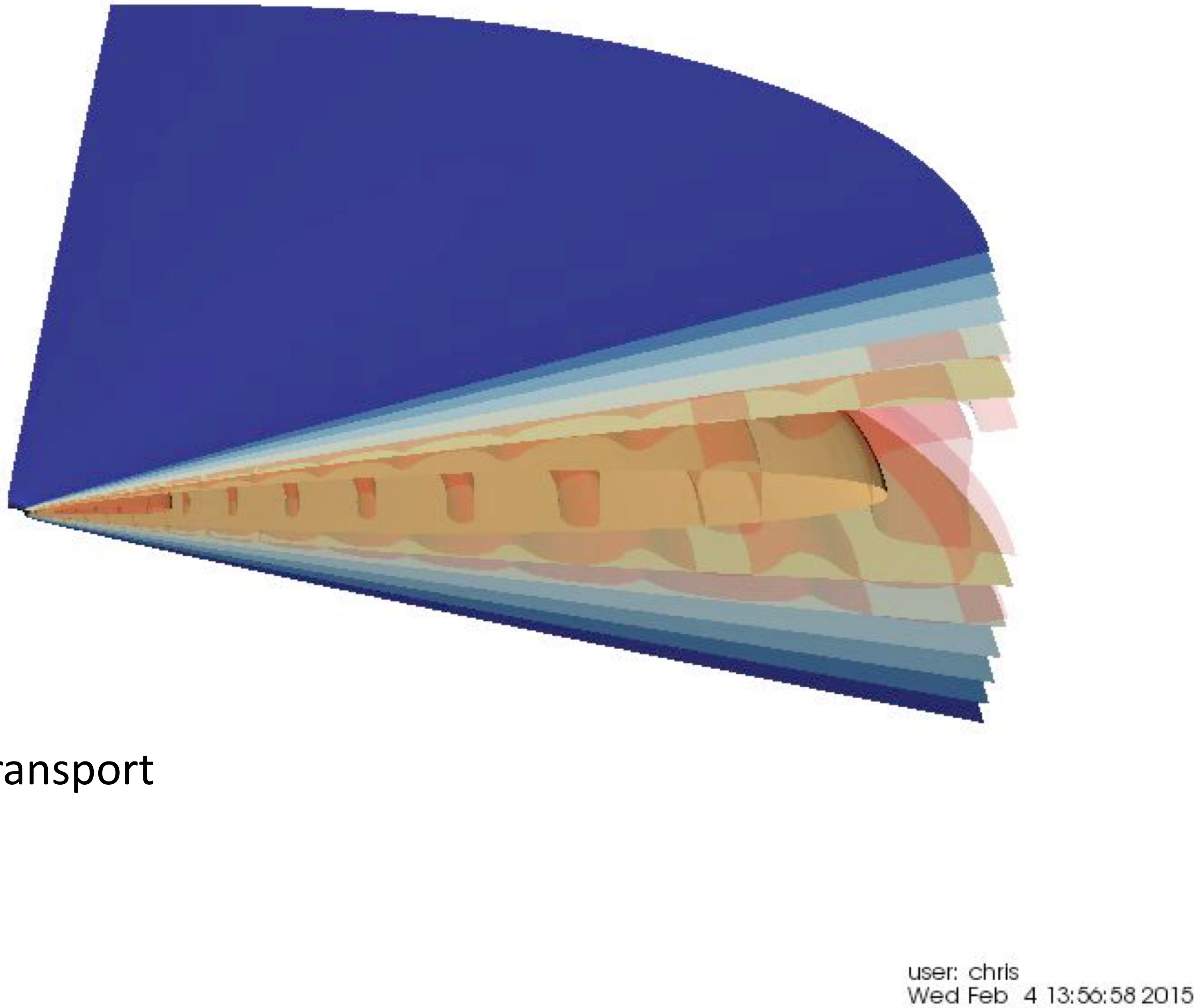
MRI believed to be critically important for accretion disks.

MRI grows rapidly… enters non-linear regime… drives turbulence.

Turbulence then drives angular momentum transport.

# “Vanilla” MHD disk:

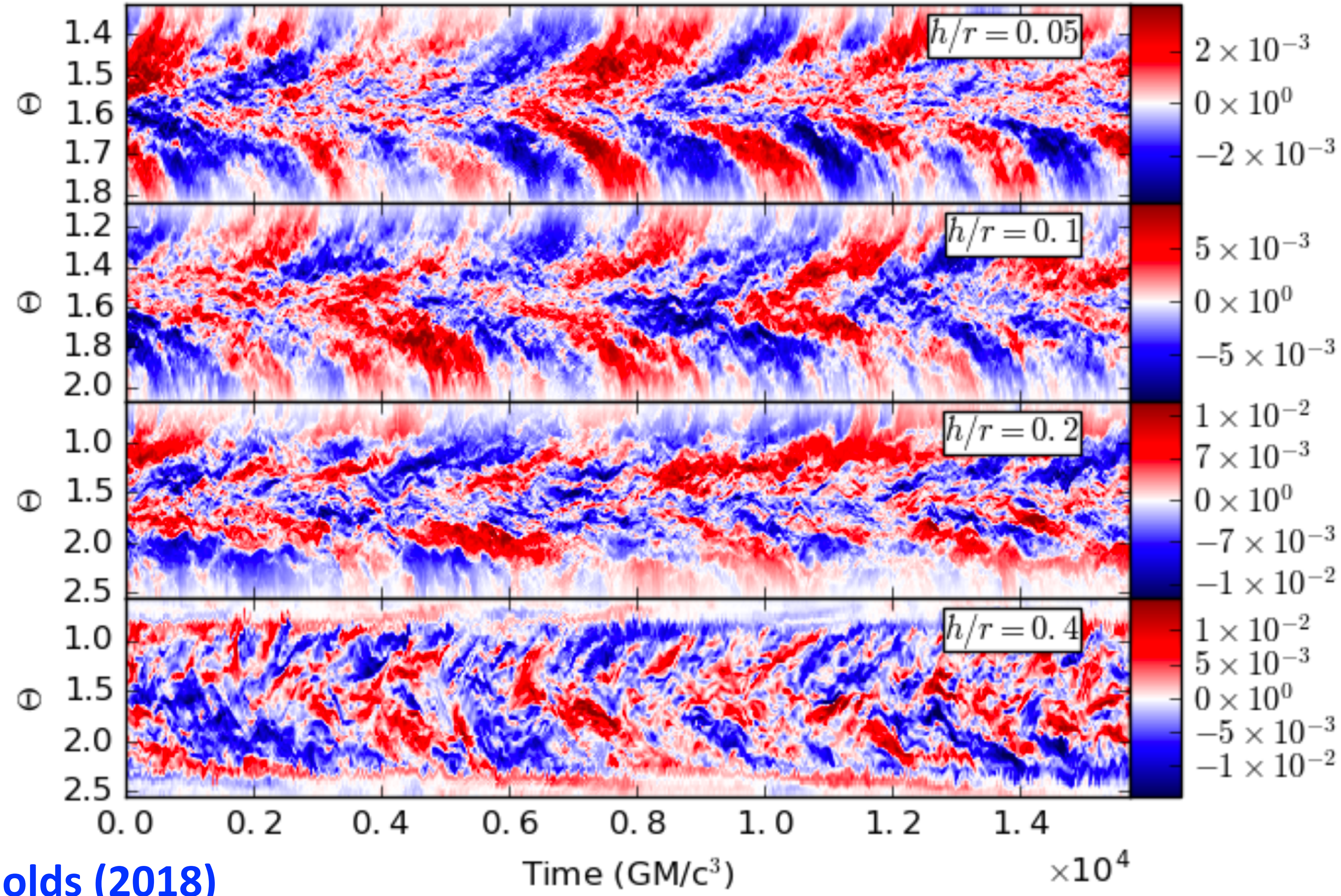
- Physics :
  - MHD + Newtonian gravity
  - Ad-hoc cooling term; maintain h/r=0.1
- Initial conditions :
  - Smooth Keplerian flow
  - Weak poloidal loops of magnetic field
- Code / algorithms :
  - PLUTOv4.2, Godunov-type code
  - Spherical polars (quarter-wedge)
  - 512 x 384 x 128 zones




Magneto-rotational instability grows $\tau_{\mathrm{mri}} \sim \Omega^{-1}$
Saturates in sustained turbulence
Correlated fluctuations in $B_r$ and $B_\phi$ → ang mtm transport

$$\frac{\langle \delta B_r \, \delta B_\phi \rangle}{4\pi} \equiv \alpha P$$

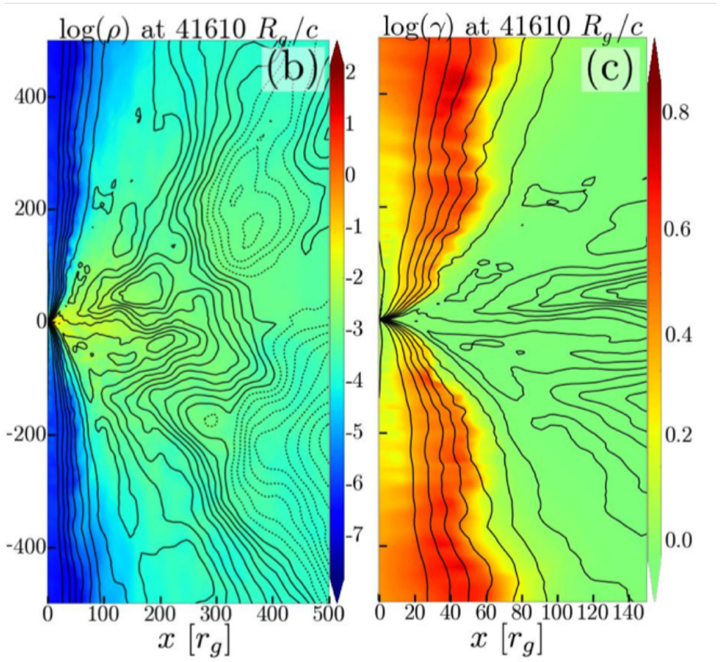
# Effect of disk thickness (3d-MHD, 32z/h, wedges, PLUTO code)…



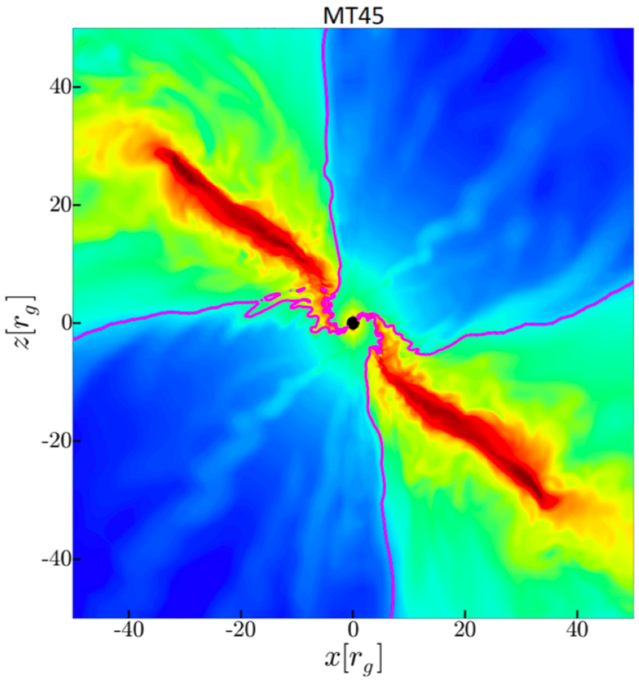

B-phi

**Hogg & Reynolds (2018)**

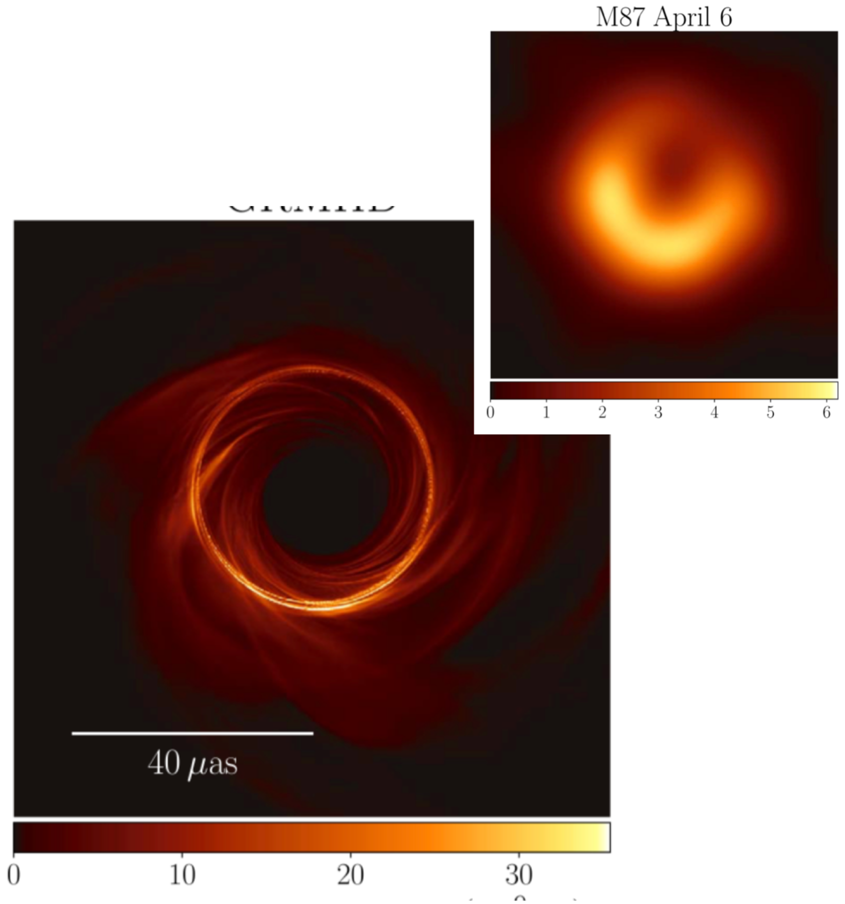
# GRMHD and black holes




BH spin powered relativistic jets
Blandford & Znajek (1977)
**Liska+ (2018)**



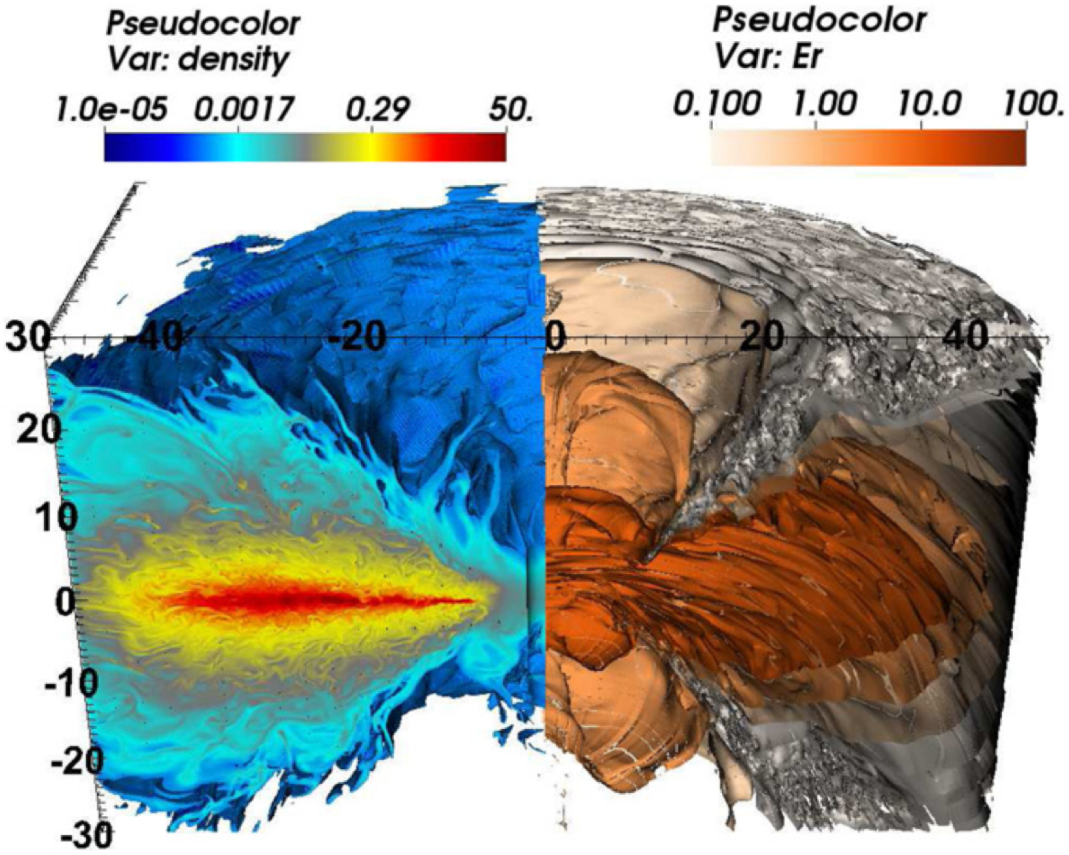

Spin-induced disk warps/tears
Bardeen & Petterson (1975)
**Liska+ (2019)**




Radiatively-inefficient accretion flows and theoretical basis for EHT analysis
Medeiros+ (2018)
Porth+ (2019)
**EHT Collab (2019)**

De Villers & Hawley (2003); Gammie+ (2003); Hirose+ (2004); Komissarov (2005); Shibata & Sekiguchi (2005); Anninos+ (2005); Tchekhovskoy+(2007); Fragile+ (2007); Hawley+ (2007); Shafee+ (2008); Avara+(2016); Schnittman+(2016); Brooks+ (2016, 2020); …. and many more

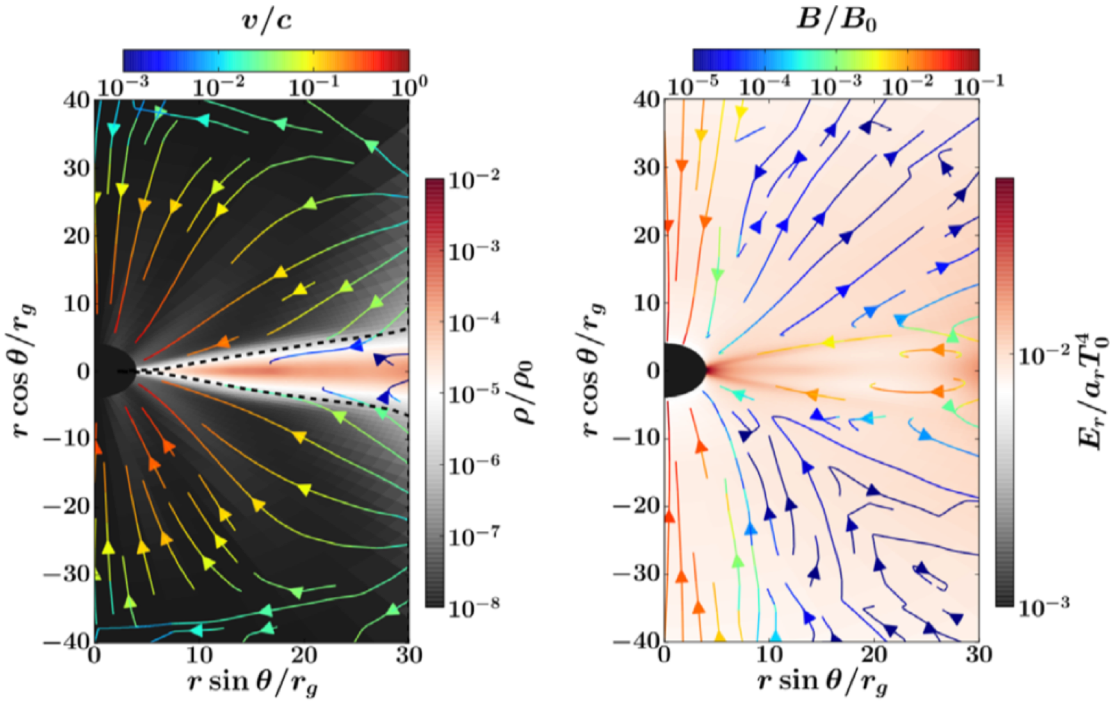
# Radiation, hyper-accretion, and rad-dominated disks




Super-Eddington BH accretion – role of radiation in vertical energy transport (Athena++)

**Jiang, Stone, Davis (2014)**

Also… Ohsuga & Mineshige (2011); Dai+ (2018)




Sub-Eddington radiation-dominated BH accretion (AGN) – role of magnetic pressure support and stabilization of thermal instability

Jiang+ (2019)