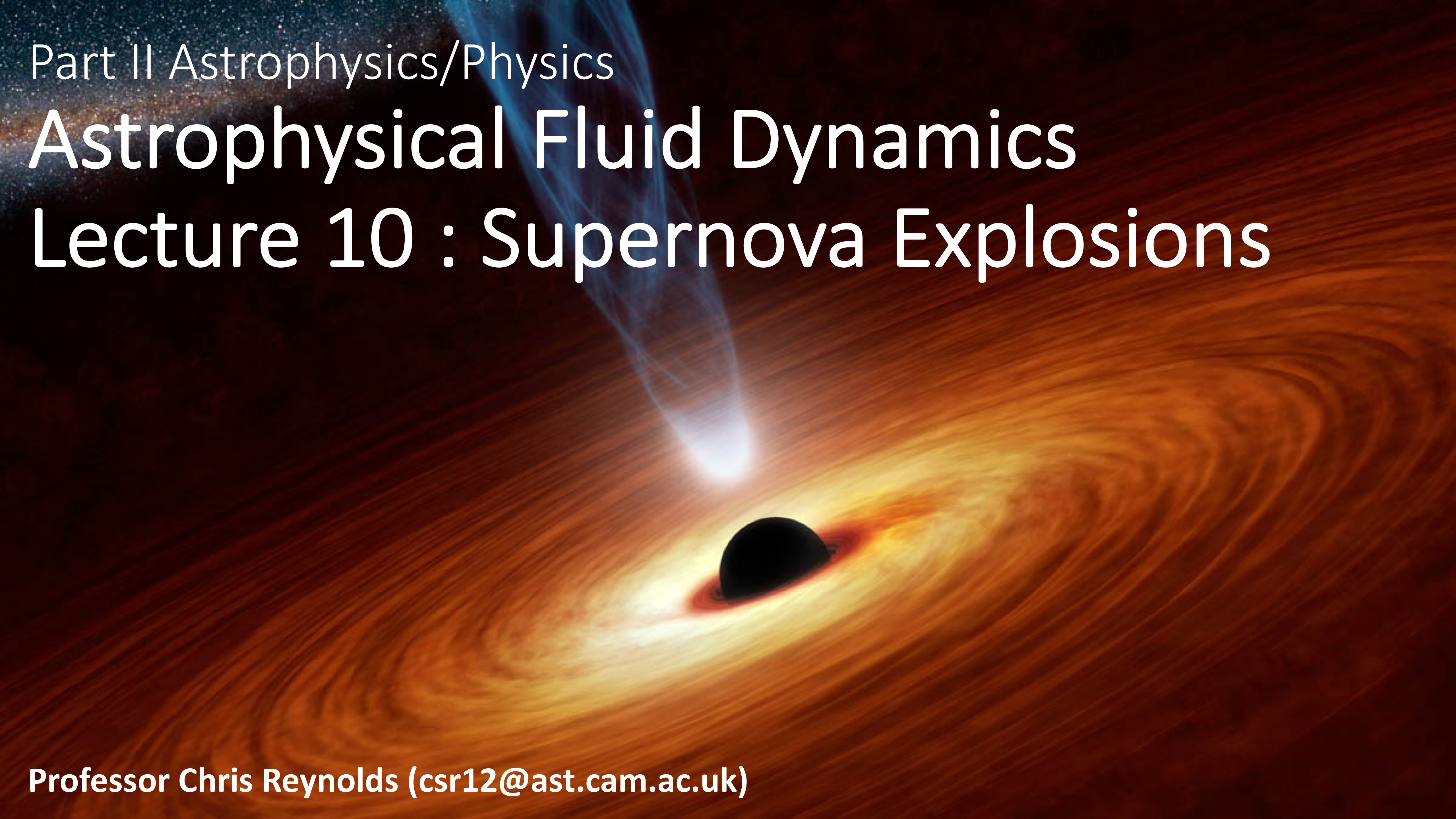


Part II Astrophysics/Physics

Astrophysical Fluid Dynamics
Lecture 10 : Supernova Explosions

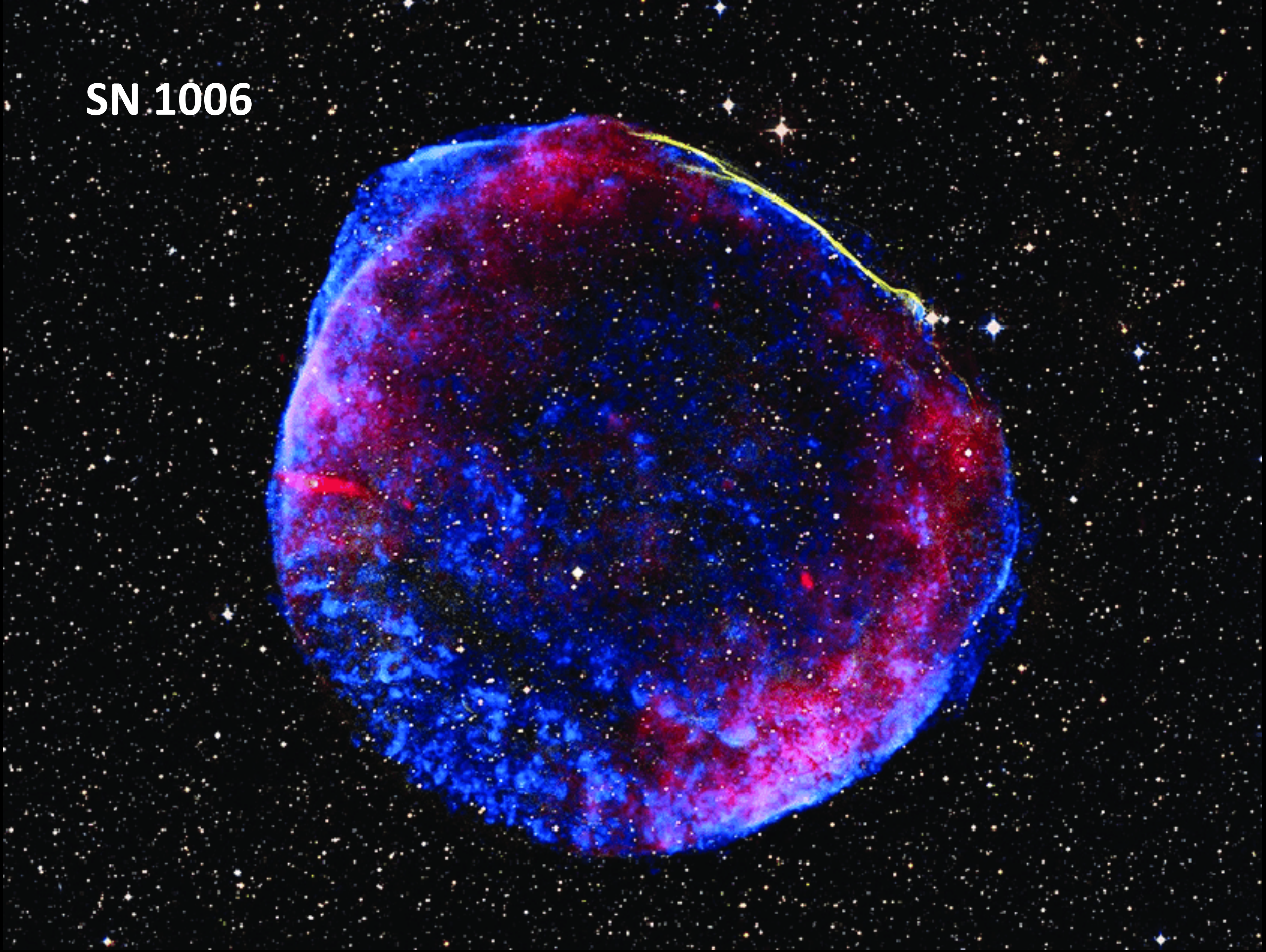
Professor Chris Reynolds (csr12@ast.cam.ac.uk)

Recap

- Developed basic equations of fluid dynamics (for ideal gas)
- Hydrostatic Equilibrium and Stellar Structure
- Sound waves – linear perturbation theorem
- Supersonic flows and the Rankine-Hugoniot jump conditions

This Lecture (Chapter F.6)

- Important application of shock theory – Supernovae!
- **How does an exploding star interact with its surroundings?**
- Simple but powerful model of interaction
- Similarity techniques (Taylor-Sedov)


SN 1006

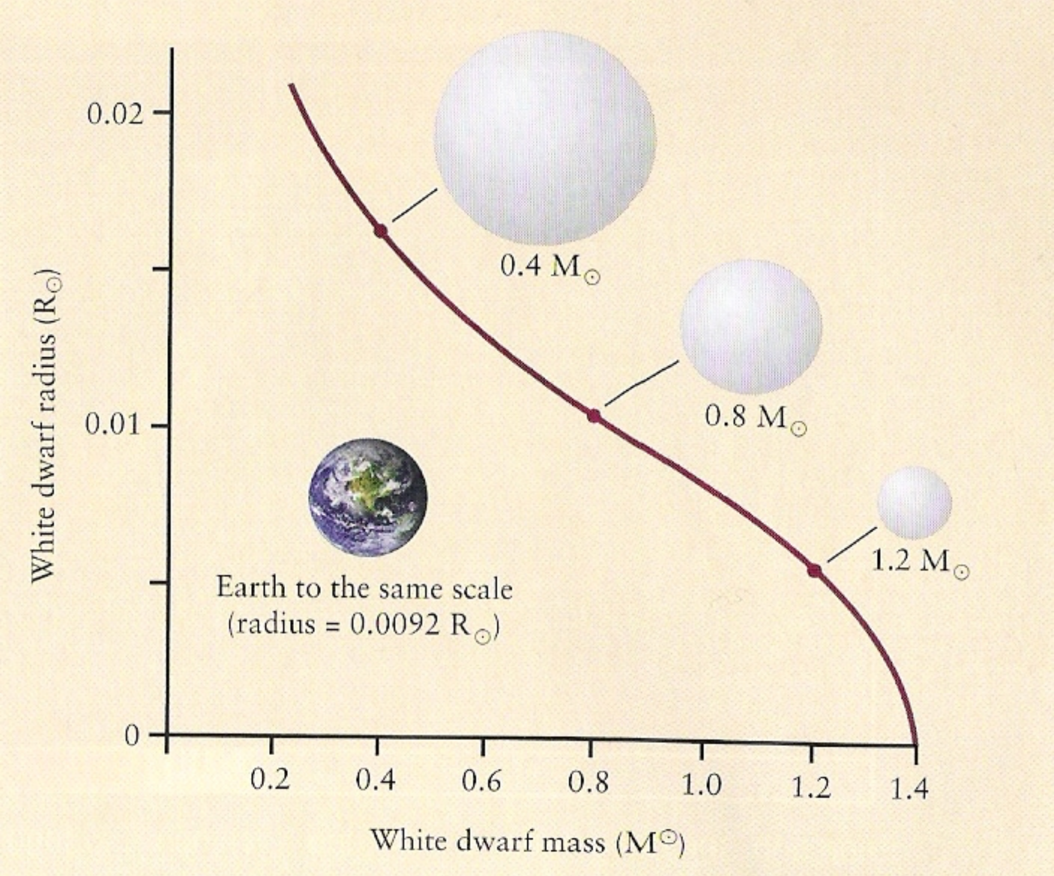
Two principal paths to a supernova…

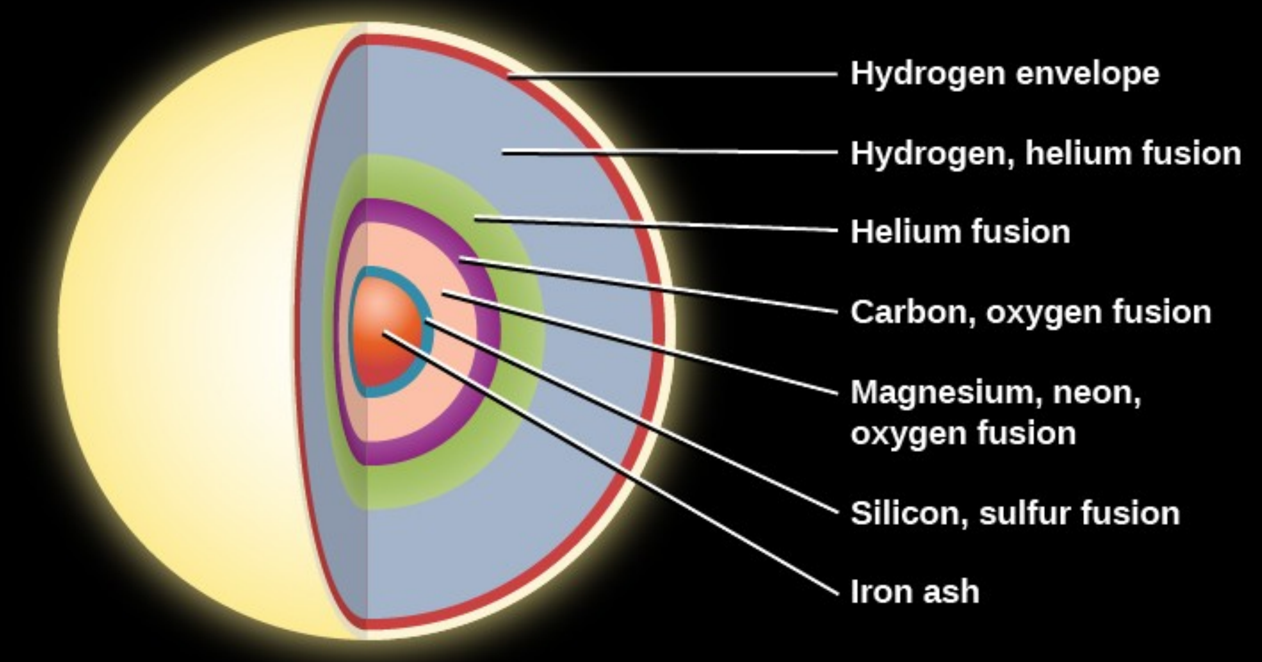
White dwarf approaches Chandrasekhar mass (merger or accretion)

- WD starts to collapse and interior heats up
- C/O undergoes thermonuclear fusion
- Release energy….

$$
\begin{aligned}
E &\approx (0.001)Mc^2 \\
&\approx 2.6 \times 10^{51}\,\mathrm{erg} \\
&= 2.6 \times 10^{44}\,\mathrm{J}
\end{aligned}
$$

… in a few seconds.





- Massive star reaches end of life
  - Iron core (and some) undergoes catastrophic gravitational collapse → neutron star or black hole
  - Release gravitational potential energy

$$E \sim \frac{GM^2}{R} \sim 5 \times 10^{53}\,\mathrm{erg} \quad (= 5 \times 10^{46}\,\mathrm{J})$$

  - But 99% of energy released as neutrinos… so ~5×10^{44}J dumped into surroundings

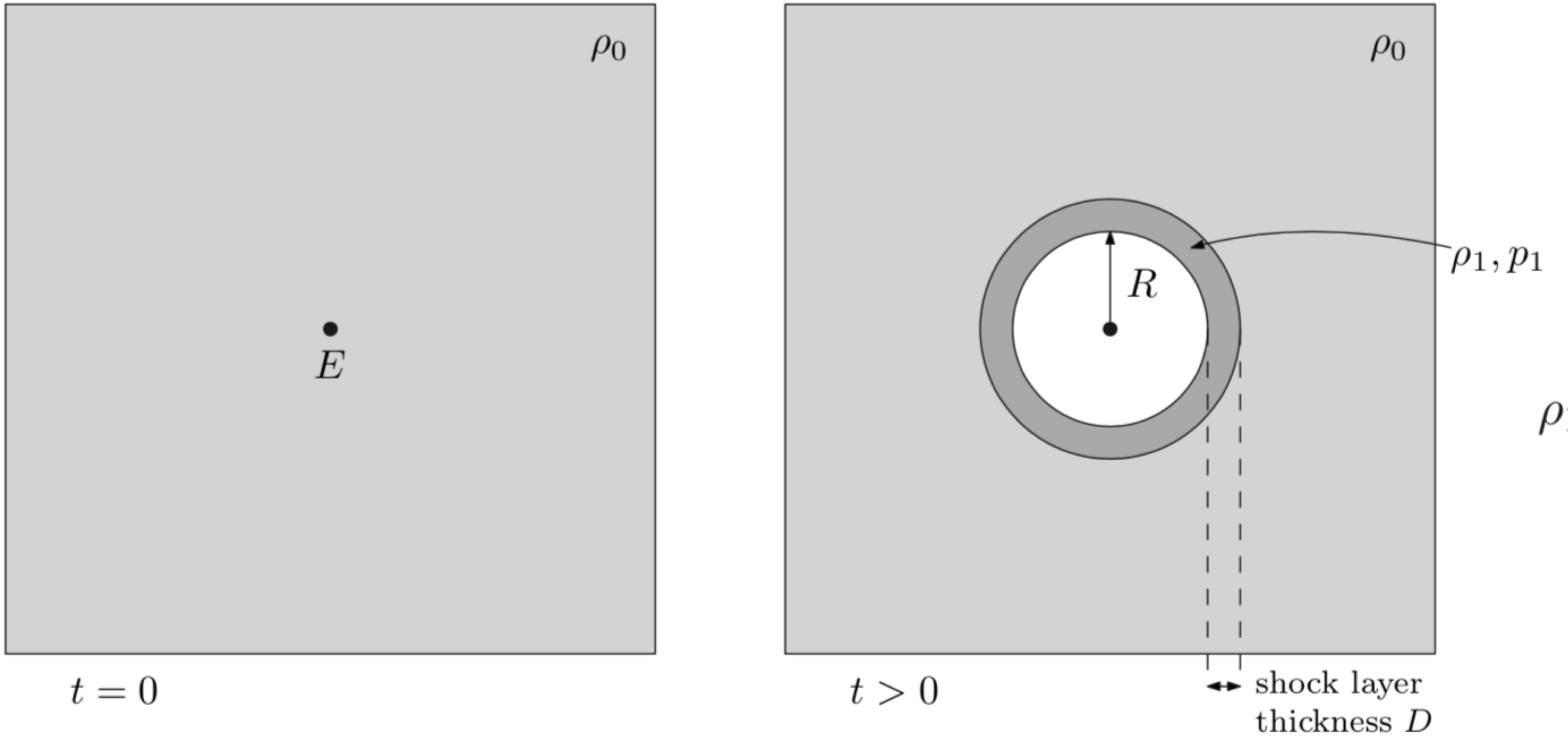
Simple model for interaction of supernova (SN) explosion with interstellar medium

Simplify the system:

- Initially uniform interstellar medium (ISM) at rest, density ρ_0
- Instantaneous point-like explosion releasing energy E
- Ignore temperature/pressure of ambient ISM (T_0=0)

Evolution:

- Energy E heats small volume of ISM to very high temperature → high pressure
- High pressure region expands, drives shock into ambient (cold) ISM
- Develop a shell of shocked ISM that sweeps outwards
- Speed of expansion of shocked shell driven by pressure discontinuity dictated by R-H relations



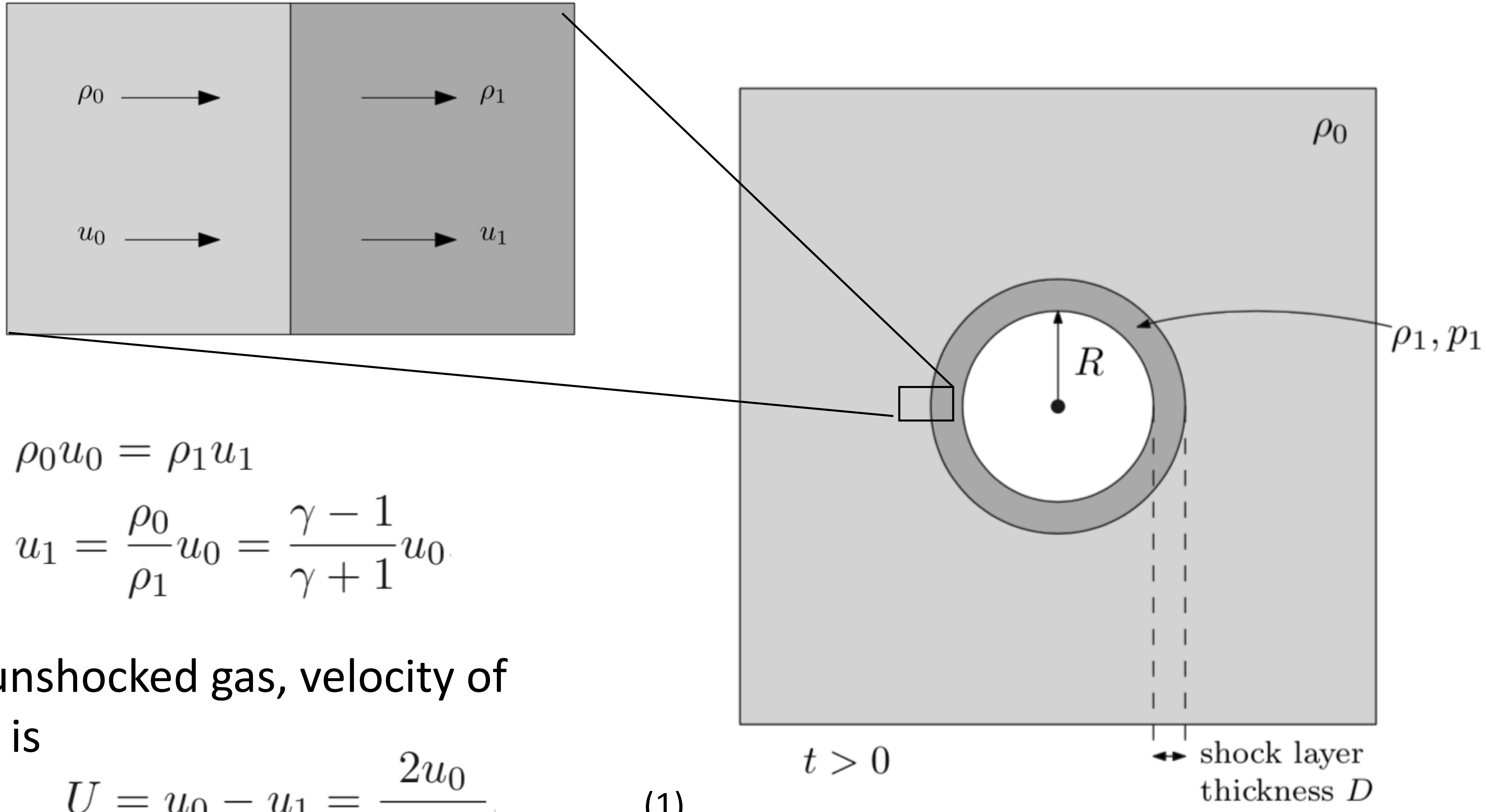
$$\rho_1 = \rho_0 \frac{\gamma + 1}{\gamma - 1}$$

Mass in shell is that swept up from sphere:

$$\Rightarrow \quad \frac{4\pi}{3}\rho_0 R^3 = 4\pi\rho_1 R^2 D \qquad \text{(assuming } D \ll R\text{)}$$

$$\Rightarrow \quad D = \frac{1}{3}\left(\frac{\gamma - 1}{\gamma + 1}\right) R \approx 0.08R \quad \text{for} \quad \gamma = 5/3$$

Let’s determine the velocity of matter in shell. Go into the frame of the shock…



$$\rho_0 u_0 = \rho_1 u_1$$

$$\Rightarrow \qquad u_1 = \frac{\rho_0}{\rho_1} u_0 = \frac{\gamma - 1}{\gamma + 1} u_0$$

Relative to unshocked gas, velocity of shocked gas is

$$U = u_0 - u_1 = \frac{2u_0}{\gamma + 1} \qquad (1)$$

Now examine dynamics of the shell. Assume that shell has uniform velocity/density.

Rate of change of momentum of shell (in some solid angle $d\Omega$) is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{4\pi}{3}\rho_0 R^3 \frac{2u_0}{\gamma+1}\right] \mathrm{d}\Omega$$

This momentum change is due to pressure on inside surface of shell due to pressure in the interior of cavity, p_{in}. We make an ansatz that $p_{\rm in} = \alpha p_1$

From 2nd R-H jump condition:

$$p_0 + \rho_0 u_0^2 = p_1 + \rho_1 u_1^2$$

$$\Rightarrow \qquad p_1 = \rho_0 u_0^2 \left[1 - \frac{\rho_1 u_1^2}{\rho_0 u_0^2}\right] \quad \text{(since } \rho_0 = 0 \text{ by assumption)}$$

$$= \rho_0 u_0^2 \left[1 - \frac{\gamma-1}{\gamma+1}\right] \quad \text{(assuming a strong shock)}$$

$$= \frac{2}{\gamma+1}\rho_0 u_0^2 \qquad (2)$$

So, looking at rate of change of momentum of shell…

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{4\pi}{3}\rho_0 R^3 \frac{2u_0}{\gamma+1}\right] &= 4\pi R^2 p_{\mathrm{in}} \\
&= 4\pi R^2 \alpha p_1 \\
&= 4\pi R^2 \frac{2}{\gamma+1}\rho_0 u_0^2
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \qquad & \frac{\mathrm{d}}{\mathrm{d}t}\left[R^3 u_0\right] = 3\alpha R^2 u_0^2 \\
\Rightarrow \qquad & \frac{\mathrm{d}}{\mathrm{d}t}\left[R^3 \dot{R}\right] = 3\alpha R^2 \dot{R}^2 \quad \text{since } u_0 \equiv \dot{R}
\end{aligned}
$$

This admits solutions of form $R \propto t^b$, and direct substitution shows that

$$
b = \frac{1}{4-3\alpha} \qquad \Rightarrow \qquad R \propto t^{1/(4-3\alpha)}, \quad u_0 \propto t^{(3\alpha-3)/(4-3\alpha)} \propto R^{3\alpha-3}
$$

Need to determine α. This comes from energy conservation for the shell+cavity…

- Kinetic energy of cavity small (very little mass)
- Kinetic energy of shell,

$$\frac{1}{2} \cdot \frac{4\pi}{3} \rho_0 R^3 U^2$$

- Internal energy of shell small (very thin so small volume)
- Internal energy of cavity

$$\frac{4\pi}{3} R^3 \underbrace{\frac{p_{\text{in}}}{\gamma - 1}}_{\rho\mathcal{E}} = \frac{4\pi}{3} R^3 \alpha \frac{p_1}{\gamma - 1}$$

Putting all together

$$\begin{aligned} E &= \frac{1}{2}\cdot\frac{4\pi}{3}\rho_0 R^3 U^2 + \frac{4\pi}{3}R^3\alpha\frac{p_1}{\gamma-1} \\ &= \frac{1}{2}\cdot\frac{4\pi}{3}\rho_0 R^3 \underbrace{\left(\frac{2u_0}{\gamma+1}\right)^2}_{①} + \frac{4\pi}{3}R^3\alpha \underbrace{\frac{2}{\gamma+1}\rho_0 u_0^2\frac{1}{\gamma-1}}_{②} \\ &= \frac{4\pi}{3}R^3 u_0^2\left[\frac{1}{2}\rho_0\frac{4}{(\gamma+1)^2} + \alpha\rho_0\frac{2}{(\gamma+1)(\gamma-1)}\right], \end{aligned}$$

$$\therefore \quad E \propto R^3 u_0^2 \propto t^{(6\alpha-3)/(4-3\alpha)}$$

So energy conservation requires $\alpha = 1/2$

$$\boxed{R \propto t^{2/5}, \quad u_0 \propto t^{-3/5}, \quad p_1 \propto t^{-6/5}}$$

Similarity solutions

How can we be more rigorous and avoid the (somewhat arbitrary) assumptions of the uniformity of the shell and the pressure ansatz for the cavity?

Similarity solutions - a powerful technique based on dimensional analysis.

Start by noting that the problem only has two parameters, *E* and ρ_0, with dimensions,

$$[E] = \frac{ML^2}{T^2}, \quad [\rho_0] = \frac{M}{L^3}$$

So, after time *t* there is only one combination of parameters that gives a length

$$\lambda = \left(\frac{Et^2}{\rho_0}\right)^{1/5}$$

We define a dimensionless parameter ξ

$$\xi \equiv \frac{r}{\lambda} = r\left(\frac{\rho_0}{Et^2}\right)^{1/5}$$

So, we know that the evolution of any variable *X(t,r)* can be expressed as

$$X = X_1(t)\tilde{X}(\xi)$$

i.e., *X(t,r*) has a fixed “shape” in radius which just gets scaled up and down by X_1 and radially stretched by factor *λ*.

<u>How to think about ξ? </u> It is neither an Eulerian (fixed point) nor a Lagrangian coordinate (fixed mass) coordinate. Instead, it is a coordinate attached to ”features” in the structure (e.g. shocks) even as those features propagate through the flow.

Can express derivatives in terms of rescaled variables…

$$\frac{\partial X}{\partial r} = X_1 \frac{\mathrm{d}\tilde{X}}{\mathrm{d}\xi} \left.\frac{\partial \xi}{\partial r}\right|_t$$

$$\frac{\partial X}{\partial t} = \tilde{X}(\xi)\frac{\mathrm{d}X_1}{\mathrm{d}t} + X_1 \frac{\mathrm{d}\tilde{X}}{\mathrm{d}\xi} \left.\frac{\partial \xi}{\partial t}\right|_r$$

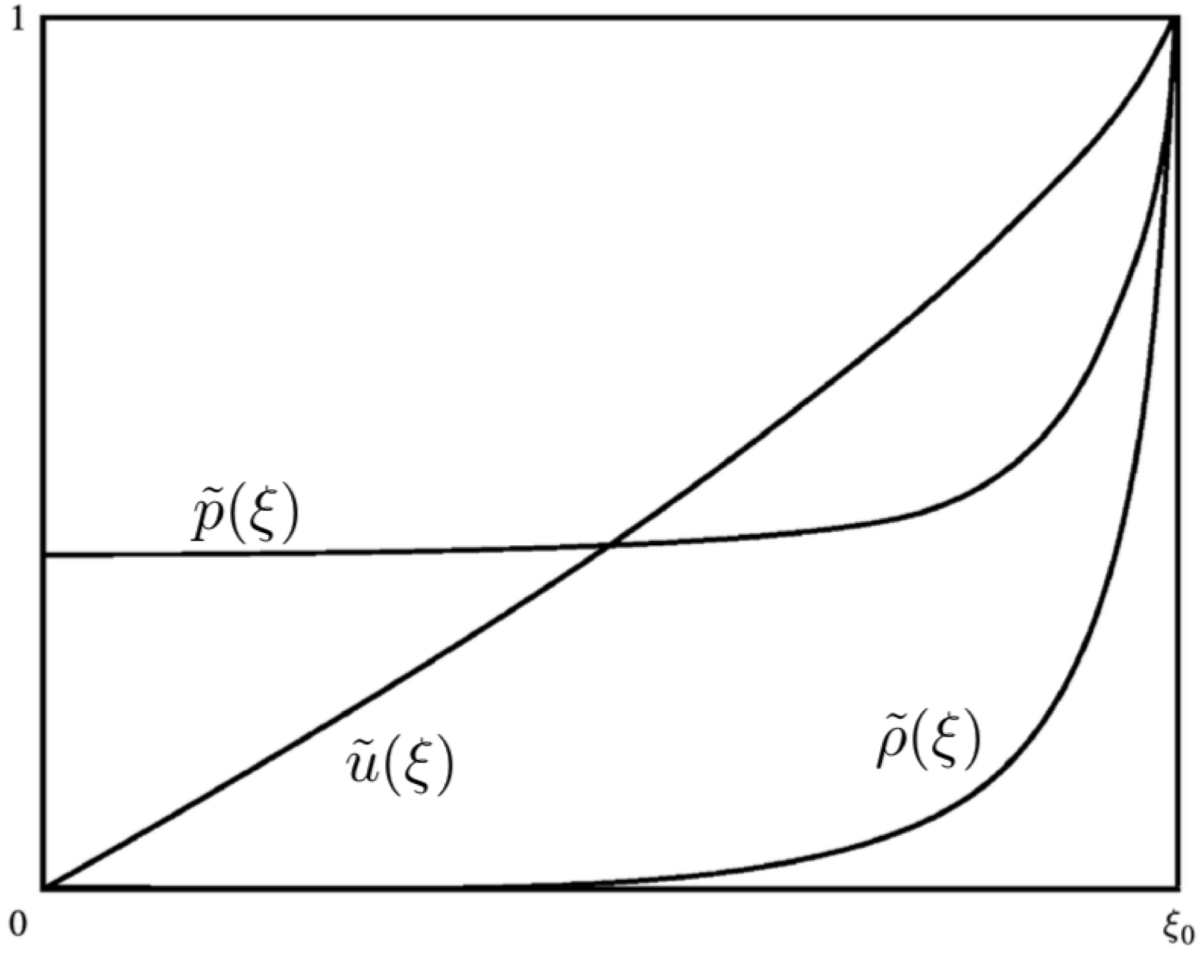
So, we can write

$$\rho(r,t) = X_\rho(t)\tilde{\rho}(\xi)$$

$$p(r,t) = X_p(t)\tilde{p}(\xi)$$

$$u(r,t) = X_u(t)\tilde{u}(\xi)$$

… substitute into the fluid equations and derive Ordinary Differential Equations for $\tilde{\rho}, \widetilde{p}, \ \tilde{u}$ as functions of ξ.



Mostly justifies assumptions of simple treatment:

- Most mass swept up in thin shell
- Post-shock pressure is multiple of p_{in}

But… shell matter not really moving at constant velocity… but arguments restored by taking appropriate weighted average.

The location of the shock is given by

$$R(t) = \xi_0 \left(\frac{E}{\rho_0}\right)^{1/5} t^{2/5}$$

$$u_0(t) = \frac{\mathrm{d}R}{\mathrm{d}t} = \frac{2}{5}\xi_0 \left(\frac{E}{\rho_0 t^3}\right)^{1/5} = \frac{2}{5}\frac{R}{t}$$

For supernovae…

$$\left.\begin{aligned} E &\approx 10^{44}\ \mathrm{J} = 10^{51}\ \mathrm{erg} \\ \rho_0 &= \rho_{\mathrm{ISM}} \approx 10^{-21}\ \mathrm{kg\ m^{-3}} \end{aligned}\quad \Rightarrow \quad \begin{aligned} R &\approx 0.3 t^{2/5}\ \mathrm{pc} \\ u_0 &\approx 10^5 t^{-3/5}\ \mathrm{km\ s^{-1}} \end{aligned}\right\} \text{where } t \text{ is measured in yrs}$$

This is valid for,

$$\begin{aligned} t &\gtrsim 100\ \mathrm{yr} \quad (\text{when } u_0 < u_{\mathrm{inj}}) \\ t &\lesssim 10^5\ \mathrm{yr} \quad (\text{after which energy losses become important}) \end{aligned}$$

The formation of a blast wave by a very intense explosion.
II. The atomic explosion of 1945

By Sir Geoffrey Taylor, F.R.S.

(*Received* 10 *November* 1949)

[Plates 7 to 9]

Photographs by J. E. Mack of the first atomic explosion in New Mexico were measured, and the radius, R, of the luminous globe or 'ball of fire' which spread out from the centre was determined for a large range of values of t, the time measured from the start of the explosion. The relationship predicted in part I, namely, that $R^{\frac{5}{2}}$ would be proportional to t, is surprisingly accurately verified over a range from $R=20$ to 185 m. The value of $R^{\frac{5}{2}}t^{-1}$ so found was used in conjunction with the formulae of part I to estimate the energy E which was generated in the explosion. The amount of this estimate depends on what value is assumed for γ, the ratio of the specific heats of air.

Two estimates are given in terms of the number of tons of the chemical explosive T.N.T. which would release the same energy. The first is probably the more accurate and is 16,800 tons. The second, which is 23,700 tons, probably overestimates the energy, but is included to show the amount of error which might be expected if the effect of radiation were neglected and that of high temperature on the specific heat of air were taken into account. Reasons are given for believing that these two effects neutralize one another.

After the explosion a hemispherical volume of very hot gas is left behind and Mack's photographs were used to measure the velocity of rise of the glowing centre of the heated volume. This velocity was found to be 35 m./sec.

Until the hot air suffers turbulent mixing with the surrounding cold air it may be expected to rise like a large bubble in water. The radius of the 'equivalent bubble' is calculated and found to be 293 m. The vertical velocity of a bubble of this radius is $\frac{2}{3}\sqrt{(g\,29300)}$ or 35·7 m./sec. The agreement with the measured value, 35 m./sec., is better than the nature of the measurements permits one to expect.

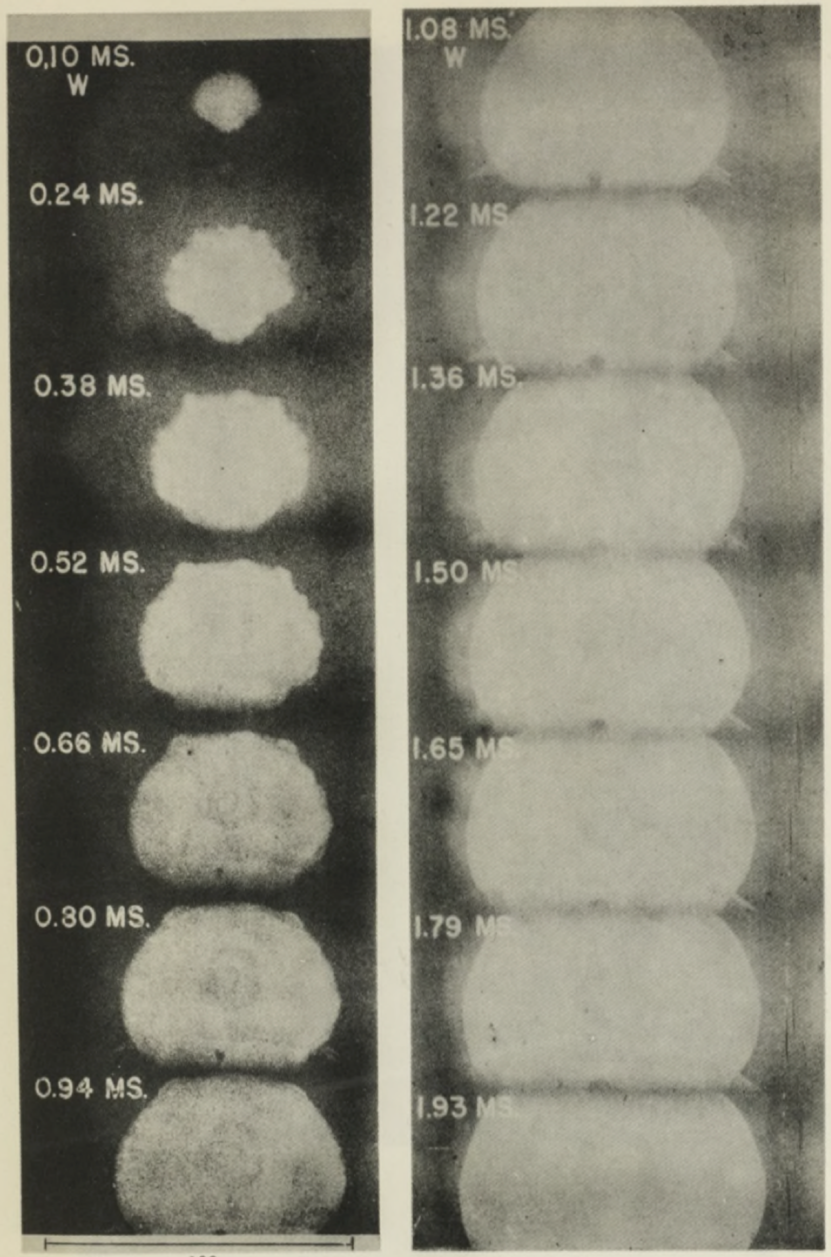


FIGURE 6. Succession of photographs of the 'ball of fire' from $t=0\cdot10$ msec. to $1\cdot93$ msec.

(Facing p. 182)

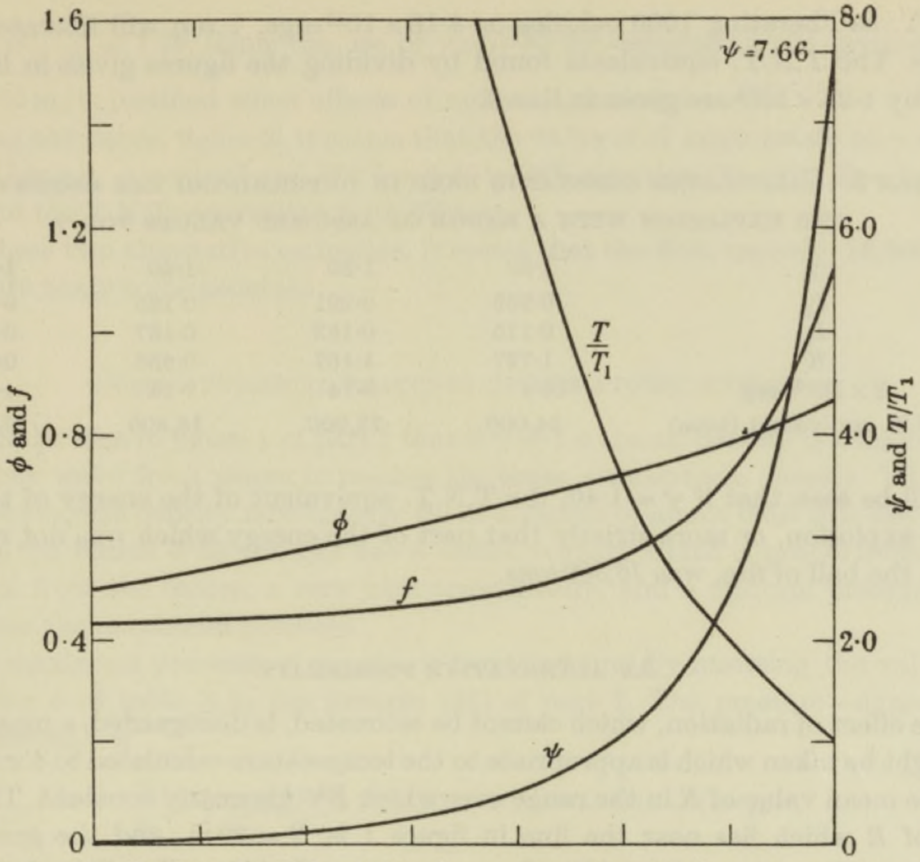


FIGURE 2. Distribution of radial velocity ϕ, pressure f, density ψ and temperature T/T_1 for $\gamma = 1\cdot30$ expressed in non-dimensional form.