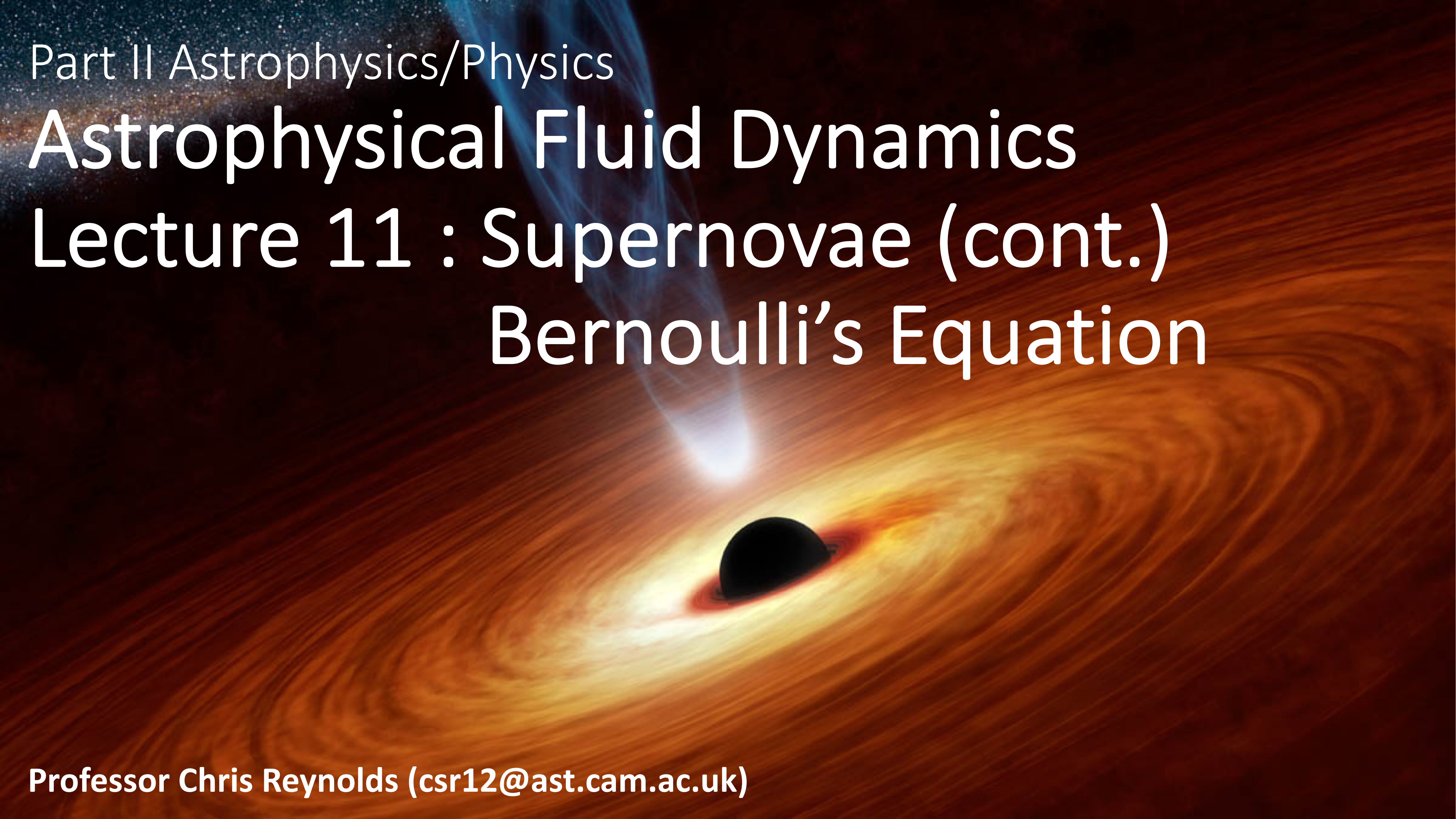


Part II Astrophysics/Physics

Astrophysical Fluid Dynamics

Lecture 11 : Supernovae (cont.)
Bernoulli’s Equation

Professor Chris Reynolds (csr12@ast.cam.ac.uk)

Recap – last couple of lectures

- Supersonic flows and shocks
    - Conservation laws → Rankine-Hugoniot jump conditions
- Theory of supernovae explosions
    - Simple analysis of explosion in a cold (T=0) ISM
    - Similarity solutions (Taylor-Sedov theory)

This Lecture

- Supernova explosions (Chapter F.6; cont.)
  - Breakdown of similarity solution
  - End stages of supernova/ISM interactions
- Steady Flows and Bernoulli's Equation (Chapter G)
  - Vorticity (G.1)
  - Bernoulli Principle (G.1)
  - Rotational and irrotational flows (G.2)
    - Kelvin Circulation Theorem

Breakdown of similarity solution

When will the similarity solution fail and the supernova explosion "stall"?

This happens when the pressure in the ambient ISM becomes comparable to that in the shocked shell, $p_1 \sim p_0$.

Previously, we found

$$p_1 = \frac{2}{\gamma + 1} \rho_0 u_0^2;$$

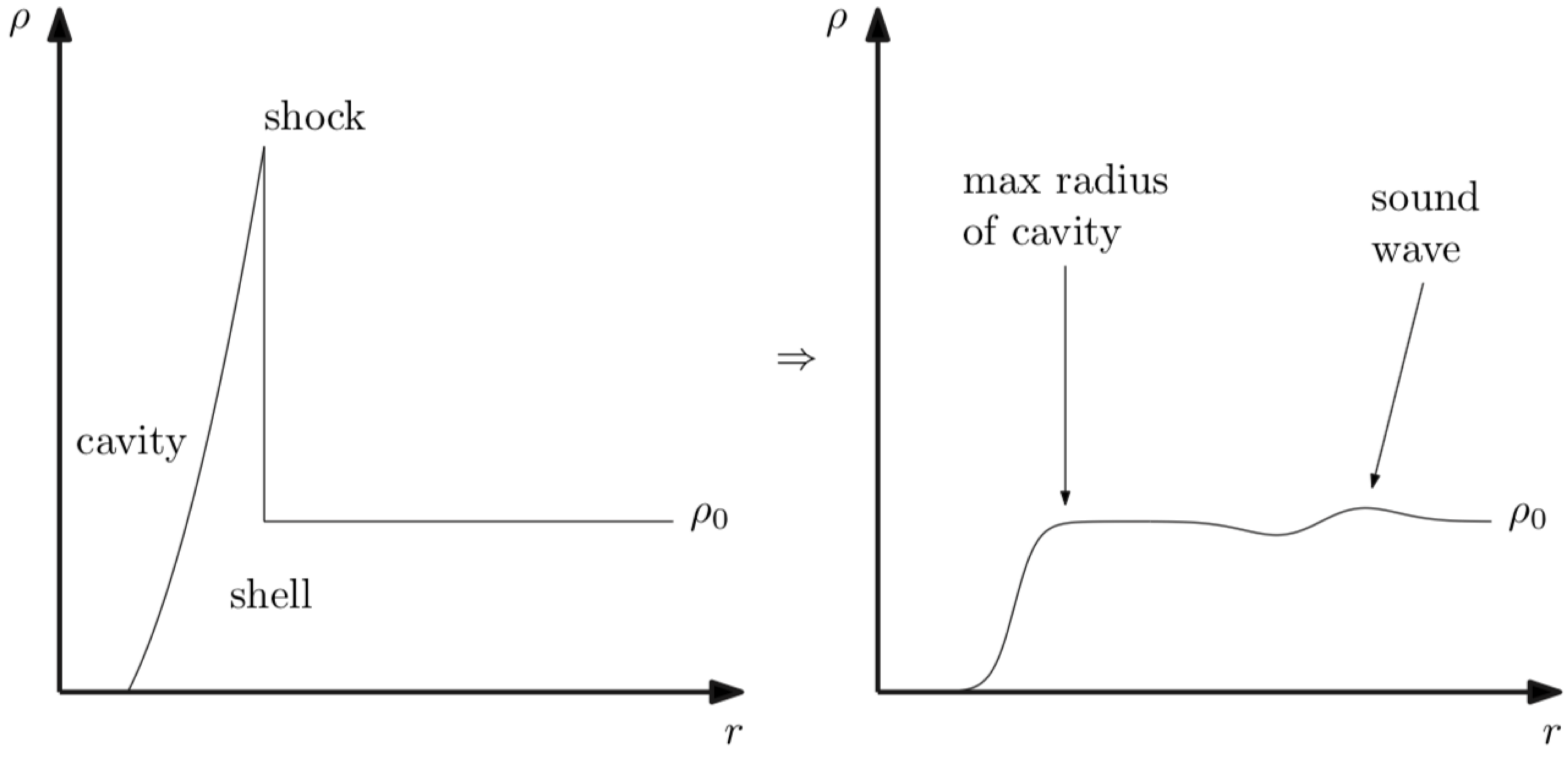
The sounds speed in the ambient medium is $c_s^2 = \frac{\gamma p_0}{\rho_0}$

So, similarity breaks down when

$$\frac{2}{\gamma + 1} \rho_0 u_0^2 \sim \frac{\rho_0 c_s^2}{\gamma}$$

$$\Rightarrow \qquad u_0 \sim c_s$$

i.e., expansion ceases to be supersonic


ρ
shock
cavity
shell
ρ0
r
⇒
ρ
max radius
of cavity
sound
wave
ρ0
r

Let's use our theory to examine this end point…

We showed that energy conservation

$$E = \frac{4\pi}{3} R^3 \left[\frac{1}{2}\rho_0 \left(\frac{2u_0}{\gamma+1} \right)^2 + \frac{\alpha}{\gamma-1} \frac{2\rho_0 u_0^2}{\gamma+1} \right] \quad \text{with } \alpha = \frac{1}{2}$$

$$= \frac{4\pi}{3} R^3 \rho_0 u_0^2 \left[\frac{2}{(\gamma+1)^2} + \frac{1}{(\gamma-1)(\gamma+1)} \right]$$

$$= \frac{4\pi}{3} R^3 \rho_0 u_0^2 \left[\frac{2(\gamma-1)+\gamma+1}{(\gamma+1)^2(\gamma-1)} \right]$$

$$= \frac{4\pi}{3} R^3 \rho_0 u_0^2 \frac{3\gamma-1}{(\gamma+1)^2(\gamma-1)}$$

$$\Rightarrow \qquad u_0^2 = \frac{(\gamma+1)(\gamma^2-1)}{3\gamma-1} \cdot \frac{3E}{4\pi\rho_0 R^3} \quad \underbrace{\sim \frac{\gamma+1}{2\gamma} c_s^2}_{\substack{\text{when blast} \\ \text{wave becomes} \\ \text{sonic and } p_1 \sim p_0}}$$

$$\Rightarrow \qquad E \sim \frac{4\pi}{3} \rho_0 R_{\max}^3 \frac{c_s^2}{2\gamma} \cdot \frac{3\gamma-1}{\gamma^2-1}$$

Compare this with the internal energy initially contained in the ISM…

$$E_{\rm init} = \frac{4\pi}{3} R_{\rm max}^3 \frac{p_0}{\gamma - 1} = \frac{4\pi}{3} R_{\rm max}^3 \rho_0 \frac{c_s^2}{\gamma(\gamma - 1)}$$

So we see that $E \sim E_{\rm init}$, i.e. the blast wave propagates until the explosion energy is comparable to the internal energy in the sphere.

This theory suggests that the time needed to reach this end-state is

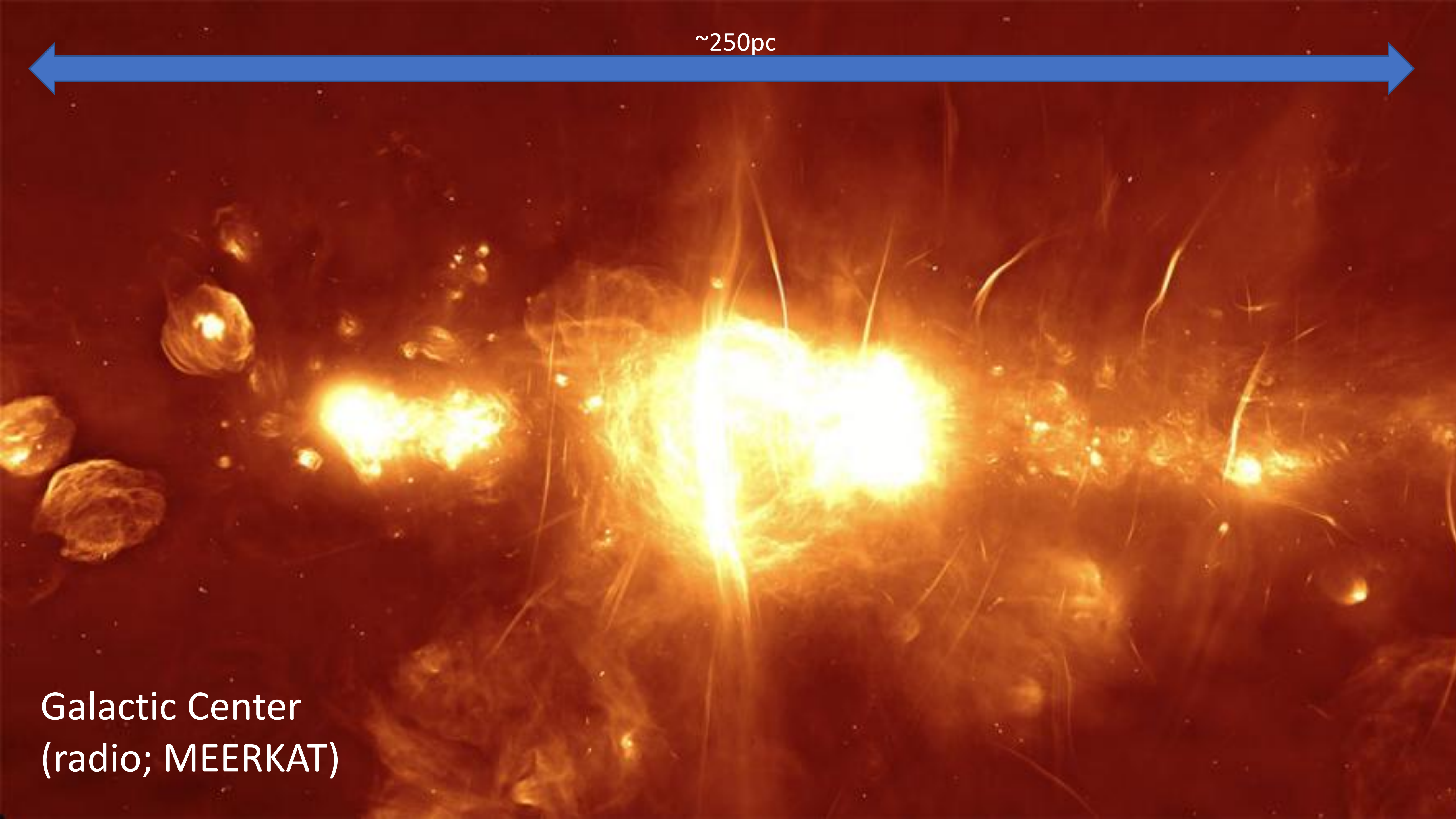
$$t_s \sim \frac{R_{\rm max}}{c_s}$$

Putting in numbers… the ambient ISM has $T \sim 10^4\ {\rm K},\ \rho \sim 10^{-21}\ {\rm kg\ m^{-3}}$

$$\Rightarrow \quad R_{\rm max} \sim \text{ few} \times 100\ {\rm pc}$$

$$t_{\rm max} \sim 10\ {\rm Myr}$$


AQUILA RIFT
Beta Arae
Upper-Scorpius OB Association
Pi Scorpii
Sigma Scorpii
Dumbbell Nebula
Lupus Dark Cloud
Rho Ophiuchi Dark Cloud
Rho Ophiuchi
Dschubba
Upper Centaurus-Lupus OB Association
Sadalsuud
Zeta Ophiuchi
S27
Graffias
G317 Dark Cloud
Rasalgethi
Beta Sagittae
Chameleon I Dark Cloud
Sadalmelik
Alpha Sagittae
Kappa Centauri
Hadar
Pi Sagittarii
Antares
Coalsack
Helix Nebula
Peacock
Shaula
Eta Centauri
Pi Herculis
Phi Sagittarii
Mimosa
Lower Centaurus-Crux OB Association
Nu Coronae Borealis
Zeta Coronae Borealis
Nunki
Acrux
IC 2602
Enif
Albireo
Alpha Arae
R Hydrae
Theta Carinae
Aspidiske
Gamma Cygni
Gamma Lupi
Beta Crucis
Zeta Cygni
Sulaphat
Spica
Rastaban
R Aquarii
Avior
Nu Carinae
Kappa Velorum
RR Lyrae
Omicron Velorum
The Sun
Achernar
LOOP I
IC 2391
SS Cygni
Scheat
Alpha Reticuli
Lambda Velorum
Alkaid
Sirius Supercluster
Chi Carinae
LOOP III
Hyades
Algenib
LOCAL BUBBLE
Furud
Beta Cephei
Elnath
Mu Leporis
Eta Aurigae
Nihal
Epsilon Leonis
Iota Aurigae
Adhara
Delta Persei
Betelgeuse
NGC 2451
Lambda Tauri
Bellatrix
Polaris
Epsilon Cassiopeiae
Pleiades
Praesepe
Xi Puppis
S184
T Tauri
Perseus OB3
Taurus Dark Cloud
HL Tauri
Gamma Cassiopeiae
Mirfak
Zeta Tauri
CW Leonis
Mira
Mirzam
Owl Nebula
Zeta Aurigae
Beta Monocerotis
Medusa Nebula
Eta Persei
LOOP II
Arneb
Mebsuta
Rigel
LINDBLAD RING
Eta Orionis

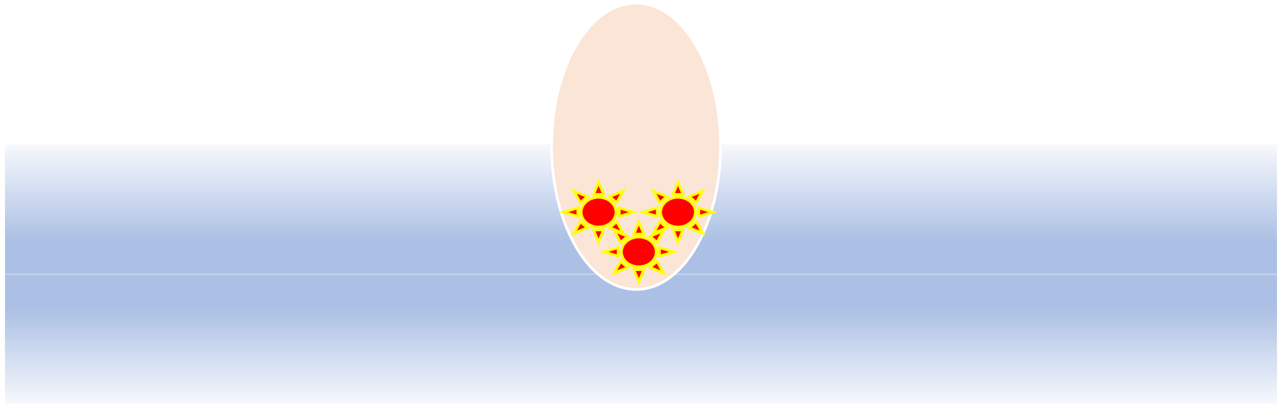

~250pc
Galactic Center
(radio; MEERKAT)

The supernova rate in our galaxy is approximately 10^{-7} SN/Myr/pc^3. So, over a duration of 10Myr we expect to find one SN in every 10^6pc^3.

So this would suggest that the entire ISM is heated by SN to 10^6K.

This is contrary to observations. The above argument fails for two reasons:

- Cooling of the shell becomes important after ~10^5yr (R~20pc) after which the shell grows more slowly ($R \sim t^{0.3}$). Ultimately, we end up with R_{max}~50pc.
- Finite height of galactic disk... Multiple SN bubbles from young star clusters merge and "blow out" of the galactic disk




M82

Chapter G : Bernoulli’s Equation and Transonic Flows

G.1 : Bernoulli's Equation

We now consider the properties of steady, barotropic flows.

This will lead us to interesting findings about transonic flows, relevant for some astrophysical accretion solutions as well as stellar winds.

Start with momentum equation:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p - \boldsymbol{\nabla}\Psi.$$

$$(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = \boldsymbol{\nabla}\left(\frac{1}{2}u^2\right) - \mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{u})$$

(vector identity)

For barotropic e-o-s:

$$\frac{\partial}{\partial x}\int \frac{\mathrm{d}p}{\rho} = \frac{\partial p}{\partial x}\frac{\mathrm{d}}{\mathrm{d}p}\int \frac{\mathrm{d}p}{\rho} = \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\Rightarrow \qquad \frac{1}{\rho}\boldsymbol{\nabla} p = \boldsymbol{\nabla}\left(\int \frac{\mathrm{d}p}{\rho}\right).$$

$$\boxed{\mathbf{w} = \boldsymbol{\nabla} \times \mathbf{u}}$$ DEFINITION OF VORTICITY

So,

$$\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\nabla}\left(\frac{1}{2}u^2\right) - \mathbf{u} \times \mathbf{w} = -\boldsymbol{\nabla}\left[\int \frac{\mathrm{d}p}{\rho} + \Psi\right] \qquad (**)$$

<u>For a steady-flow,</u> we take dot product of this with **u** to get

$$\mathbf{u} \cdot \boldsymbol{\nabla}\left[\frac{1}{2}u^2 + \int \frac{\mathrm{d}p}{\rho} + \Psi\right] = 0$$

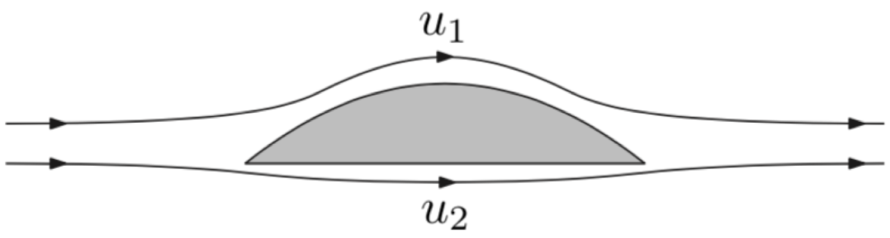
So, the quantity

$$H = \frac{1}{2}u^2 + \int \frac{\mathrm{d}p}{\rho} + \Psi$$

is constant along a streamline. This is **<u>Bernoulli's Principle.</u>**

Examples:

Aircraft wing (apocryphal)

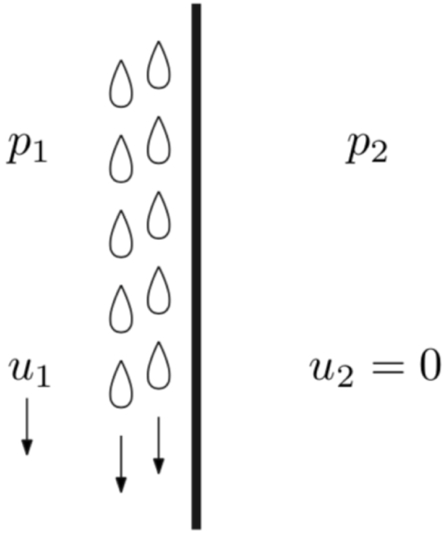


$$u_1 > u_2 \quad \Rightarrow \quad p_1 < p_2 \quad \text{from } H$$

$$\Rightarrow \quad \text{pressure difference}$$

$$\Rightarrow \quad \text{lift force.}$$

Shower curtain



$$\Rightarrow \quad p_1 < p_2$$

$$\Rightarrow \quad \text{curtain blows inwards.}$$

G.2 : Rotational & Irrotational Flows

Definitions:

$$\mathbf{w} = 0 \text{ everywhere} \qquad \Rightarrow \text{ irrotational}$$
$$\mathbf{w} \neq 0 \qquad \Rightarrow \text{ rotational}$$

From eqn (**), we see that a steady-state, barotropic flow that is also irrotational has

$$\boldsymbol{\nabla} H = 0$$

So *H* = constant through the flow (not just along a streamline).

For a general barotropic flow (rotational and not necessarily steady-state), then equation (**) becomes

$$\frac{\partial \mathbf{u}}{\partial t} = -\boldsymbol{\nabla} H + \mathbf{u} \times \mathbf{w}$$

Take curl of this equation:

$$\frac{\partial}{\partial t} \underbrace{(\boldsymbol{\nabla} \times \mathbf{u})}_{\mathbf{w}} = \underbrace{-\boldsymbol{\nabla} \times (\boldsymbol{\nabla} H)}_{\equiv 0} + \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{w})$$

$$\Rightarrow \quad \boxed{\frac{\partial \mathbf{w}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{w})} \qquad \text{HELMHOLTZ'S EQN.}$$

Notes:

- If **w** = 0 initially, it will remain so thereafter (not true when viscosity included). In that case, we can infer that there exists a potential function Φ_u such that

$$\mathbf{u} = -\boldsymbol{\nabla} \Phi_u$$

If flow is also incompressible, then we have $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ and so $\nabla^2 \Phi_u = 0$

- For a time-dependent rotational flow, the Helmholtz equation

$$\frac{\partial \mathbf{w}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{w})$$

implies that

$$\frac{\mathrm{D}}{\mathrm{D}t} \int_S \mathbf{w} \cdot \mathrm{d}\mathbf{S} = 0$$

(S is a surface that is carried along in the flow)

(proof of this in the PDF notes).

Mental picture… think of “lines of vorticity” that get carried around in the flow.
Close analogy with lines of magnetic field.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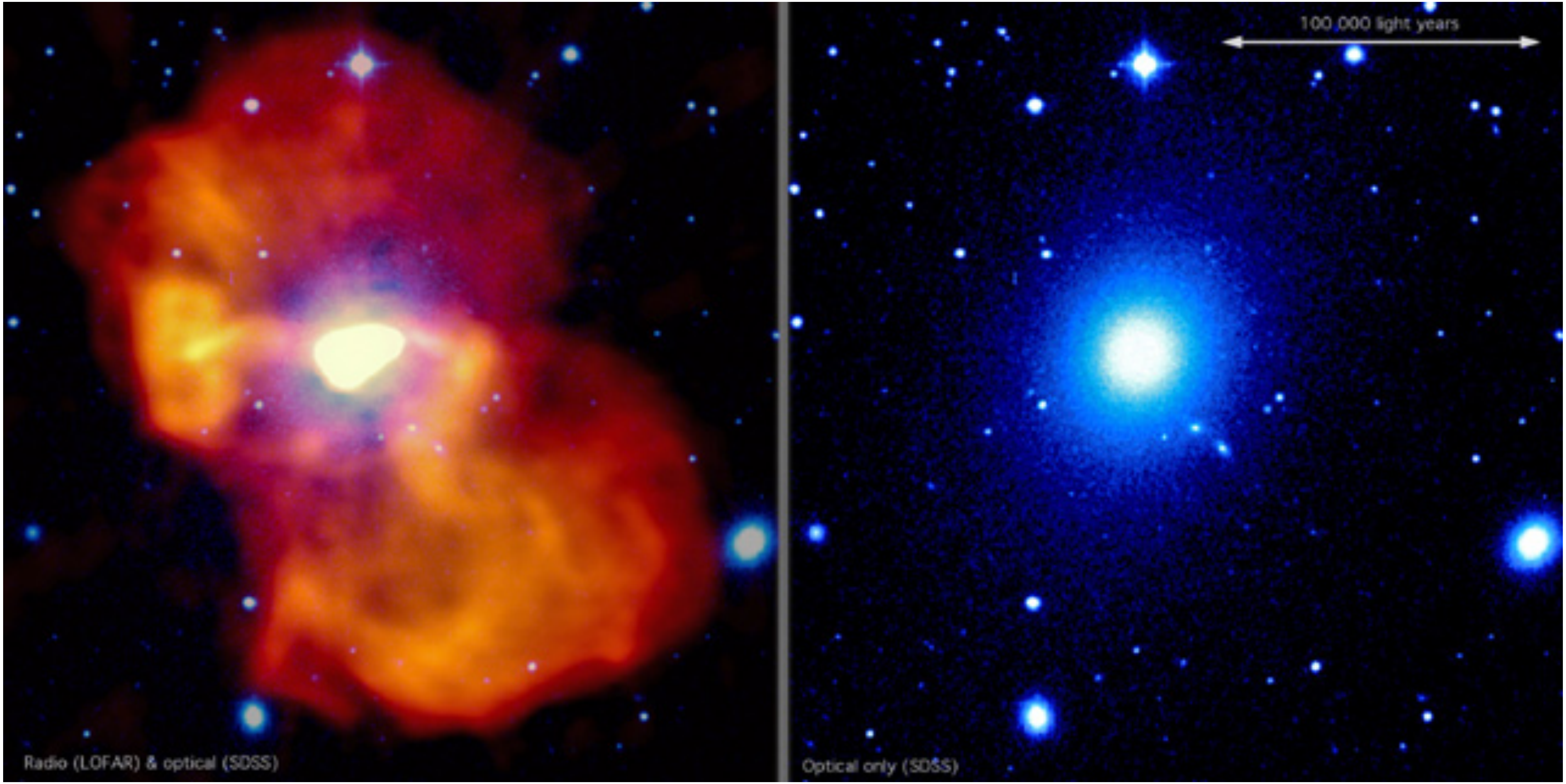
Radio (LOFAR) & optical (SDSS)
100,000 light years
Optical only (SDSS)