Part II Astrophysics/Physics

Astrophysical Fluid Dynamics
Lecture 16: Thermal Instability

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Recap – last two lectures

- Fluid instabilities!
- Convective instability – stability of a hydrostatic equilibrium
  - Schwarzschild criterion… unstable if entropy decreasing upwards
- Gravitational instability – stability to gravitational collapse
  - Jeans analysis… always unstable on sufficiently large scales
- Interface instabilities
  - Rayleigh-Taylor instability… unstable if dense fluid sits above less dense fluid
  - Kelvin-Helmholtz instability… unstable if relative motion along interface

- Generally, instability is driven by some available source of “free energy”

This Lecture

- Conclude our current discussion of fluid instabilities
- Thermal instability
  - Problem set-up and analysis
  - Dispersion relation
  - The Field Instability Criterion
  - Possible instability of Field stable systems

H.4 : Thermal Instability

Examine stability of medium in thermal equilibrium (heating=cooling) to perturbations of temperature.

Obviously, this will involve perturbing the energy equation:

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot [(E+p)\mathbf{u}] = \rho \frac{\partial \Psi}{\partial t} - \rho \dot{Q}_{\rm cool} \qquad \textsc{Energy Equation}$$

In fact, to examine thermal instability it will be convenient to derive an alternative form of the energy equation that involves the entropy-like quantity $K = p/\rho^{\gamma}$.

For ideal gas:

$$p = K\rho^\gamma \qquad \Rightarrow \qquad \mathrm{d}p = \rho^\gamma\,\mathrm{d}K + K\gamma\rho^{\gamma-1}\,\mathrm{d}\rho$$
$$= \rho^\gamma\,\mathrm{d}K + \frac{\gamma p}{\rho}\,\mathrm{d}\rho \qquad ①$$

$$p = \frac{\mathcal{R}_*}{\mu}\rho T \qquad \Rightarrow \qquad \mathrm{d}p = \frac{\mathcal{R}_*}{\mu}T\,\mathrm{d}\rho + \frac{\mathcal{R}_*}{\mu}\rho\,\mathrm{d}T$$
$$= \frac{p}{\rho}\,\mathrm{d}\rho + \frac{\mathcal{R}_*}{\mu}\rho\,\mathrm{d}T \qquad ②$$

So…

$$\rho^\gamma\,\mathrm{d}K + \gamma\frac{p}{\rho}\,\mathrm{d}\rho = \frac{p}{\rho}\,\mathrm{d}\rho + \frac{\mathcal{R}_*}{\mu}\rho\,\mathrm{d}T$$
$$\Rightarrow \qquad \rho^\gamma\,\mathrm{d}K = (1-\gamma)\frac{p}{\rho}\,\mathrm{d}\rho + \frac{\mathcal{R}_*}{\mu}\rho\,\mathrm{d}T$$
$$\Rightarrow \qquad \mathrm{d}K = \rho^{1-\gamma}(1-\gamma)\underbrace{\left[\frac{p}{\rho^2}\,\mathrm{d}\rho + \frac{\mathcal{R}_*}{\mu(1-\gamma)}\,\mathrm{d}T\right]}_{-\text{đ}Q}$$

$$\text{đ}Q = p\,\mathrm{d}V + \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}T}\,\mathrm{d}T \qquad \text{(unit mass)}$$

$$\text{đ}Q = p\,\mathrm{d}(1/\rho) + C_V\,\mathrm{d}T$$
$$= -\frac{p}{\rho^2}\,\mathrm{d}\rho - \frac{\mathcal{R}_*}{\mu(1-\gamma)}\,\mathrm{d}T$$

since we have $(\gamma-1)C_V = \mathcal{R}_*/\mu$

Hence,

$$\mathrm{d}K = -(1-\gamma)\rho^{1-\gamma}\,đQ \qquad \text{for fluid element}$$

$$\Rightarrow \qquad \frac{\mathrm{D}K}{\mathrm{D}t} = -(\gamma-1)\rho^{1-\gamma}\dot{Q}$$

$$\Rightarrow \qquad \frac{1}{K}\frac{\mathrm{D}K}{\mathrm{D}t} \equiv \frac{\mathrm{D}}{\mathrm{D}t}(\ln K) = -(\gamma-1)\frac{\rho}{p}\dot{Q}$$

$$\boxed{\frac{1}{K}\frac{\mathrm{D}K}{\mathrm{D}t} = -(\gamma-1)\frac{\rho\dot{Q}}{p}}$$

ENTROPY FORM OF THE ENERGY EQUATION

$$\frac{1}{K}\left(\frac{\partial K}{\partial t} + \mathbf{u}\cdot\nabla K\right) = -(\gamma-1)\frac{\rho\dot{Q}}{p}$$

Back to thermal instability… consider the following system and its equilibrium:

- Ideal gas
- No gravitational field
- Static, thermal equilibrium with uniform density ρ_0 and pressure p_0

$$\mathbf{u}_0 = \mathbf{0}, \dot{Q}_0 = 0, \underbrace{\boldsymbol{\nabla} p_0 = 0, \boldsymbol{\nabla} \rho_0 = 0}_{\boldsymbol{\nabla} K_0 = 0} \qquad \text{where } p = K\rho^\gamma$$

Governing equations:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0$$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = -\boldsymbol{\nabla} p$$

$$\frac{1}{K}\left(\frac{\partial K}{\partial t} + \mathbf{u} \cdot \nabla K\right) = -(\gamma - 1)\frac{\rho \dot{Q}}{p}$$

Introduce perturbations

$$\rho \to \rho_0 + \Delta\rho$$

$$p \to p_0 + \Delta p$$

$$\mathbf{u} \to \Delta\mathbf{u}$$

$$K \to K_0 + \Delta K$$

$$p = K\rho^\gamma \qquad \Rightarrow \qquad \Delta p = \rho_0^\gamma \Delta K + \gamma \frac{p_0}{\rho_0}\Delta\rho \qquad (1)$$

… and linearize equations

$$\frac{\partial \Delta\rho}{\partial t} + \rho_0 \boldsymbol{\nabla}\cdot(\Delta\mathbf{u}) = 0 \qquad (2)$$

$$\rho_0 \frac{\partial \Delta\mathbf{u}}{\partial t} = -\boldsymbol{\nabla}(\Delta p) \qquad (3)$$

$$\frac{\partial \Delta K}{\partial t} = -\frac{\gamma - 1}{\rho_0^{\gamma-1}}\Delta\dot{Q}$$

$$\Delta\dot{Q} = \left.\frac{\partial \dot{Q}}{\partial p}\right|_\rho \Delta p + \left.\frac{\partial \dot{Q}}{\partial \rho}\right|_p \Delta\rho$$

For convenience, write this as

$$\frac{\partial \Delta K}{\partial t} = -A^* \Delta p - B^* \Delta \rho \quad \text{with} \quad A^* = \frac{\gamma - 1}{\rho_0^{\gamma-1}} \left.\frac{\partial \dot{Q}}{\partial p}\right|_\rho , \qquad B^* = \frac{\gamma - 1}{\rho_0^{\gamma-1}} \left.\frac{\partial \dot{Q}}{\partial \rho}\right|_p \qquad (4)$$

Seek plane wave solutions:

$$\Delta p = p_1 e^{\mathbf{i}\mathbf{k}\cdot\mathbf{x}+qt}$$

$$\Delta \rho = \rho_1 e^{\mathbf{i}\mathbf{k}\cdot\mathbf{x}+qt}$$

$$\Delta \mathbf{u} = \mathbf{u}_1 e^{\mathbf{i}\mathbf{k}\cdot\mathbf{x}+qt}$$

$$\Delta K = K_1 e^{\mathbf{i}\mathbf{k}\cdot\mathbf{x}+qt}$$

(2) $\Rightarrow$ $$q\rho_1 + \rho_0 \mathbf{i}\mathbf{k} \cdot \mathbf{u}_1 = 0$$

(3) $\Rightarrow$ $$q\rho_0 \mathbf{u}_1 = -\mathbf{i}\mathbf{k} p_1$$

(4) $\Rightarrow$ $$qK_1 = -A^* p_1 - B^* \rho_1$$

(1) $\Rightarrow$ $$p_1 = \rho_0^\gamma K_1 + \frac{\gamma p_0}{\rho_0} \rho_1$$

Eliminate perturbation amplitudes:

$$\frac{A^* q}{k^2} - \frac{B^*}{q} = -\left(\frac{q^2}{k^2} + \gamma \frac{p_0}{\rho_0}\right) \frac{1}{\rho_0^\gamma}$$

$$\Rightarrow \qquad \underbrace{q^3 + A^* \rho_0^\gamma q^2 + k^2 \gamma \frac{p_0}{\rho_0} q - B^* k^2 \rho_0^\gamma}_{\text{cubic in } q, \text{ call } E(q)} = 0$$

This has at least one real root.

System is unstable if this root is positive, *q*>0. This is the case if *B*>0.*

$$\therefore \qquad B^* = \frac{\gamma - 1}{\rho_0^{\gamma-1}} \left.\frac{\partial \dot{Q}}{\partial \rho}\right|_p > 0 \qquad \text{(instability)}$$

$$\Rightarrow \qquad \left.\frac{\partial \dot{Q}}{\partial \left(\frac{\mu p}{\mathcal{R}_* T}\right)}\right|_p > 0$$

$$\Rightarrow \qquad -\frac{T^2}{p} \left.\frac{\partial \dot{Q}}{\partial T}\right|_p > 0$$

(note that there was an error at this point in an earlier version of the PDF notes)

This finally gives us the Field's Instability criterion:

$$\boxed{\text{unstable if} \qquad \left.\frac{\partial \dot{Q}}{\partial T}\right|_p < 0} \qquad \textsc{Field Criterion}$$

Intuitive… unstable if net cooling decreases for a positive temperature fluctuation. Full analysis informs us that it's the change in cooling at constant pressure that's most important.

All modes (i.e. all k's) are unstable if the system is Field unstable.

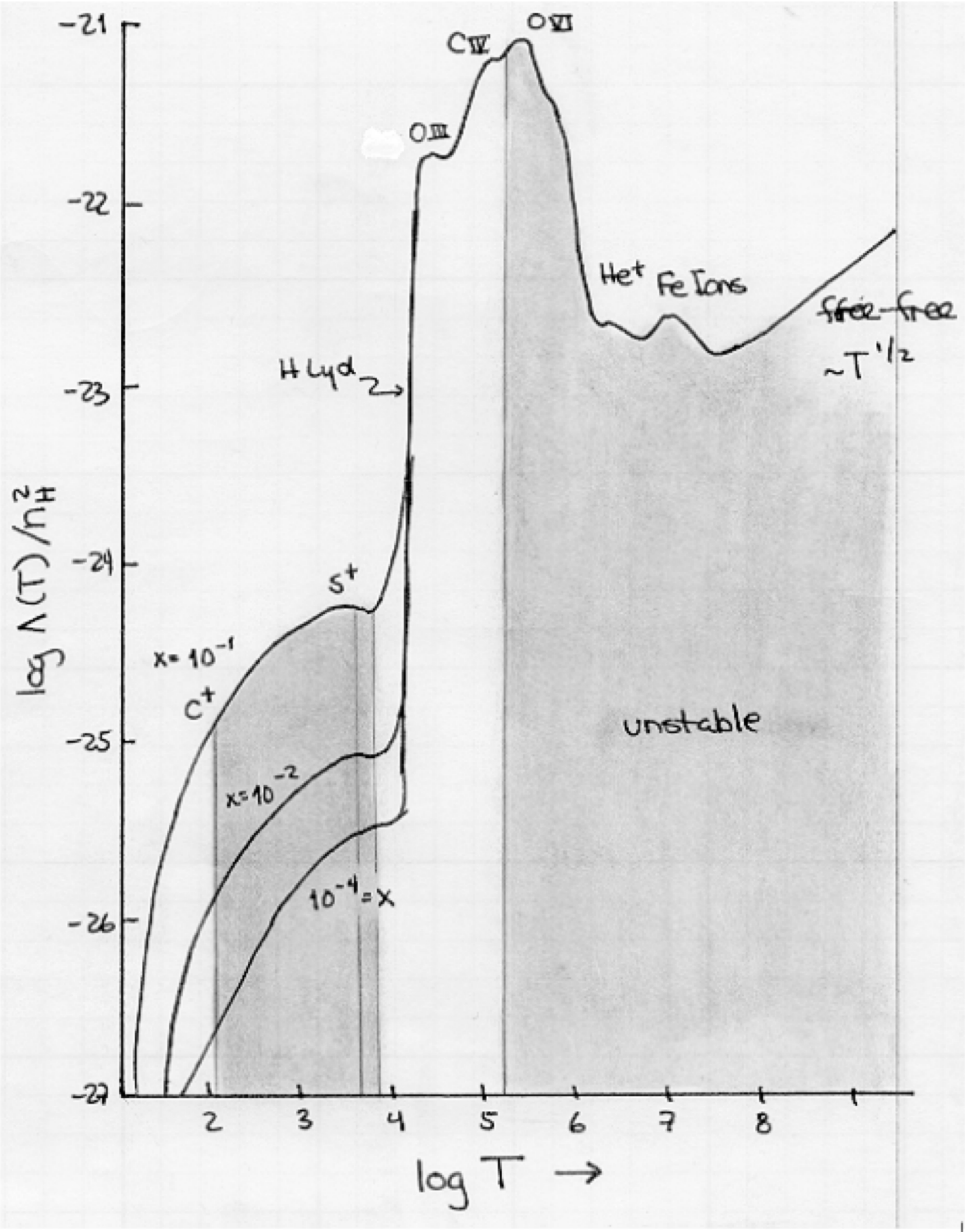
Example : Adopt form of cooling from Lecture 5…

$$\dot{Q} = A\rho T^{\alpha} - H$$
$$= \frac{A\mu}{\mathcal{R}_*} p T^{\alpha-1} - H$$

Bremsstrahlung has α=0.5, so Field unstable!

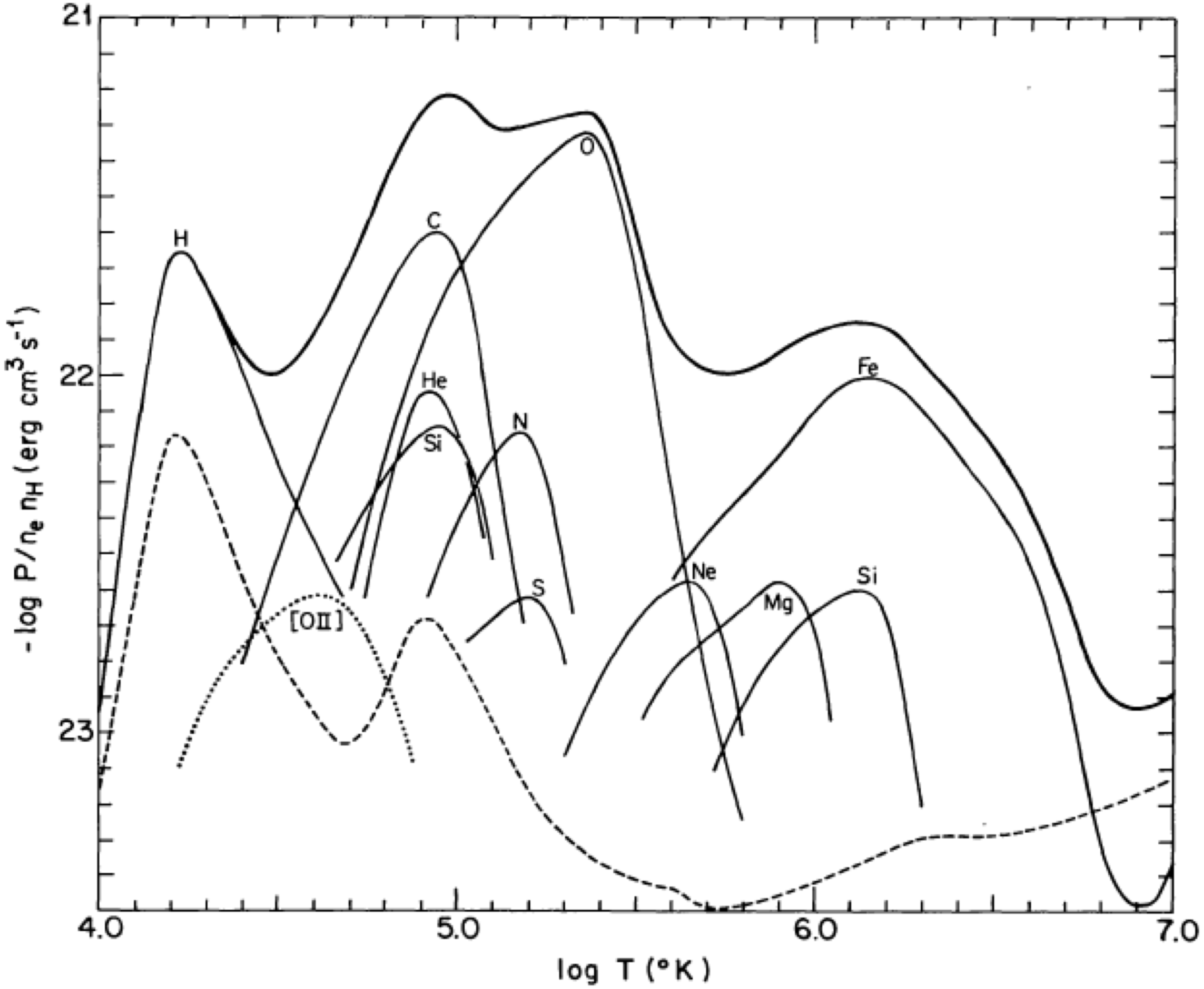
$$\Rightarrow \qquad \left.\frac{\partial \dot{Q}}{\partial T}\right|_p = (\alpha - 1)\frac{A\mu p}{\mathcal{R}_*} T^{\alpha-2}$$

Field unstable, $\left.\partial \dot{Q} \big/ \partial T\right|_p < 0$ if $\alpha < 1$.



Dalgarno & McCray, 1972

atomic and metal line cooling



Gaetz & Salpeter, 1983

In fact, can still have unstable modes even if system is Field stable.

To see this, return to dispersion relation:

$$q^3 + A^* \rho_0^\gamma q^2 + k^2 \gamma \frac{p_0}{\rho_0} q - B^* k^2 \rho_0^\gamma = 0$$

Consider very long wavelength modes, so that k is small. Then this gives

$$q^2 (q + A^* \rho_0^\gamma) \approx 0$$

$$\Rightarrow \qquad q \approx -A^* \rho_0^\gamma$$

So, instability (*q*>0) if

$$A^* < 0$$

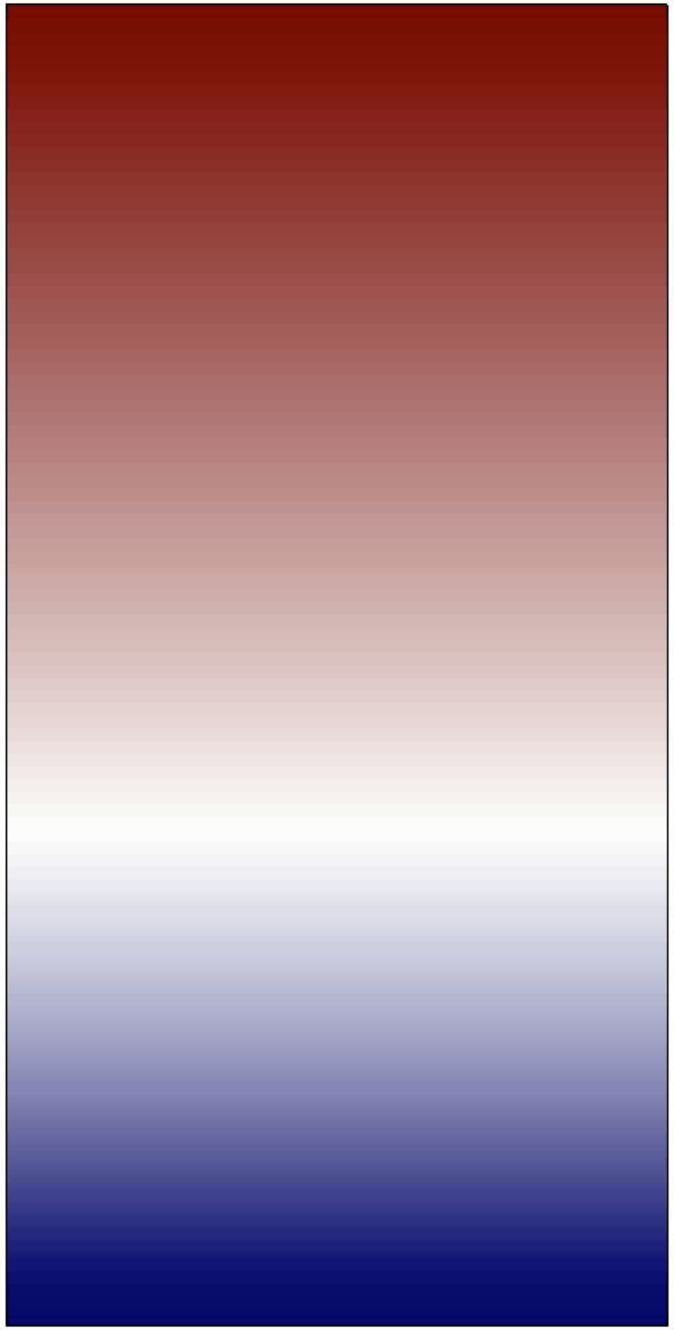
$$\Rightarrow \quad \left. \frac{\partial \dot{Q}}{\partial T} \right|_\rho < 0$$

Interpretation: instability if net cooling rate decreases when temperature increases

- Short wavelength perturbations : readily brought into pressure equilibrium by sound waves, so it's response of $\dot{Q}$ at constant pressure that matters. **Field instability.**
- Long wavelength perturbations : insufficient time for sound waves to equalize pressure, so it's response of $\dot{Q}$ at constant density that matters (sometimes called **isochoric thermal instability**).

Note about gravity:

- Buoyancy interactions with thermal instability; a powerful stabilizing effect.
- Thermal instability can become subtle question of the functional dependence of the cooling/heating balance...



Kunz et al. (2012)


Perseus cluster (Jay Ga