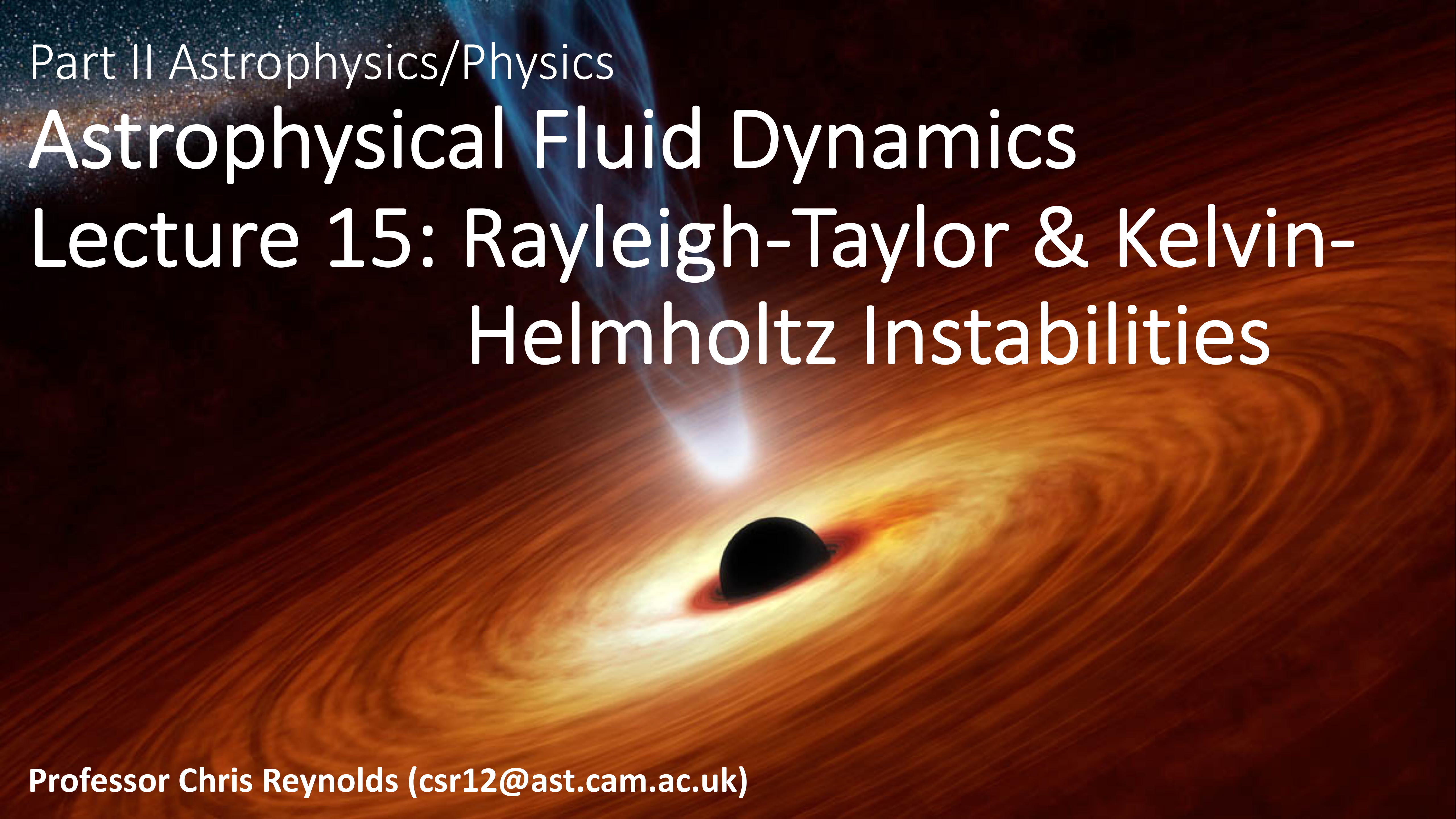

Part II Astrophysics/Physics

Astrophysical Fluid Dynamics

Lecture 15: Rayleigh-Taylor & Kelvin-Helmholtz Instabilities

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Recap – last lecture

- Started discussion of fluid instabilities
- Convective instability – heuristic (fluid element) approach
  - Schwarzschild criterion… instability if entropy decreases upwards

$$\frac{\mathrm{d}K}{\mathrm{d}z} < 0 \qquad \text{(instability)} \qquad\qquad \frac{\mathrm{d}T}{\mathrm{d}z} < \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\frac{\mathrm{d}p}{\mathrm{d}z} \qquad \text{(instability)}$$

- Gravitational instability – linear perturbation theory
  - Gravitationally modified sound waves
  - Instability for wavelengths greater than

$$\lambda_J = \frac{2\pi}{k_J} = \sqrt{\frac{\pi c_s^2}{G\rho_0}}$$

This Lecture

- Fluid Instabilities (cont)
- Briefly revisiting convective instability
- Instability of an interface of two fluids (Chapter H.3)
  - Instability of an interface
  - Problem set up and derivation of dispersion relation
  - Surface waves, Rayleigh-Taylor instability, Kelvin-Helmholtz instability

Revisiting convective instability

Fluid element approach uncovered Schwarzschild criterion… stable if

$$\frac{\mathrm{d}T}{\mathrm{d}z} < \left(1 - \frac{1}{\gamma}\right) \frac{T}{p} \frac{\mathrm{d}p}{\mathrm{d}z} \qquad \text{(instability)}$$

BUT, why wasn't this uncovered by our linear stability analysis when we examined sound waves in a stratified atmosphere?

Ans : that analysis restricted itself to waves that had $\mathbf{k} = k\hat{\mathbf{z}}$

- Thus, all fluid elements displaced upwards simultaneously
- Approach of comparing density of displaced fluid element to "unperturbed" background doesn't make sense.

To examine convective instability with linear theory, need to include perturbations with components in the horizontal direction.

Can then derive a cumbersome dispersion relation that has roots corresponding to the sound waves, and another root corresponding to convective stability.

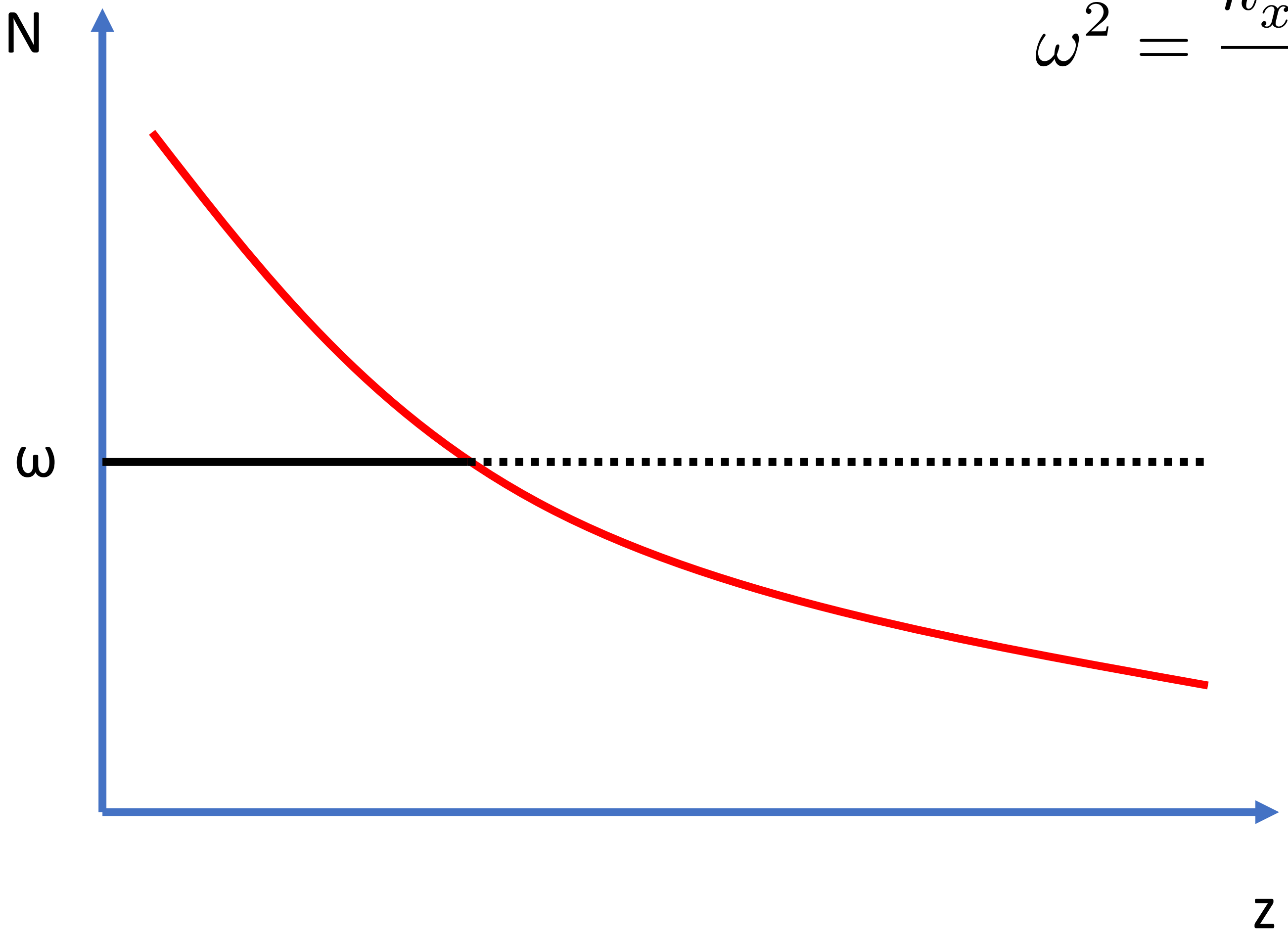
We can “filter out” the sound waves from out analysis (isolating the convection problem) by applying the *Boussinesq approximation*… neglect density perturbations except where they are multiplied by g.

Resulting dispersion relation: if $\mathbf{g} = -g\hat{\mathbf{z}}$, then

$$\omega^2 = \frac{k_x^2 + k_y^2}{k^2} N^2$$

$$N^2 = \frac{g}{T}\left[\frac{\mathrm{d}T}{\mathrm{d}z} - \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\frac{\mathrm{d}p}{\mathrm{d}z}\right] \qquad \text{Brunt-Väisälä Frequency}$$

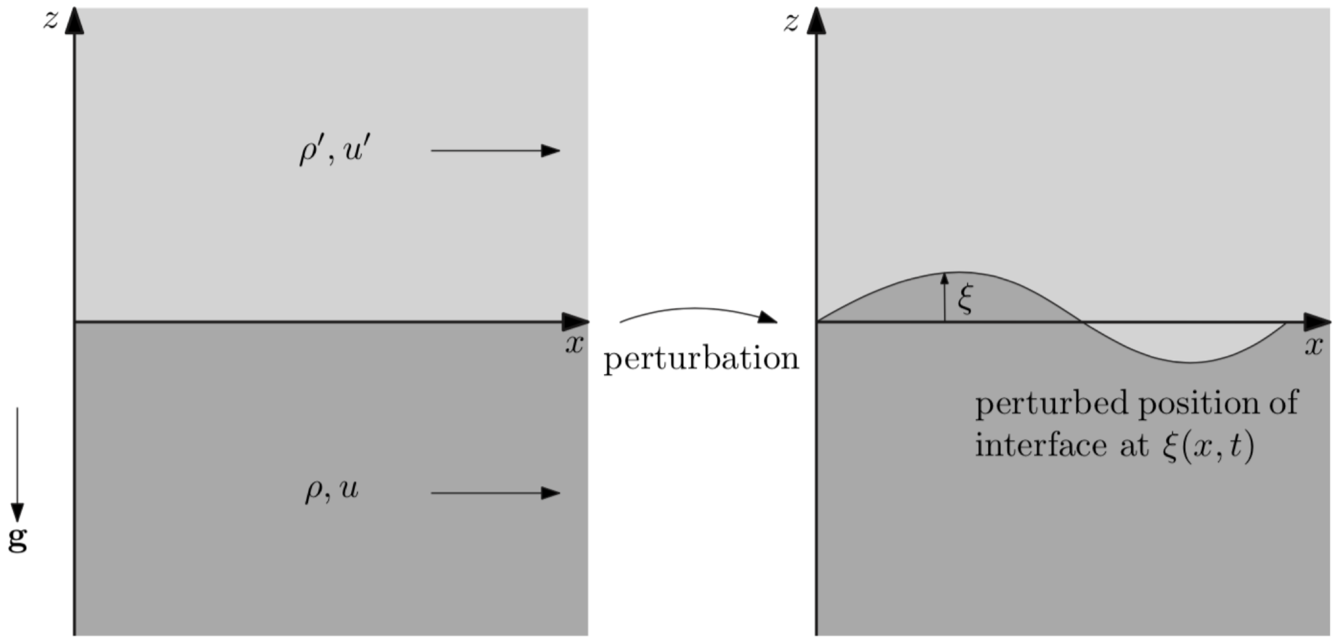
If N decreases upwards (common situation), this can lead to trapping of internal gravity waves.



$$\omega^2 = \frac{k_x^2 + k_y^2}{k^2} N^2$$

H.3: Stability of fluid interfaces

Examine the stability of an interface with discontinuous change of density or tangential velocity.



For convenience, let's assume:

- Constant gravity
- Ideal fluid
- Irrotational flow
- Incompressible
- 2-dimensional

$$\mathbf{u} = -\nabla\Phi$$

$$\nabla^2\Phi = 0$$

Write down momentum equation for each fluid:

$$
\begin{aligned}
&\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \mathbf{g} \\
\Rightarrow\quad & -\boldsymbol{\nabla}\frac{\partial \Phi}{\partial t} + \boldsymbol{\nabla}\left(\frac{1}{2}u^2\right) = -\underbrace{\boldsymbol{\nabla}\left(\frac{p}{\rho}\right)}_{\substack{\text{since } \rho=\text{const.}\\ \text{within each fluid}}} - \boldsymbol{\nabla}\Psi \\
\Rightarrow\quad & \boldsymbol{\nabla}\left[-\frac{\partial \Phi}{\partial t} + \frac{1}{2}u^2 + \frac{p}{\rho} + \Psi\right] = 0 \\
\Rightarrow\quad & -\frac{\partial \Phi}{\partial t} + \frac{1}{2}u^2 + \frac{p}{\rho} + \Psi = F(t) \qquad (*)
\end{aligned}
$$

Let unperturbed fluid velocities in lower and upper fluids be *U* and *U’* in *x*-direction. We can write split the velocity potentials into unperturbed and perturbed pieces:

$$
\begin{aligned}
\Phi_{\text{low}} &= -Ux + \phi \\
\Phi_{\text{up}} &= -U'x + \phi'
\end{aligned}
\qquad \text{where} \qquad \nabla^2\phi = \nabla^2\phi' = 0 \qquad (\text{since } \boldsymbol{\nabla}\cdot\mathbf{u} = 0) \qquad ①
$$

The ϕ and ϕ' fields are sourced by displacements of the boundary $\xi(x,t)$.

We have:

$$u_z = \frac{\mathrm{D}\xi}{\mathrm{D}t} \quad \Rightarrow \quad \left.\begin{aligned} -\frac{\partial \phi}{\partial z} &= \frac{\partial \xi}{\partial t} + U\frac{\partial \xi}{\partial x} \\ -\frac{\partial \phi'}{\partial z} &= \frac{\partial \xi}{\partial t} + U'\frac{\partial \xi}{\partial x} \end{aligned}\right\} \quad \text{to first order} \qquad ②$$

Look for plane wave solutions:

$$\xi = Ae^{\mathrm{i}(kx-\omega t)} \qquad \nabla^2\phi = 0 \quad \Rightarrow \quad -k^2 + k_z^2 = 0$$

$$\phi = Ce^{\mathrm{i}(kx-\omega t)+k_z z} \qquad \Rightarrow \quad k_z = |k|$$

since $\phi \to 0$ as $z \to -\infty$

$$\phi' = C'e^{\mathrm{i}(kx-\omega t)+k_z' z}$$

$$\nabla^2\phi' = 0 \quad \Rightarrow \quad -k^2 + {k_z'}^2 = 0$$

$$\Rightarrow \quad k_z' = -|k|$$

since $\phi' \to 0$ as $z \to \infty$.

So,

$$\phi = Ce^{\mathrm{i}(kx-\omega t)+kz}$$

$$\phi' = C'e^{\mathrm{i}(kx-\omega t)-kz}$$

$$-\frac{\partial\phi}{\partial z} = \frac{\partial\xi}{\partial t} + U\frac{\partial\xi}{\partial x}$$

$$-\frac{\partial\phi'}{\partial z} = \frac{\partial\xi}{\partial t} + U'\frac{\partial\xi}{\partial x}.$$

Substitute into (2):

$$-kC = -\mathrm{i}\omega A + \mathrm{i}UkA = \mathrm{i}(kU-\omega)A \qquad ③$$

$$kC' = \mathrm{i}(kU'-\omega)A \qquad ④$$

Got three unknowns and two equations… need one more equation. We haven't yet used the fact that the **pressure is continuous** across interface. Let's use (*) to get

$$p = -\rho\left(-\frac{\partial\phi}{\partial t} + \frac{1}{2}u^2 + g\xi\right) + \rho F(t)$$

$$p' = -\rho'\left(-\frac{\partial\phi'}{\partial t} + \frac{1}{2}u'^2 + g\xi\right) + \rho' F'(t)$$

$$-\frac{\partial\Phi}{\partial t} + \frac{1}{2}u^2 + \frac{p}{\rho} + \Psi = F(t)$$

Set equal at interface z=0:

$$\rho\left(-\frac{\partial\phi}{\partial t}+\frac{u^2}{2}+g\xi\right)=\rho'\left(-\frac{\partial\phi'}{\partial t}+\frac{u'^2}{2}+g\xi\right)+K(t) \qquad ⑤$$

$$K\equiv\rho F(t)-\rho' F'(t)$$

Can examine conditions at infinite to determine that K(t) is actually a constant:

$$\rho F(t)-\rho' F'(t)=\underbrace{\frac{1}{2}U^2\rho-\frac{1}{2}U'^2\rho'}_{\substack{\text{conditions at }\infty\\ \text{and so a constant}}}$$

To make progress on (5), need to determine u^2:

$$\mathbf{u}=-\boldsymbol{\nabla}\Phi=-\boldsymbol{\nabla}(-Ux+\phi)=U\hat{\mathbf{x}}-\boldsymbol{\nabla}\phi$$

$$\Rightarrow \qquad u^2=U^2-2U\frac{\partial\phi}{\partial x} \qquad \text{(dropping 2nd order terms)}$$ and $$u'^2=U'^2-2U'\frac{\partial\phi'}{\partial x}$$

Substitute into (5):

$$\rho\left(-\frac{\partial\phi}{\partial t}+\frac{u^2}{2}+g\xi\right)=\rho'\left(-\frac{\partial\phi'}{\partial t}+\frac{u'^2}{2}+g\xi\right)+K(t)$$

$$\rho\left(-\frac{\partial\phi}{\partial t}+\cancel{\frac{1}{2}U^2}-U\frac{\partial\phi}{\partial x}+g\xi\right)=\rho'\left(-\frac{\partial\phi'}{\partial t}+\cancel{\frac{1}{2}U'^2}-U'\frac{\partial\phi'}{\partial x}+g\xi\right)+\underbrace{\cancel{\frac{1}{2}U^2\rho}-\cancel{\frac{1}{2}U'^2\rho'}}_{K}$$

$$\Rightarrow\qquad\rho\left(-\frac{\partial\phi}{\partial t}-U\frac{\partial\phi}{\partial x}+g\xi\right)=\rho'\left(-\frac{\partial\phi'}{\partial t}-U'\frac{\partial\phi'}{\partial x}+g\xi\right)$$

$$\Rightarrow\qquad\rho\mathrm{i}\omega C-\rho U\mathrm{i}kC+\rho gA=\rho'\mathrm{i}\omega C'-\rho'U'\mathrm{i}kC'+\rho'gA$$

$$\Rightarrow\qquad\rho(kU-\omega)C+\mathrm{i}\rho gA=\rho'(kU'-\omega)C'+\mathrm{i}\rho'gA$$

Finally, we can combine this with eqns (3) and (4) to eliminate *C, C′* and *A*.

$$\boxed{\rho(kU-\omega)^2+\rho'(kU'-\omega)^2=kg(\rho-\rho')}$$

$$-kC=-\mathrm{i}\omega A+\mathrm{i}UkA=\mathrm{i}(kU-\omega)A$$

$$kC'=\mathrm{i}(kU'-\omega)A$$

This is the dispersion relation for our problem:

$$\boxed{\rho(kU-\omega)^2+\rho'(kU'-\omega)^2=kg(\rho-\rho')}$$

Case I : Two fluids at rest (*U=U'=0*) with $\rho' < \rho$ (heavy fluid on bottom)

$$\omega^2(\rho+\rho')=kg(\rho-\rho')$$

$$\Rightarrow \qquad \omega^2=k\frac{g(\rho-\rho')}{\rho+\rho'}$$

So, if $k \in \mathbb{R}$ then $\omega \in \mathbb{R}$… so oscillatory/wave solutions.

Phase speed

$$\frac{\omega}{k}=\pm\sqrt{\frac{g}{k}\frac{\rho-\rho'}{\rho+\rho'}}=\underbrace{f(k)}_{\substack{\text{waves are}\\ \text{dispersive}}}$$

These are **<u>surface gravity waves.</u>**

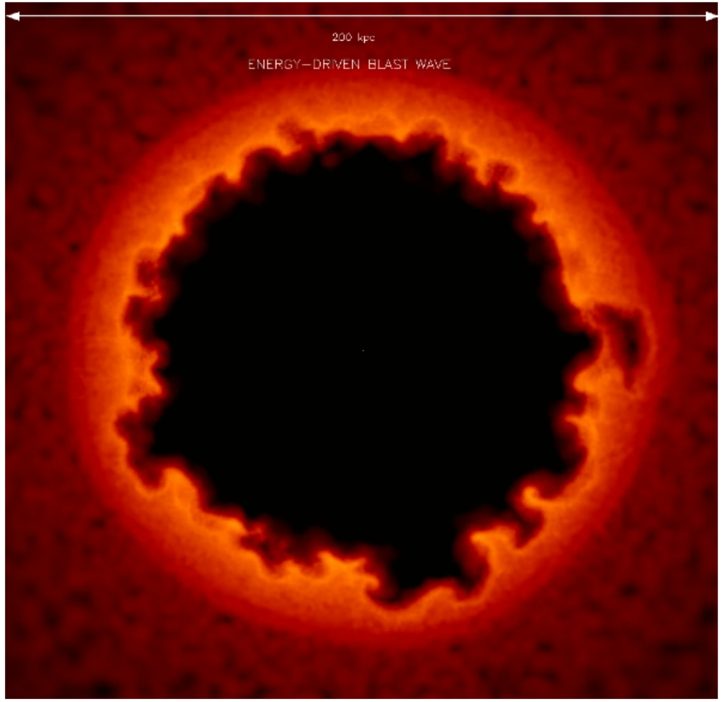
Case II : Two fluids at rest ($U=U'=0$) with $\rho < \rho'$ (heavy fluid on top)

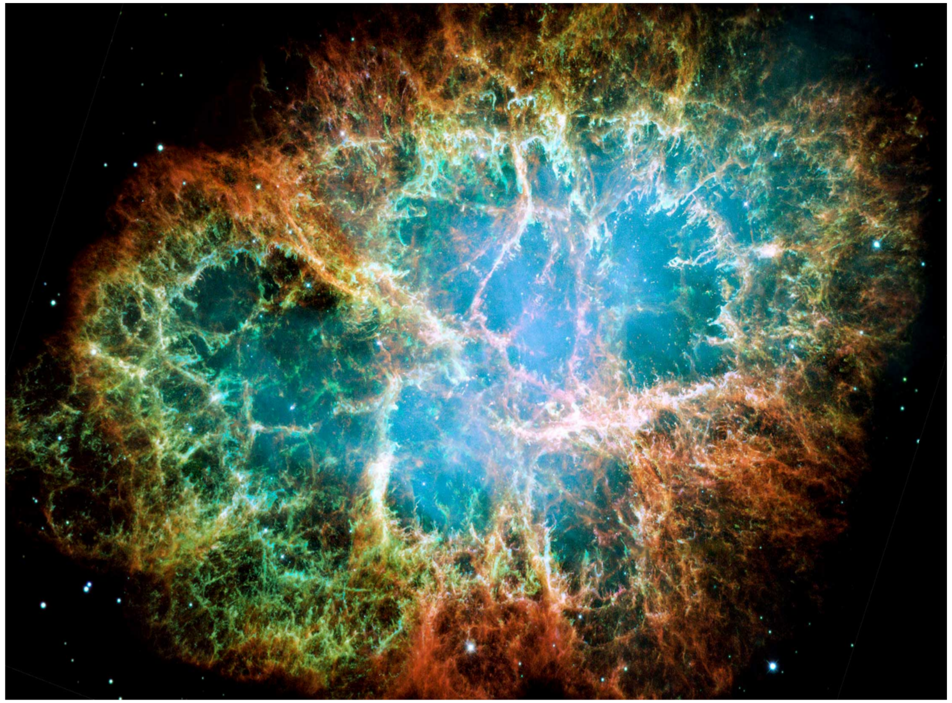
$$\omega^2 = k\frac{g(\rho-\rho')}{\rho+\rho'} \qquad \Rightarrow \qquad \frac{\omega}{k} = \pm \mathrm{i}\sqrt{\frac{g}{k}\frac{\rho'-\rho}{\rho+\rho'}}$$

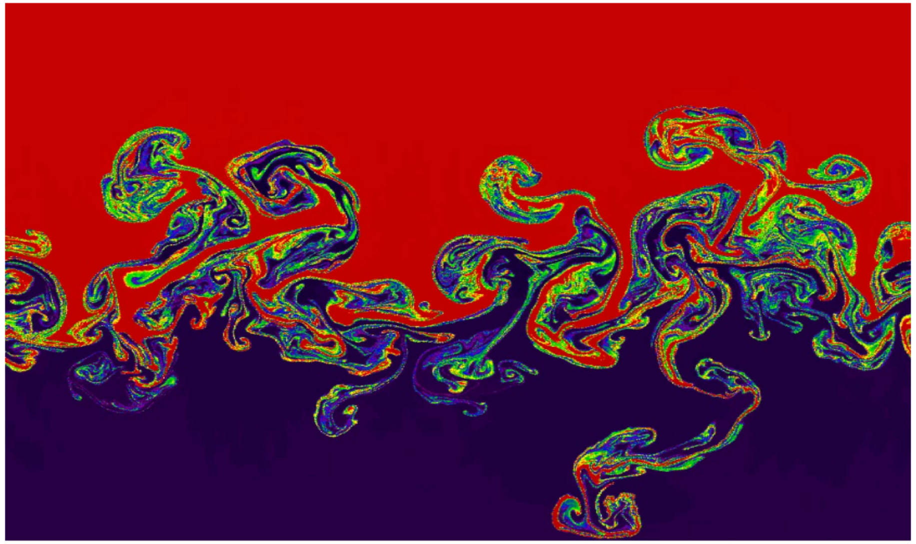
So, if $k \in \mathbb{R}$ then ω is pure imaginary… so exponentially growing/decaying solutions.

This is the **Rayleigh-Taylor Instability.**

RT instability







Case III : Fluids in motion with different velocities, $\rho' < \rho$ so RT stable…

$$\rho(kU - \omega)^2 + \rho'(kU' - \omega)^2 = kg(\rho - \rho')$$

$$\Rightarrow \qquad \frac{\omega}{k} = \frac{\rho U + \rho' U'}{\rho + \rho'} \pm \sqrt{\frac{g}{k}\frac{\rho - \rho'}{\rho + \rho'} - \frac{\rho\rho'(U - U')^2}{(\rho + \rho')^2}}$$

$$\frac{g}{k}\frac{\rho - \rho'}{\rho + \rho'} - \frac{\rho\rho'(U - U')^2}{(\rho + \rho')^2} < 0 \qquad \text{(instability)}$$

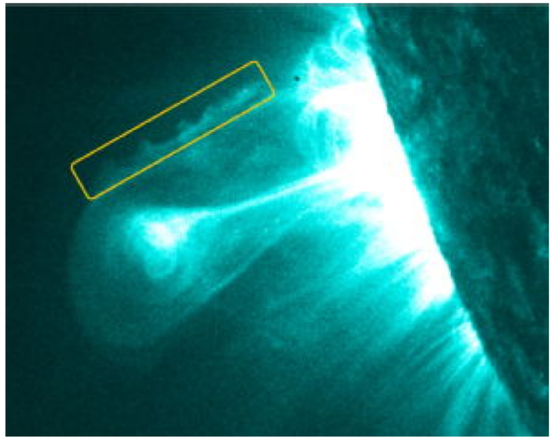
If g=0, any relative motion gives instability… the **Kelvin-Helmholtz Instability.**

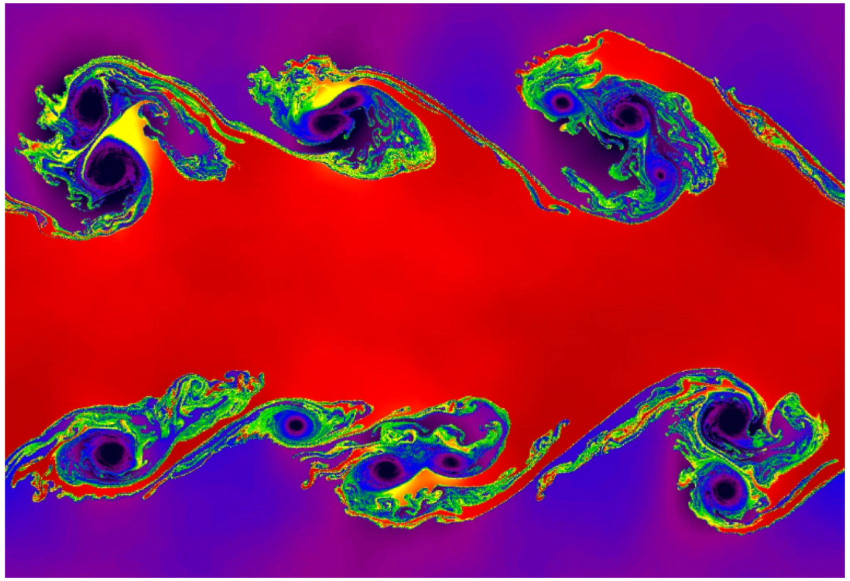
If g≠0, unstable modes have

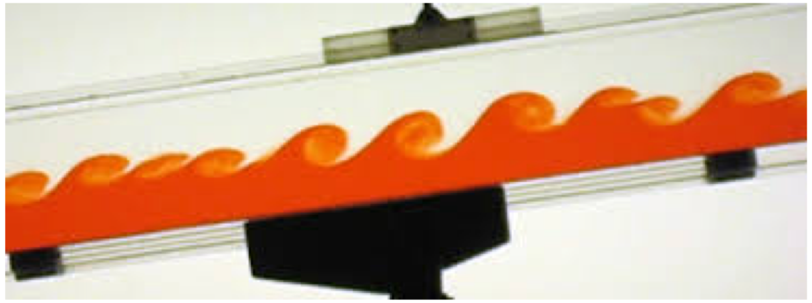
$$k > \frac{(\rho^2 - \rho'^2)g}{\rho\rho'(U - U')^2}$$

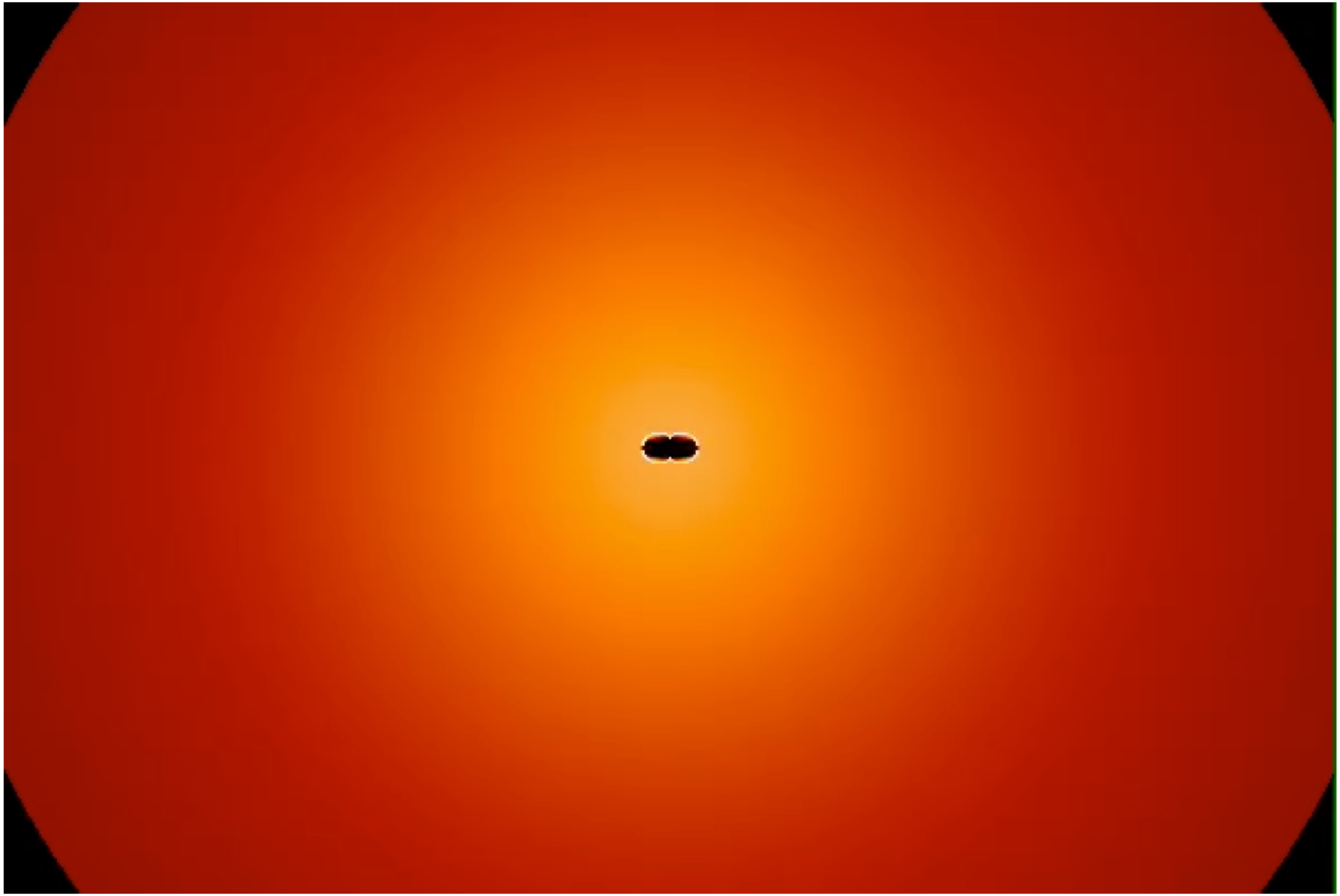
So, sufficiently long-wavelength modes are stabilized by gravity.

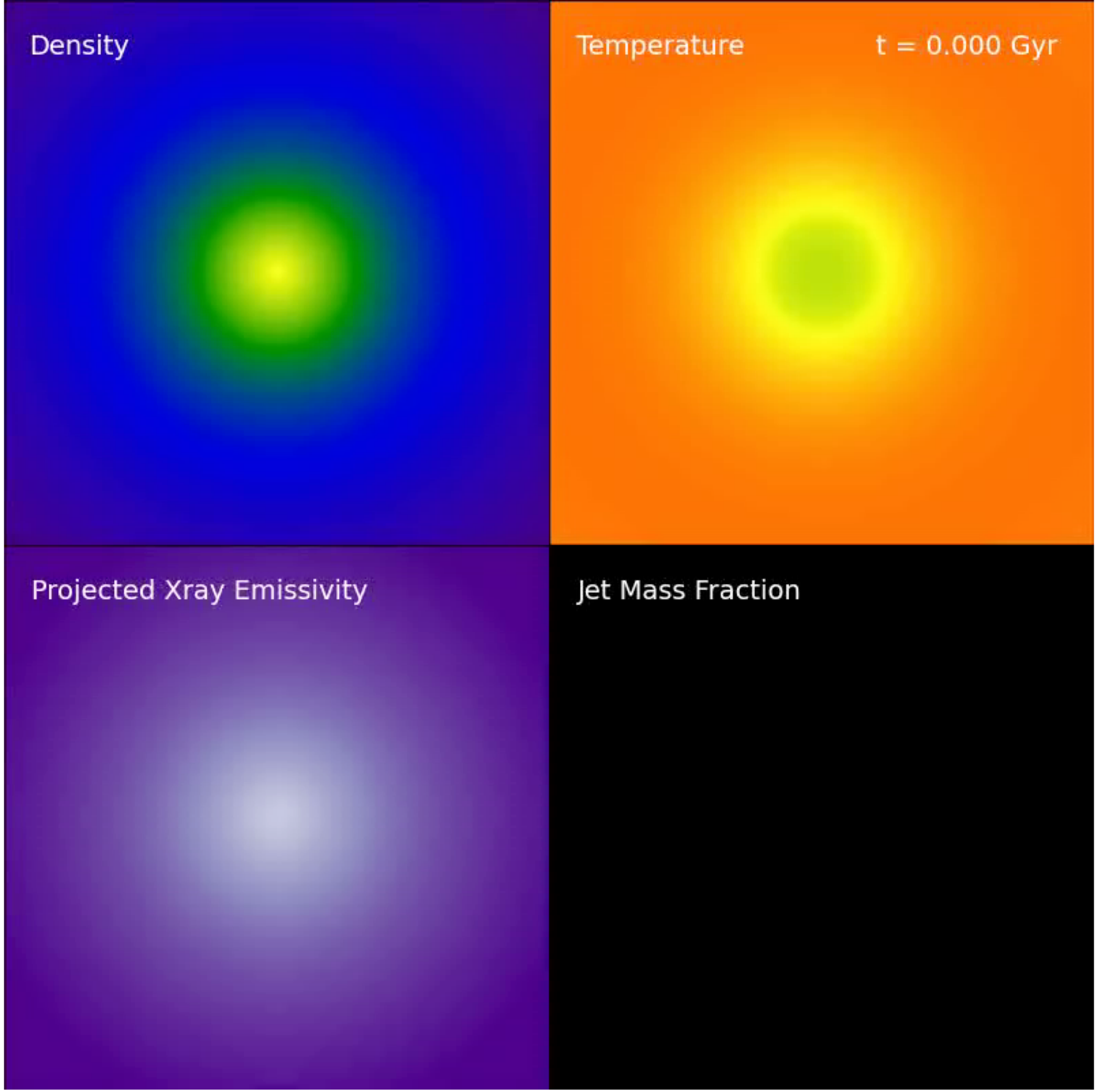
KH instability











Yang & Reynolds (2016)