Part II Astrophysics/Physics

Astrophysical Fluid Dynamics
Lecture 14: Convective & Gravitational Instability

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Recap – taking stock

- Formulation of basic fluid equations
- Hydrostatic equilibrium and stellar structure
- Sound waves – linear perturbation theory of simple configurations
- Shocks and supersonic flows – Rankine-Hugoniot relations
- Steady-state transonic flows – critical-point systems
    - Nozzles (jets), accretion flows and winds.

This Lecture – Fluid Instabilities (Chapter H)

- Convective instability (H.1)
  - Heuristic (fluid element) approach
  - Conditions for instability
  - Internal gravity waves
- Gravitational instability (H.2)
  - Jeans analysis (linear perturbation theory)
  - Jeans mass and Jeans length

- By end of this lecture, you can do…
  - All of Examples Sheet 1, 2 and 3.

Chapter H : Fluid Instabilities

Suppose we have a fluid in a state of equilibrium…

- flow obeys all relevant fluid equations with $\partial/\partial t=0$.
- can be dynamical equilibrium with $\mathbf{u} \neq 0$.

Now introduce (small) perturbation into the flow…

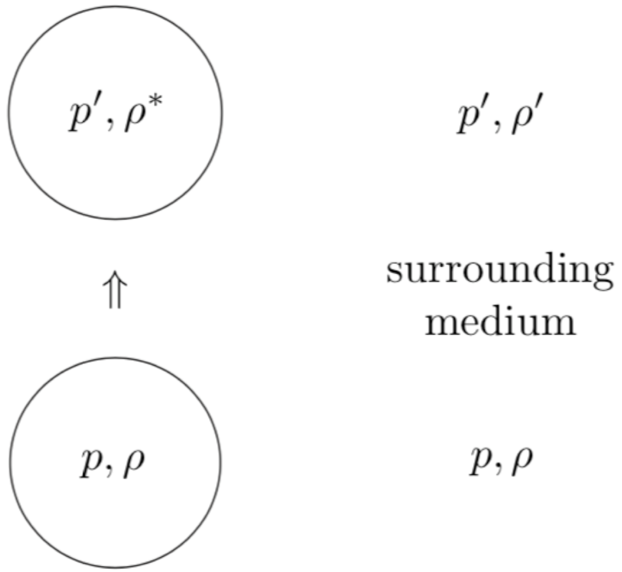
- Perturbation grows in time $\Longrightarrow$ configuration (linearly) unstable
- Perturbation decays in time or undergoes SHM $\Longrightarrow$ configuration stable
- (Perturbation undergoes growing oscillations $\Longrightarrow$ configuration overstable)

H.1 : Convective Instability

Convective instability concerns the stability of a hydrostatic equilibrium.

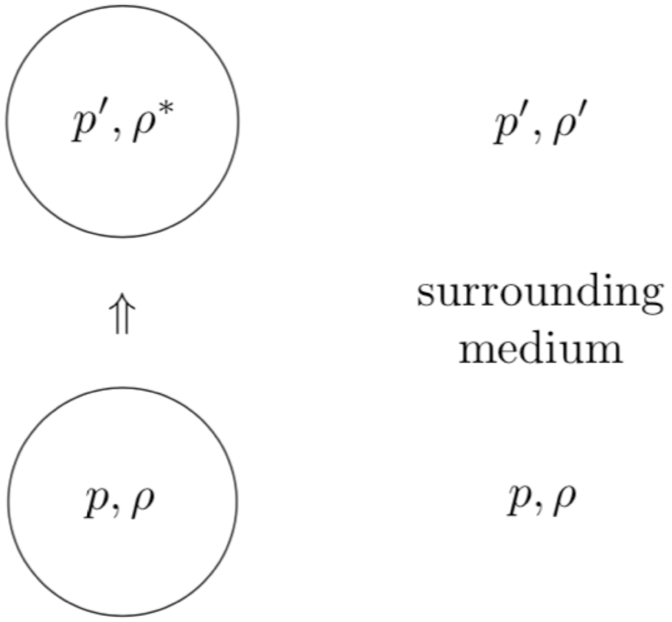
We can understand instability without doing full formal perturbation analysis.

- Consider an ideal gas in hydrostatic equilibrium
- Now perturb a fluid element upwards



- Perturbation is slow enough that fluid element remains in pressure balance with surroundings.
- Perturbation is fast enough that the fluid element cannot exchange heat with surroundings
- … so density evolves adiabatically.

Stability depends on new value of density…



$$\left.\begin{aligned} p &= K\rho^\gamma \\ p' &= K\rho^{*\gamma} \end{aligned}\right\} \quad \Rightarrow \quad \rho^* = \rho \left(\frac{p'}{p}\right)^{1/\gamma}$$

$\rho^* < \rho' \quad \Rightarrow$ perturbed element buoyant

$\Rightarrow$ system unstable;

$\rho^* > \rho' \quad \Rightarrow$ perturbed element sinks back

$\Rightarrow$ system stable.

- For displaced fluid element:

$$p' = p + \frac{\mathrm{d}p}{\mathrm{d}z}\delta z$$

$$\Rightarrow \qquad \rho^* = \rho\left(\frac{p + \frac{\mathrm{d}p}{\mathrm{d}z}\delta z}{p}\right)^{1/\gamma}$$
$$= \rho\left(1 + \frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}z}\delta z\right)^{1/\gamma}$$
$$\approx \rho + \frac{\rho}{p\gamma}\frac{\mathrm{d}p}{\mathrm{d}z}\delta z$$

- For background atmosphere

$$\rho' = \rho + \frac{\mathrm{d}\rho}{\mathrm{d}z}\delta z$$

So, unstable if

$$\rho + \frac{\rho}{p\gamma}\frac{\mathrm{d}p}{\mathrm{d}z}\delta z < \rho + \frac{\mathrm{d}\rho}{\mathrm{d}z}\delta z$$
$$\Rightarrow \qquad \frac{\rho}{p\gamma}\frac{\mathrm{d}p}{\mathrm{d}z} < \frac{\mathrm{d}\rho}{\mathrm{d}z}$$
$$\Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}z}\ln p < \gamma\frac{\mathrm{d}}{\mathrm{d}z}\ln\rho$$
$$\Rightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}z}\left(\ln p\rho^{-\gamma}\right) < 0$$
$$\Rightarrow \qquad \frac{\mathrm{d}K}{\mathrm{d}z} < 0 \qquad \text{(instability)}$$

i.e. system is convective unstable if entropy decreases in the upwards direction… **Schwarzschild criterion**

Can express the Schwarzschild criterion in terms of temperature gradients:

$$K = p\rho^{-\gamma} = (\text{const.})p^{1-\gamma}T^{\gamma}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\ln K = (1-\gamma)\frac{\mathrm{d}}{\mathrm{d}z}\ln p + \gamma\frac{\mathrm{d}}{\mathrm{d}z}\ln T < 0$$

$$\Rightarrow \qquad \frac{\mathrm{d}T}{\mathrm{d}z} < \left(1-\frac{1}{\gamma}\right)\frac{T}{p}\frac{\mathrm{d}p}{\mathrm{d}z} \qquad \text{(instability)}$$

So we are Schwarzschild **stable** if

$$\boxed{\frac{\mathrm{d}T}{\mathrm{d}z} > \left(1-\frac{1}{\gamma}\right)\frac{T}{p}\frac{\mathrm{d}p}{\mathrm{d}z}}$$

(of course, dp/dz<0 given the need for hydrostatic equilibrium)

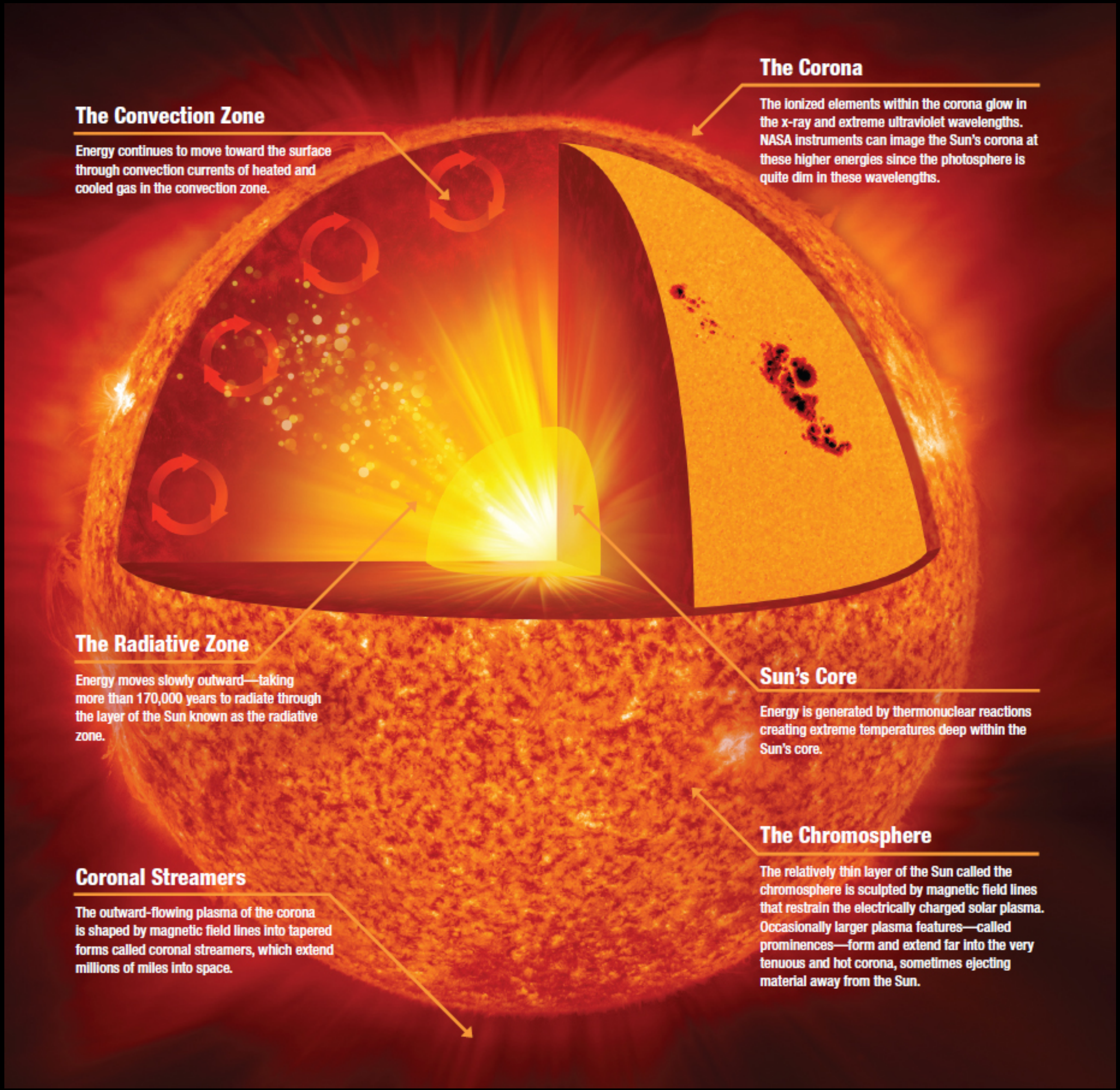
Positive temperature gradients are always stable (since $\gamma > 1$).

Can tolerate a moderate temperature inversion, but instability sets in if dT/dz becomes too negative

2-d simulation of convection

3-d simulation of solar convection

Time-lapse of solar surface


The Convection Zone
Energy continues to move toward the surface through convection currents of heated and cooled gas in the convection zone.
The Corona
The ionized elements within the corona glow in the x-ray and extreme ultraviolet wavelengths. NASA instruments can image the Sun's corona at these higher energies since the photosphere is quite dim in these wavelengths.
The Radiative Zone
Energy moves slowly outward—taking more than 170,000 years to radiate through the layer of the Sun known as the radiative zone.
Sun's Core
Energy is generated by thermonuclear reactions creating extreme temperatures deep within the Sun's core.
The Chromosphere
The relatively thin layer of the Sun called the chromosphere is sculpted by magnetic field lines that restrain the electrically charged solar plasma. Occasionally larger plasma features—called prominences—form and extend far into the very tenuous and hot corona, sometimes ejecting material away from the Sun.
Coronal Streamers
The outward-flowing plasma of the corona is shaped by magnetic field lines into tapered forms called coronal streamers, which extend millions of miles into space.

If the atmosphere is convectively stable, the fluid element will just undergo oscillations

$$\rho^* \frac{\mathrm{d}^2}{\mathrm{d}t^2}\delta z = -g\left(\rho^* - \rho'\right) \qquad \text{(Eqn of motion for the fluid element)}$$

$$\Rightarrow \qquad (\rho + \cancel{\delta\rho})^{\text{small}} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\delta z = -g\left[\frac{\rho}{T}\frac{\mathrm{d}T}{\mathrm{d}z} - \left(1 - \frac{1}{\gamma}\right)\frac{\rho}{p}\frac{\mathrm{d}p}{\mathrm{d}z}\right]\delta z$$

$$\Rightarrow \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\delta z = -\frac{g}{T}\left[\frac{\mathrm{d}T}{\mathrm{d}z} - \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\frac{\mathrm{d}p}{\mathrm{d}z}\right]\delta z$$

So, SHM with frequency

$$N^2 = \frac{g}{T}\left[\frac{\mathrm{d}T}{\mathrm{d}z} - \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\frac{\mathrm{d}p}{\mathrm{d}z}\right] \qquad \text{Brunt-Väisälä Frequency}$$

These are called **internal gravity waves** (restoring force is gravity).

We will return to internal gravity waves after we've gained a little more experience with perturbation theory.

H.2 : Gravitational Instability

This concerns the stability of a medium against gravitational collapse. Consider:

- Uniform medium that is initially static
- Barotropic equation of state
- Gravitational field generated by medium

Follow the standard "linear perturbation analysis playbook"…

Step 1 : Define the equilibrium

$$p = p_0, \;\; \text{const.}$$
$$\rho = \rho_0, \;\; \text{const.}$$
$$\mathbf{u} = \mathbf{0}$$

Step 2 : write down governing equations for the system

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0$$
$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p - \boldsymbol{\nabla}\Psi$$
$$\nabla^2 \Psi = 4\pi G \rho$$

Step 3 : introduce small perturbations

$$p = p_0 + \Delta p$$
$$\rho = \rho_0 + \Delta \rho$$
$$\mathbf{u} = \Delta \mathbf{u}$$
$$\Psi = \Psi_0 + \Delta \Psi$$

Step 4 : Linearize the equations (i.e. expand and neglect all 2nd order and higher terms).

$$\frac{\partial \Delta\rho}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot (\Delta \mathbf{u}) = 0 \qquad ①$$

$$\frac{\partial \Delta \mathbf{u}}{\partial t} = -\frac{\mathrm{d}p}{\mathrm{d}\rho}\frac{1}{\rho_0}\boldsymbol{\nabla}(\Delta\rho) - \boldsymbol{\nabla}(\Delta\Psi) = -c_s^2 \frac{\boldsymbol{\nabla}(\Delta\rho)}{\rho_0} - \boldsymbol{\nabla}(\Delta\Psi) \qquad ②$$

$$\nabla^2(\Delta\Psi) = 4\pi G \Delta\rho \qquad ③$$

Step 5 : Assume plane wave form for perturbations and substitute into linear eqns

$$\Delta\rho = \rho_1 e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - \omega t)}$$

$$\Delta\Psi = \Psi_1 e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - \omega t)}$$

$$\Delta\mathbf{u} = \mathbf{u}_1 e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - \omega t)}.$$

$$① \Rightarrow \quad -\omega\rho_1 + \rho_0 \mathbf{k} \cdot \mathbf{u}_1 = 0 \qquad ④$$

$$② \Rightarrow \quad -\rho_0 \omega \mathbf{u}_1 = -c_s^2 \mathbf{k} \rho_1 - \rho_0 \mathbf{k} \Psi_1 \qquad ⑤$$

$$③ \Rightarrow \quad -k^2 \Psi_1 = 4\pi G \rho_1. \qquad ⑥$$

Step 6 : solve algebraic system to derive dispersion relation $f(\omega, k) = 0$.

$$④ + ⑤ \quad \Rightarrow \quad \rho_1 \omega^2 = k^2(\rho_1 c_s^2 + \rho_0 \Psi_1)$$
$$= k^2 \rho_1 c_s^2 - 4\pi G \rho_0 \rho_1 \qquad \text{from } ⑥$$
$$\Rightarrow \quad \omega^2 = c_s^2 \left(k^2 - \frac{4\pi G \rho_0}{c_s^2}\right).$$

Define the Jeans wavenumber (inverse of the Jeans length) $k_J^2 = 4\pi G \rho_0 / c_s^2$

Then,

$$\boxed{\omega^2 = c_s^2 (k^2 - k_J^2)}$$

1. For $k \gg k_J$… we have $\omega^2 = c_s^2 k^2$, so normal dispersion-free sound waves.
2. For $k \gtrsim k_J$… we have gravitationally-modified sound waves (group vel. < c_s)
3. For $k < k_J$… if $k \in \mathbb{R}$, then $\omega^2 < 0$ so $\omega = \pm i\tilde{\omega}$ with $\tilde{\omega} \in \mathbb{R}$. So,

$$e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - \omega t)} = e^{\tilde{\omega} t} e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$$

Thus, perturbation grows exponentially… **gravitational instability.**

Maximal stable wavelength is

$$\lambda_J = \frac{2\pi}{k_J} = \sqrt{\frac{\pi c_s^2}{G\rho_0}} \qquad \textsc{Jeans Length}$$

Associated mass scale is

$$M_J \sim \rho_0 \lambda_J^3 \qquad \textsc{Jeans Mass}$$

So system will undergo **gravitational collapse** if its mass exceeds Jeans mass.

Consider a collapsing system, and suppose that the collapse is isothermal (cooling and heating processes balance). Noting that $M_J \propto c_s^3 \rho_0^{-1/2} \propto (T^3/\rho_0)^{1/2}$, we see

⇒ Jeans mass decreases as system collapses

⇒ system undergoes **gravitational fragmentation.**

Key physics in star and galaxy formation.

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