Part II Astrophysics/Physics

Astrophysical Fluid Dynamics
Lecture 18: Viscosity & Accretion Disks

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Recap – last lecture

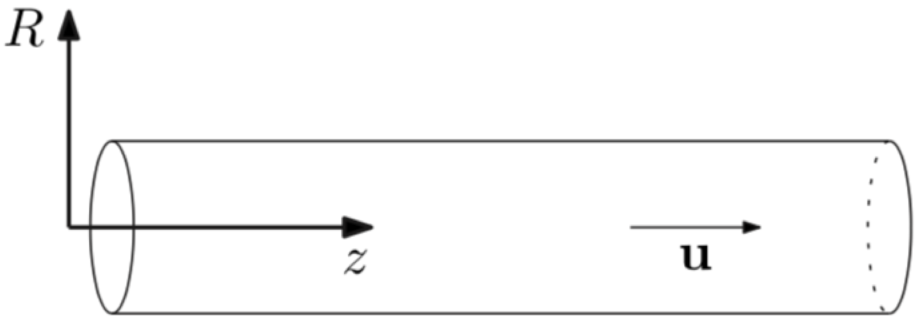
- Introduced notion of viscosity
  - Diffusion of momentum through fluid due to the particle nature of fluid
  - Leading order finite-λ correction
- Viscous stress tensor and the Navier-Stokes equations
- Diffusion of vorticity in a viscous flow
- Energy dissipation in a viscous flow (2nd law of TD $\Rightarrow$ positivity of η)

This Lecture – more viscosity

- Steady state viscous flow in a pipe (Chapter I.5)
  - Analysis of laminar state
  - Transition to turbulence
- Accretion disks (Chapter I.6)
  - Some basic principles
  - Accretion disks as viscous systems
  - Modeling geometrically-thin accretion disks

I.5 : Steady-state viscous flow in a pipe

As an illustration of an explicit calculation involving viscous flows, we examine the important case of **steady-state**, **laminar, incompressible, viscous** flow through a pipe with constant **circular** cross-section, **neglecting gravity**…



$$\frac{\partial}{\partial t} = 0$$

$$u_R = u_\phi = 0, \quad u_z = u_z(R) \neq 0$$

$$\cancelto{0}{\frac{\partial \mathbf{u}}{\partial t}} + \cancelto{0}{\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}} = -\frac{1}{\rho}\boldsymbol{\nabla}p - \cancelto{0}{\boldsymbol{\nabla}\Psi} + \nu\left(\nabla^2\mathbf{u} + \frac{1}{3}\boldsymbol{\nabla}(\cancelto{0}{\boldsymbol{\nabla}\cdot\mathbf{u}})\right)$$

steady symmetry no gravity incompressible

So, only surviving terms are:

$$\nu \nabla^2 \mathbf{u} = \frac{1}{\rho} \boldsymbol{\nabla} p$$

Examine R and ϕ components:

$$u_R = u_\phi = 0 \qquad \Rightarrow \qquad \frac{\partial p}{\partial R} = \frac{\partial p}{\partial \phi} = 0$$

Examine z-component:

$$\underbrace{\frac{1}{\rho}\frac{\partial p}{\partial z}}_{\substack{\text{function of} \\ z\text{ only}}} = \underbrace{\nu \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial u_z}{\partial R}\right)}_{\text{function of } R \text{ only}} = \underbrace{-\frac{1}{\rho}\frac{\Delta p}{l}}_{\substack{\text{constant, written} \\ \text{in terms of global} \\ \text{pressure gradient}}}$$

Integrate:

$$u = -\frac{\Delta p}{4\rho\nu l} R^2 + a \ln R + b$$

(*a,b* constants)

$$u = -\frac{\Delta p}{4\rho\nu l}R^2 + a\ln R + b$$

Fix a, b via boundary condition.

At R=0, regularity (finite u) demands a=0

At $R=R_0$, set u=0 (no slip boundary condition). So,

$$\Rightarrow \qquad u = \frac{\Delta p}{4\nu\rho l}(R_0^2 - R^2)$$

Can now calculate mass flux:

$$Q = \int_0^{R_0} 2\pi\rho u R\,\mathrm{d}R = \frac{\pi}{8}\frac{\Delta p}{\nu l}R_0^4$$

So, mass flux completely determined by the pressure gradient, radius of pipe, and coefficient of kinematic viscosity.

In fact, flow will become **turbulent** (not steady, $u_R \neq 0, u_\varphi \neq 0$, motions on large range of scales), above a critical Reynolds number.

$$Re = \frac{uR_0}{\nu}$$

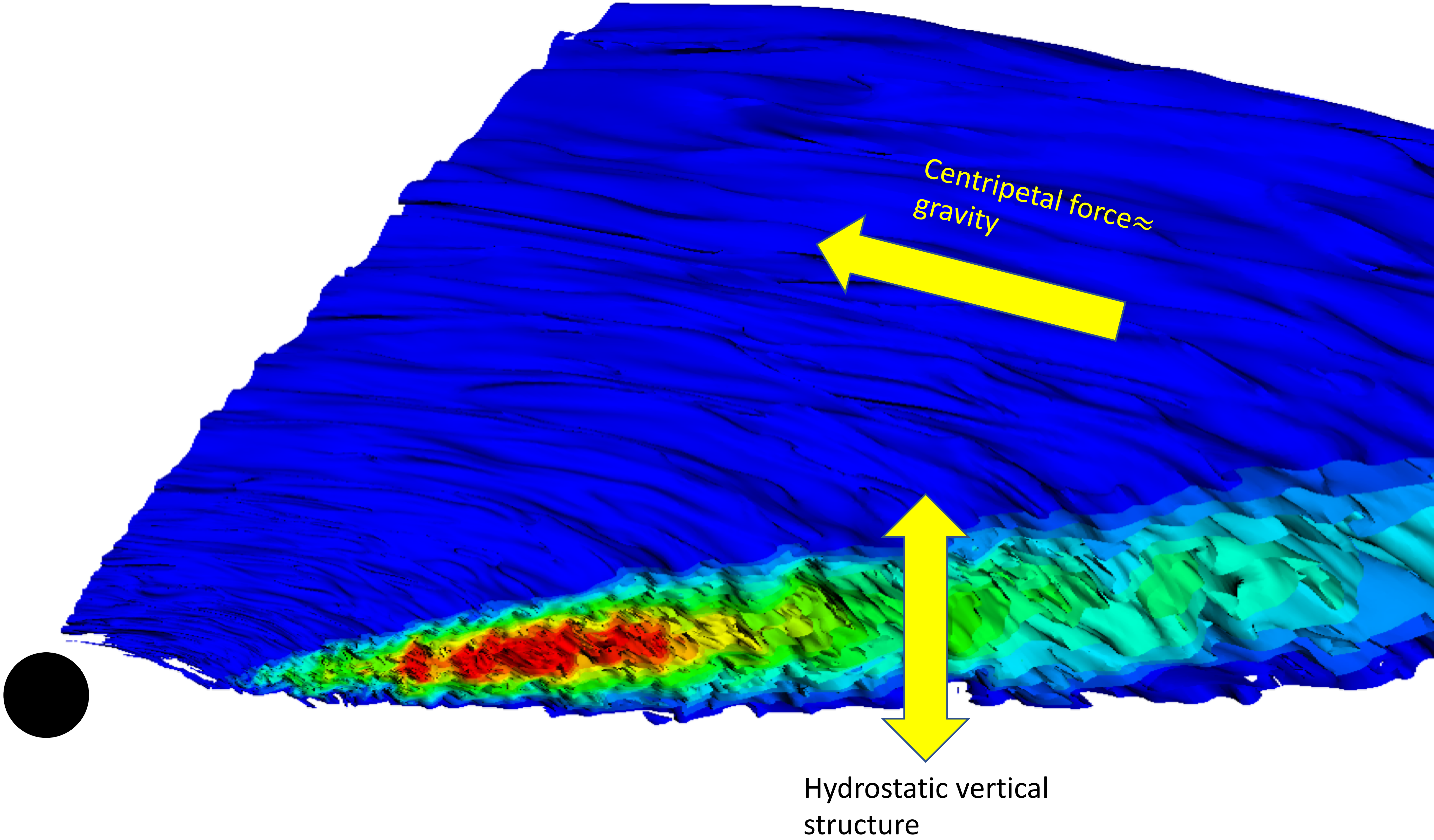
I.6 : Accretion Disks

The theory of accretion disks is one of the most important applications of the Navier-Stokes equations (indeed of fluid dynamics) in astrophysics.

Consider gas flowing on bound orbit towards a gravitating object (star, black hole...).

Almost always, the gas will have appreciable angular momentum about the central object. Then…

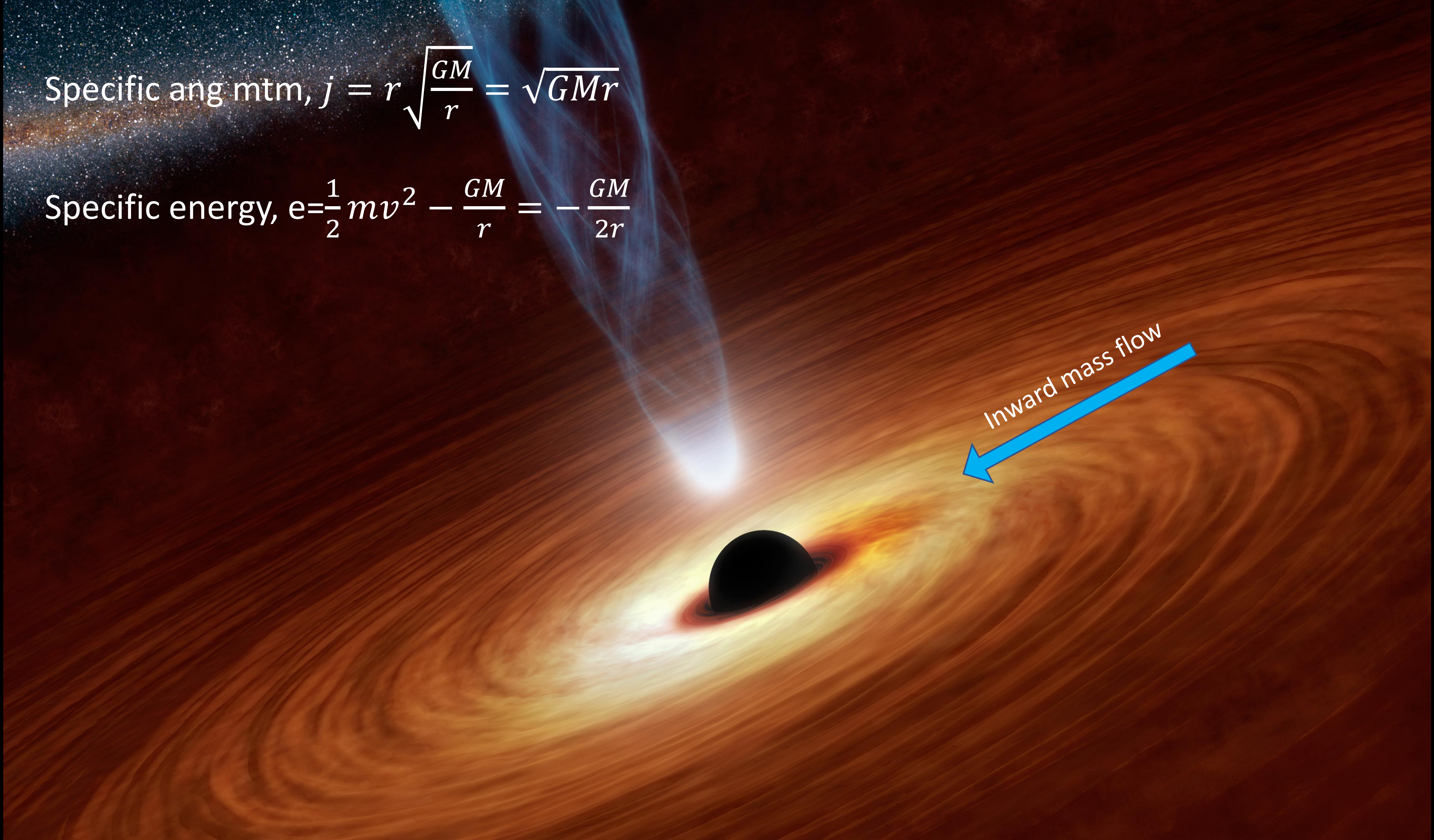
- Gas will settle into plane defined by overall angular momentum vector (gas streamers will intersect/shock and damp out out-of-plane motions).
- Within plane, gas will tend to settle onto circular orbits (a fluid element’s lowest total energy configuration for a given specific angular momentum)

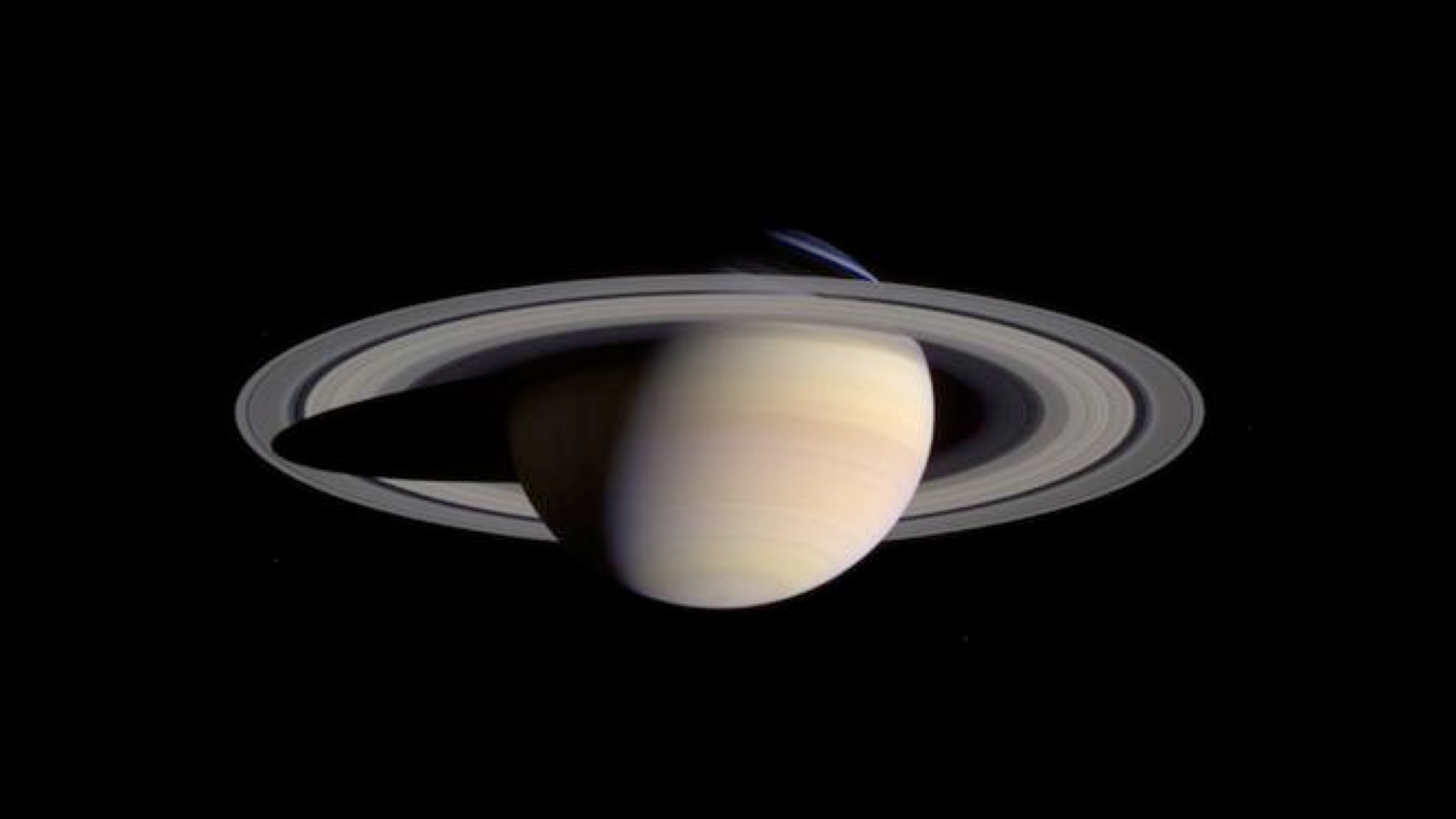


Centripetal force≈
gravity
Hydrostatic vertical
structure

Specific ang mtm, $j = r\sqrt{\frac{GM}{r}} = \sqrt{GMr}$

Specific energy, e=$\frac{1}{2}mv^2 - \frac{GM}{r} = -\frac{GM}{2r}$





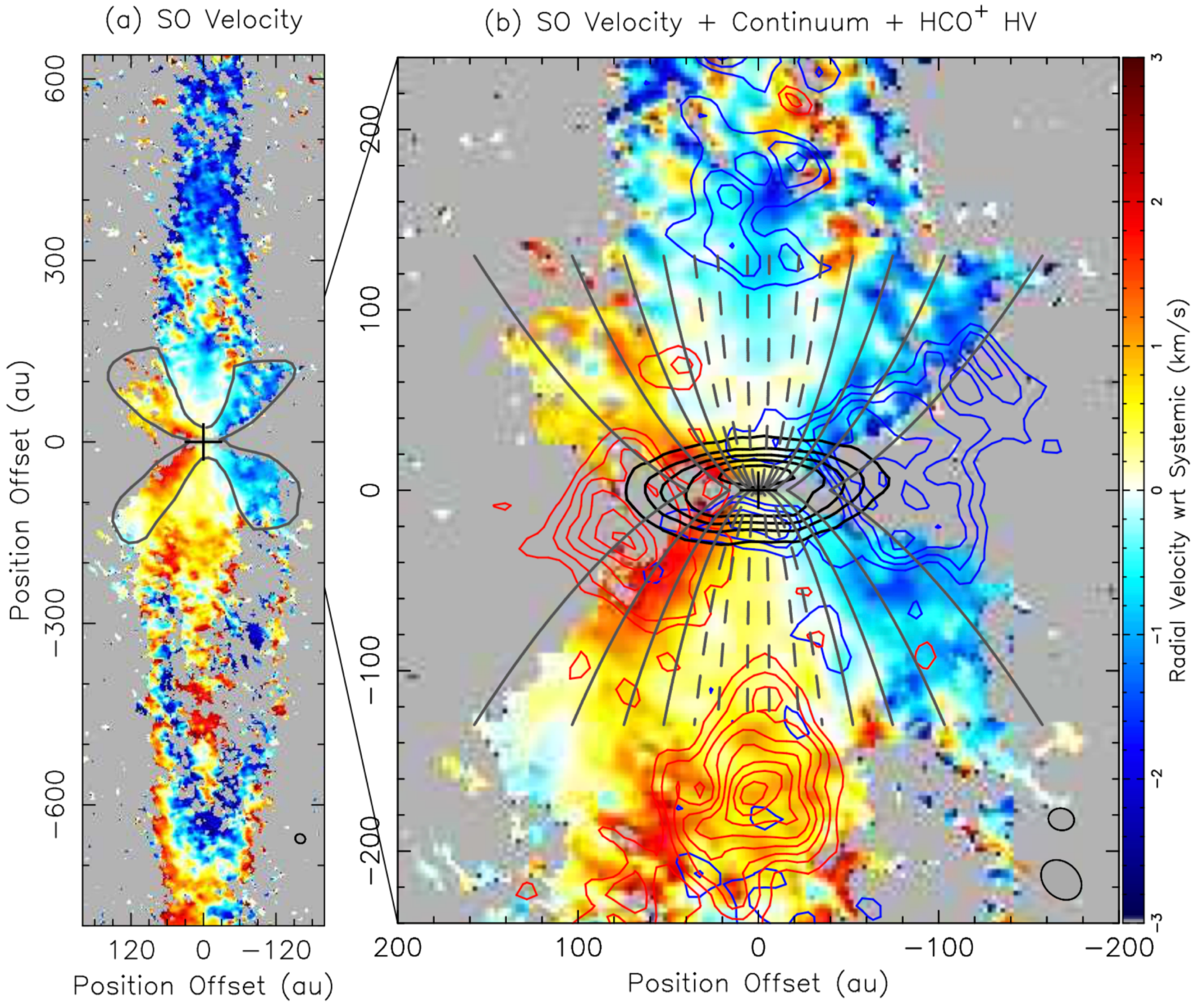
Accretion requires fluid elements to lose angular momentum.

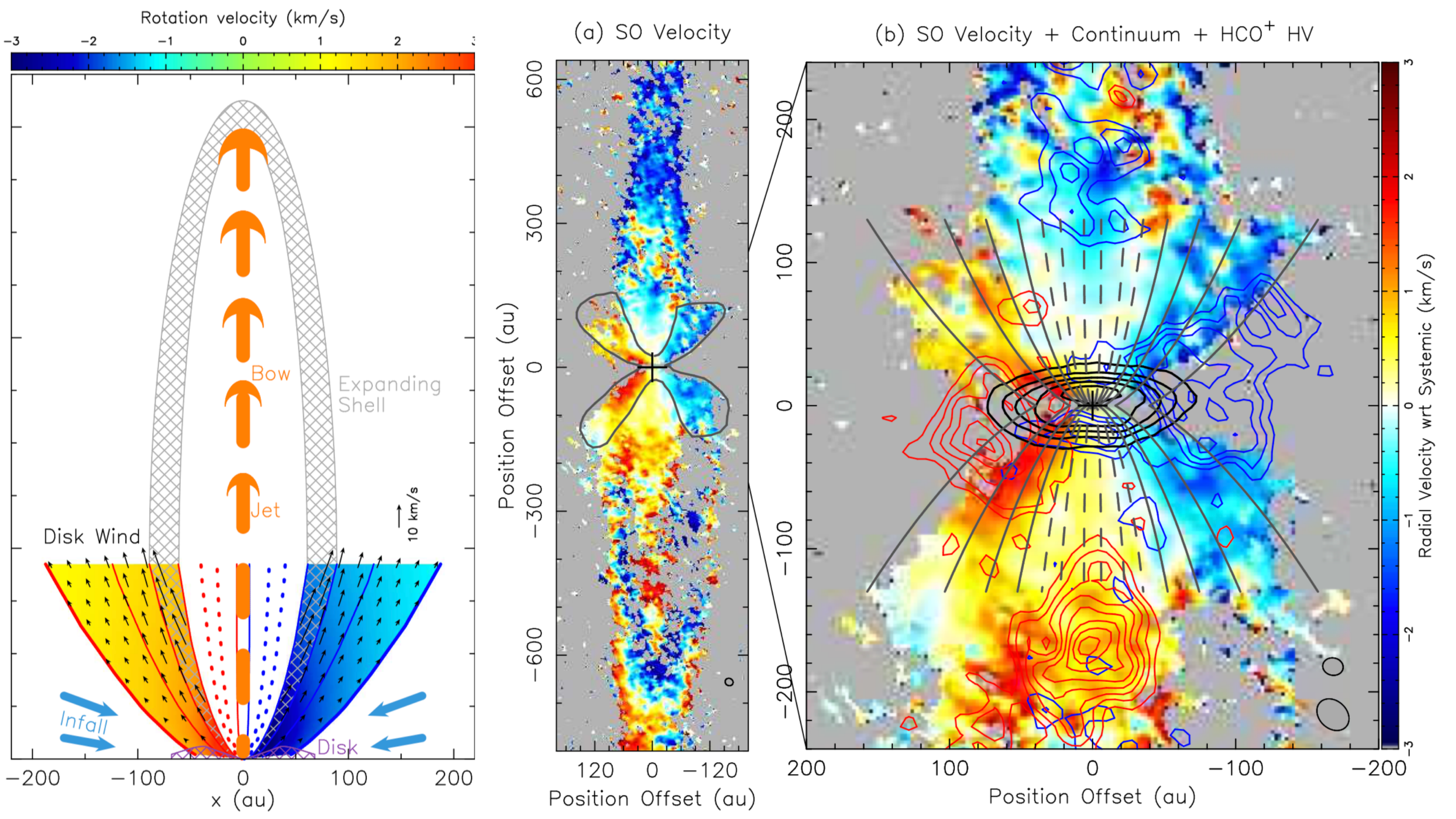
Two general pictures for how this can happen…

- Winds : Fluid elements in disk pass their angular momentum (via magnetic forces) to material that carries it away in a wind.
- Internal viscosity : Fluid elements in disk pass their angular momentum to fluid elements further out via action of shear “viscosity”.

ALMA observations of rotating wind from the protostellar disk in HH212

(Lee at al. 2021, https://arxiv.org/abs/2101.03293v1)




Rotation velocity (km/s)
−3
−2
−1
0
1
2
3
Bow
Expanding Shell
Jet
10 km/s
Disk Wind
Infall
Disk
−200
−100
0
100
200
x (au)
(a) SO Velocity
Position Offset (au)
600
300
0
−300
−600
120
0
−120
Position Offset (au)
(b) SO Velocity + Continuum + HCO+ HV
200
100
0
−100
−200
200
100
0
−100
−200
Position Offset (au)
Radial Velocity wrt Systemic (km/s)
3
2
1
0
−1
−2
−3

Relative role of wind and internal viscosity in angular momentum transport is uncertain and may vary from system to system (or even time-to-time in given systems).

Here, we focus on models based on internal viscosity.

Let's set up a simple model of a geometrically-thin accretion disk. We will use a cylindrical polar coordinate system (R, φ, z). Assume:

- Axisymmetry, $\partial/\partial\varphi = 0$,
- Hydrostatic equilibrium in z-direction, $u_z = 0$,
- Flow close to Keplerian, i.e., centripetal acceleration due to gravity,
- Zero bulk viscosity.

Continuity equation in cylindrical polars:

$$\frac{\partial \rho}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\rho u_R) + \underbrace{\cancelto{0}{\frac{1}{R}\frac{\partial}{\partial \phi}(\rho u_\phi)}}_{\text{axisymmetry}} + \underbrace{\cancelto{0}{\frac{\partial}{\partial z}(\rho u_z)}}_{\text{hydrostatic eqm.}} = 0$$

$$\Rightarrow \qquad \frac{\partial \rho}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\rho u_R) = 0$$

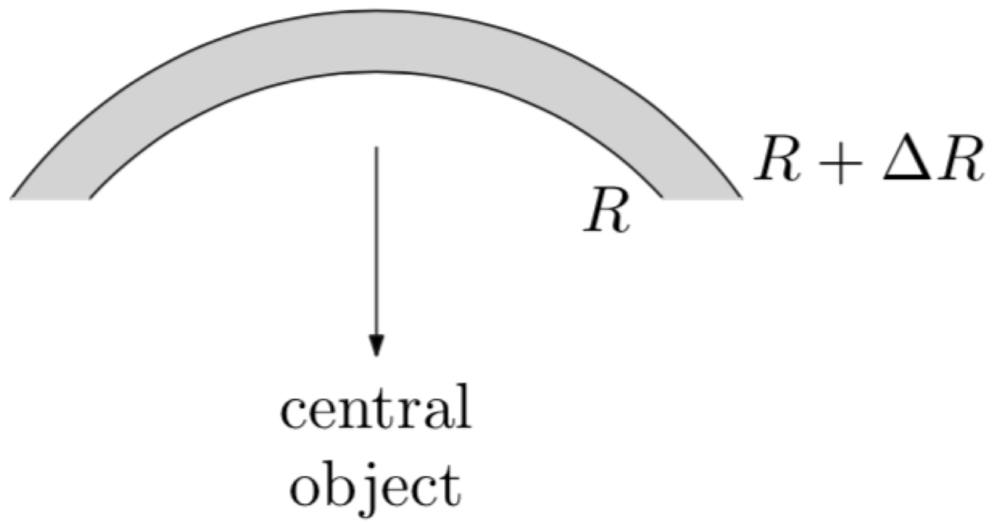
Useful to work with surface density, **Σ**:

$$\Sigma \equiv \int_{-\infty}^{\infty} \rho \, \mathrm{d}z$$

Integrating continuity equation over z, we have

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\Sigma u_R) = 0 \qquad ①$$

Can get same result by thinking of the disk as a set of (Eulerian) rings:



$$\begin{matrix}\text{rate of change}\\\text{of mass in}\\\text{the annulus}\end{matrix} = \begin{matrix}\text{flux into}\\\text{annulus}\end{matrix} + \begin{matrix}\text{flux out}\\\text{of annulus}\end{matrix}$$

$$\Rightarrow \qquad \frac{\partial}{\partial t}(2\pi R \Delta R \Sigma) = 2\pi R \Sigma(R) u_R(R) - 2\pi(R+\Delta R)\Sigma(R+\Delta R)u_R(R+\Delta R)$$

$$\Rightarrow \qquad R\frac{\partial \Sigma}{\partial t} = -\left[\frac{(R+\Delta R)\Sigma(R+\Delta R)u_R(R+\Delta R) - R\Sigma(R)u_R(R)}{\Delta R}\right]$$

$$\Rightarrow \qquad R\frac{\partial \Sigma}{\partial t} = -\frac{\partial}{\partial R}(R\Sigma u_R) \qquad \text{taking } \Delta R \to 0$$

Now look at conservation of angular momentum.

In principle, can take polar-form of Navier-Stokes equation and apply same assumptions. The phi-component reads:

$$\frac{\partial u_\phi}{\partial t} + u_R \frac{\partial u_\phi}{\partial R} + \frac{u_\phi}{R}\frac{\partial u_\phi}{\partial \phi} + u_z \frac{\partial u_\phi}{\partial z} + \frac{u_\phi u_R}{R} =$$

$$-\frac{1}{R\rho}\frac{\partial p}{\partial \phi} + \nu\left(\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial u_\phi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 u_\phi}{\partial \phi^2} + \frac{\partial^2 u_\phi}{\partial z^2} - \frac{u_\phi}{R^2}\right) + \nu\frac{2}{R^2}\frac{\partial u_R}{\partial \phi}$$

$$\begin{aligned}
&= \nu\left(\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial u_\phi}{\partial R}\right) - \frac{u_\phi}{R^2}\right) \\
&= \frac{1}{R}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}(u_\phi/R)\right) \\
&= \frac{1}{R}\frac{\partial}{\partial R}\left(R^2\frac{\partial \Omega}{\partial R}\right)
\end{aligned}$$

so, momentum carried through rotating flow due to gradients in angular velocity Ω

For thin disks, it is conceptually and mathematically easier to use the ring approach:

$$\begin{array}{c}\text{rate of change}\\ \text{of ang. mtm.}\end{array} = \begin{array}{c}\text{ang. mtm.}\\ \text{of mass}\\ \text{entering ring}\end{array} - \begin{array}{c}\text{ang. mtm.}\\ \text{of mass}\\ \text{leaving ring}\end{array} + \begin{array}{c}\text{net torque on ring}\\ \text{(viscous, magnetic, etc.)}\end{array}$$

$$\Rightarrow \qquad \frac{\partial}{\partial t}(2\pi R\Delta R\Sigma R^2\Omega) = f(R) - f(R+\Delta R) + G(R+\Delta R) - G(R)$$

$$f(R) \equiv 2\pi R\Sigma u_R \Omega R^2$$

(ang mtm advection)

$$G(R) = 2\pi R\nu\Sigma R\frac{\partial\Omega}{\partial R}R = 2\pi R^3\nu\Sigma\frac{\mathrm{d}\Omega}{\mathrm{d}R}$$

(viscous torques)

$$\therefore \qquad \frac{\partial}{\partial t}(R\Sigma u_\phi) = -\frac{1}{R}\frac{\partial}{\partial R}(\Sigma R^2 u_\phi u_R) + \frac{1}{R}\frac{\partial}{\partial R}\left(\nu\Sigma R^3\frac{\mathrm{d}\Omega}{\mathrm{d}R}\right) \qquad ②$$

Summary:

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\Sigma u_R) = 0 \qquad \text{①}$$

$$\frac{\partial}{\partial t}(R\Sigma u_\phi) = -\frac{1}{R}\frac{\partial}{\partial R}(\Sigma R^2 u_\phi u_R) + \frac{1}{R}\frac{\partial}{\partial R}\left(\nu \Sigma R^3 \frac{\mathrm{d}\Omega}{\mathrm{d}R}\right) \qquad \text{②}$$

Now, flow is approximately Keplerian, so $\partial u_\phi / \partial t = 0$

$$R u_\phi \frac{\partial \Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(\Sigma R^2 u_\phi u_R) = \frac{1}{R}\frac{\partial}{\partial R}\left(\nu \Sigma R^3 \frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)$$

$$\Rightarrow \qquad - u_\phi \frac{\partial}{\partial R}(R\Sigma u_R) + \frac{1}{R}\frac{\partial}{\partial R}\left(\Sigma R^2 u_\phi u_R\right) = \frac{1}{R}\frac{\partial}{\partial R}\left(\nu \Sigma R^3 \frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)$$

$$\Rightarrow \qquad \cancel{- u_\phi \frac{\partial}{\partial R}(R\Sigma u_R)} + \cancel{\frac{u_\phi R}{R}\frac{\partial}{\partial R}(R\Sigma u_R)} + \Sigma u_R \frac{\partial}{\partial R}(u_\phi R) = \frac{1}{R}\frac{\partial}{\partial R}\left(\nu \Sigma R^3 \frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)$$

$$\Rightarrow \qquad R\Sigma u_R \frac{\partial}{\partial R}(R\Omega^2) = \frac{\partial}{\partial R}\left(\nu \Sigma R^3 \frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)$$

$$\Rightarrow \qquad u_R = \frac{\frac{\partial}{\partial R}\left(\nu \Sigma R^3 \frac{\mathrm{d}\Omega}{\mathrm{d}R}\right)}{R\Sigma \frac{\partial}{\partial R}(R\Omega^2)}$$

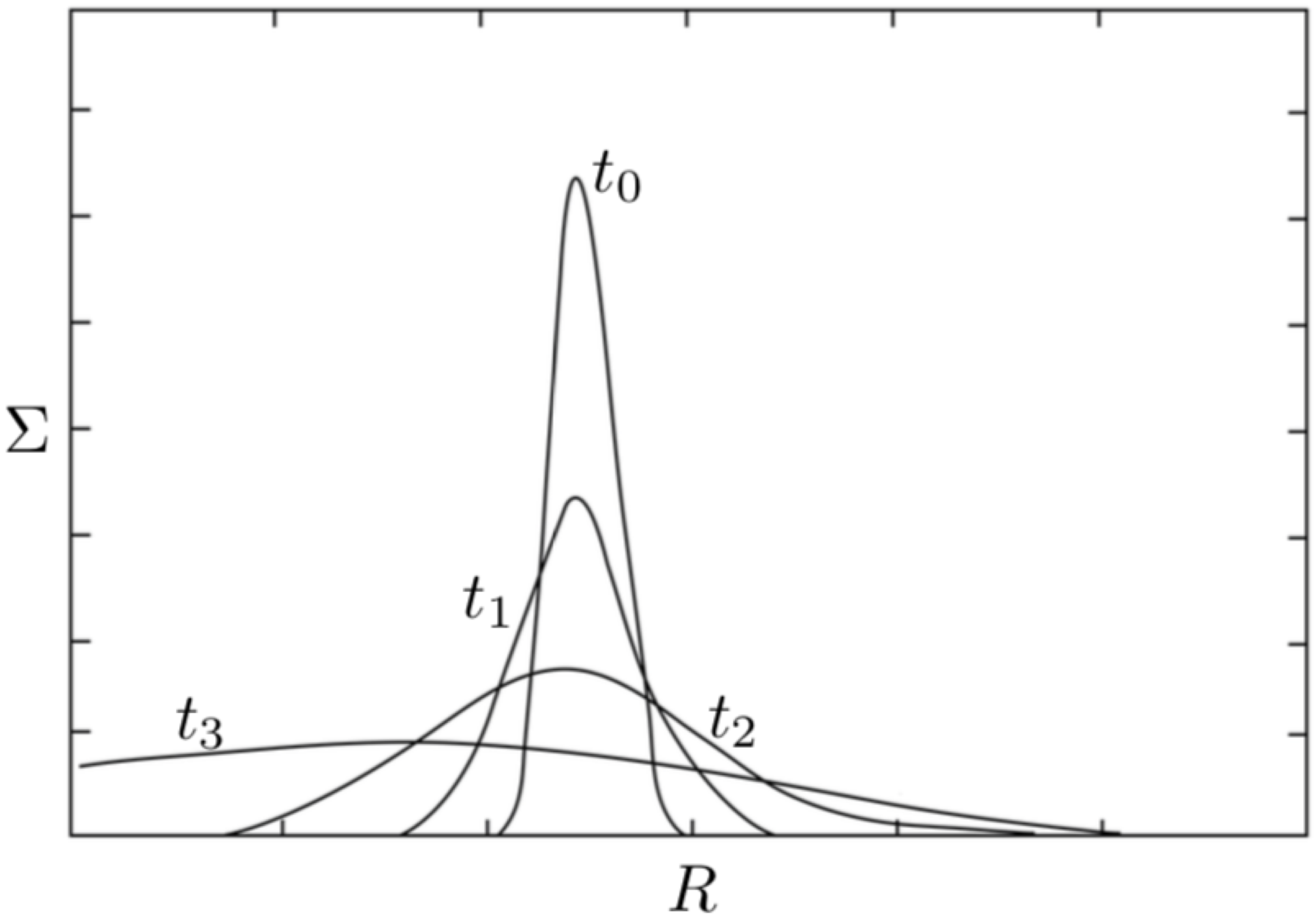
Substitute u_R into (1), and specialize to Newtonian gravity of point source

$$\boxed{\frac{\partial \Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}(\nu \Sigma R^{1/2})\right]}$$

Surface density obeys a diffusion-like equation.

Notes

1. In general, $\nu = \nu(R, \Sigma, T, \ldots)$ and so this is a non-linear diffusion equation. Reduces to linear equation if $\nu = \nu(R)$.
2. Vertical structure only enters via the temperature dependence of ν. So the geometrically-thin assumptions allows the radial and vertical problems to be mostly decoupled.
3. Diffusion-like nature of this equation shows that an initial ring of matter will broaden and "slump" towards the central object.



Σ
t_0
t_1
t_2
t_3
R