Part II Astrophysics/Physics

Astrophysical Fluid Dynamics
Lecture 8 : Sound Waves

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Recap

- Developed basic equations of fluid dynamics (for ideal gas)
- Hydrostatic Equilibrium and Stellar Structure

This Lecture

- Start looking at dynamics
- Sound waves (Chapter F.1)
  - Intro to perturbation analysis for fluids
  - Sound waves in a uniform medium
    - Dispersion free waves
    - Isothermal and adiabatic sound speeds
  - Sound waves in a stratified medium
    - Dispersive waves
    - Acoustic cutoffs and acoustic shocking

Sound waves in a uniform medium

Consider an equilibrium consisting of uniform density, pressure, **u**=0 (no gravity).

$$\rho = \rho_0 \qquad \text{(uniform \& constant)}$$
$$p = p_0 \qquad \text{(uniform \& constant)}$$
$$\mathbf{u} = \mathbf{0}$$

Now introduce small perturbations (Lagrangian, so for given fluid element):

$$p = p_0 + \Delta p$$
$$\rho = \rho_0 + \Delta \rho$$
$$\mathbf{u} = \Delta \mathbf{u}$$

Relationship between Lagrangian and Eulerian perturbations

$$\underbrace{\delta\rho}_{\substack{\text{Eulerian}\\ \text{pert.}}} = \underbrace{\Delta\rho}_{\substack{\text{Lagrangian}\\ \text{pert.}}} - \underbrace{\boldsymbol{\xi}\cdot\boldsymbol{\nabla}\rho_0}_{\substack{\text{Element displacement dot}\\ \text{Gradient of unpert. state}}}$$

So in case, no distinction between Lagrangian and Eulerian perturbations...

Now we introduction perturbation into continuity and momentum equation and keep first order terms.

Start with continuity equation:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0$$

Apply perturbation and neglect 2nd order terms:

$$\frac{\partial}{\partial t}(\rho_0 + \Delta\rho) + \boldsymbol{\nabla} \cdot [(\rho_0 + \Delta\rho)\Delta\mathbf{u}] = 0$$

$$\Rightarrow \quad \cancelto{0}{\frac{\partial \rho_0}{\partial t}} + \frac{\partial \Delta\rho}{\partial t} + \cancelto{0}{\boldsymbol{\nabla}\rho_0 \cdot \Delta\mathbf{u}} + \cancelto{\text{2nd order}}{\boldsymbol{\nabla}(\Delta\rho) \cdot \Delta\mathbf{u}} + \rho_0 \boldsymbol{\nabla} \cdot (\Delta\mathbf{u}) + \cancelto{\text{2nd order}}{\Delta\rho \boldsymbol{\nabla} \cdot (\Delta\mathbf{u})} = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial t}(\Delta\rho) + \rho_0 \boldsymbol{\nabla} \cdot (\Delta\mathbf{u}) = 0 \qquad ①$$

Similarly, for the momentum equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p$$

$$\frac{\partial}{\partial t}(\Delta \mathbf{u}) = -\frac{1}{\rho_0}\boldsymbol{\nabla}(\Delta p)$$

$$\Rightarrow \quad \frac{\partial}{\partial t}(\Delta \mathbf{u}) = -\left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0} \frac{\boldsymbol{\nabla}(\Delta \rho)}{\rho_0}. \qquad \text{assuming barotropic EoS} \qquad ②$$

Take time-derivative of (1):

$$\begin{aligned} \frac{\partial^2}{\partial t^2}(\Delta \rho) &= -\rho_0 \frac{\partial}{\partial t}[\boldsymbol{\nabla} \cdot (\Delta \mathbf{u})] \\ &= -\rho_0 \boldsymbol{\nabla} \cdot \left[\frac{\partial}{\partial t}(\Delta \mathbf{u})\right] \\ &= \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0} \nabla^2(\Delta \rho). \end{aligned}$$

So we have

$$\boxed{\frac{\partial^2(\Delta\rho)}{\partial t^2} = \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0} \nabla^2(\Delta\rho).}$$

Wave Equation

Consider plane-waves

$$\Delta\rho = \Delta\rho_0 e^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)}$$

$$\Rightarrow \qquad (-\mathrm{i}\omega)^2\Delta\rho_0 = \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0} (\mathrm{i}k)^2\, \Delta\rho_0$$

$$\Rightarrow \qquad \omega^2 = \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0} k^2$$

So we have dispersionless waves with phase velocity $v_p = \omega/k$

$$c_s = \sqrt{\left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_{\rho=\rho_0}}$$

Sound Speed as the Derivative of $p(\rho)$

Example : Isothermal case (effective heat transport through gas that keeps temperature constant)

$$c_s^2 = \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_T$$

$$p = \frac{\mathcal{R}_*}{\mu}\rho T$$

$$\Rightarrow \qquad c_{s,\mathrm{I}} = \sqrt{\frac{\mathcal{R}_* T}{\mu}}$$

Example : Adiabatic case(no heat exchange between fluid elements)

$$c_s^2 = \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_S$$

$$p = K\rho^\gamma$$

$$\Rightarrow \qquad \left.\frac{\mathrm{d}p}{\mathrm{d}\rho}\right|_S = \gamma K \rho^{\gamma-1} = \frac{\gamma p}{\rho}$$

$$c_{s,\mathrm{A}} = \sqrt{\frac{\gamma \mathcal{R}_* T}{\mu}}$$

Differ by factor γ

Important to note that thermal structure of perturbations does not need to be that same as that of the background structure… e.g., can have adiabatic perturbations in an isothermal background atmosphere

Let's look at relationship between density and velocity perturbations:

$$\Delta\rho = \Delta\rho_0 e^{\mathrm{i}(kx-\omega t)}$$

$$\Delta u = \Delta u_0 e^{\mathrm{i}(kx-\omega t)}$$

Substitute into (1)

$$-\mathrm{i}\omega\Delta\rho + \rho_0 \mathrm{i}k\Delta u = 0$$

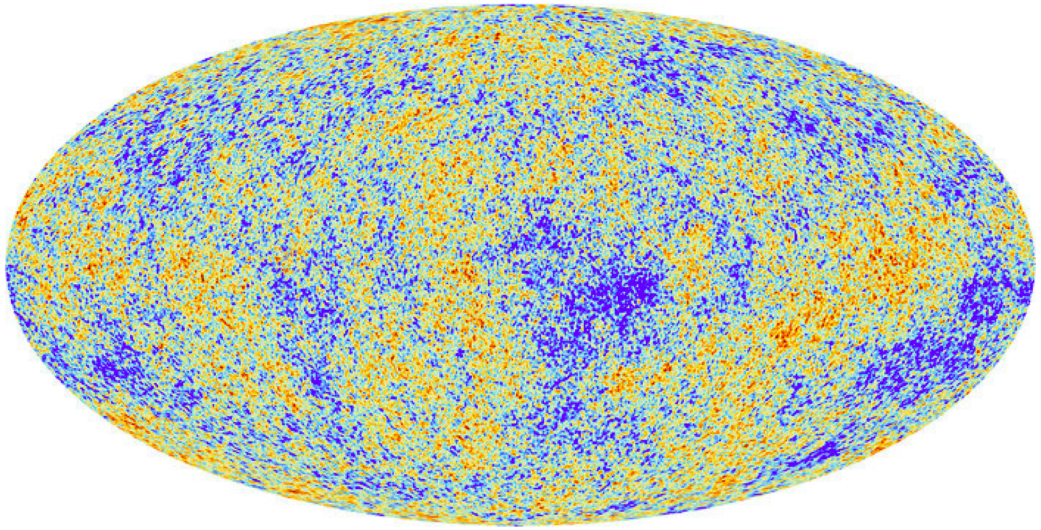
$$\Rightarrow \qquad \Delta u = \frac{\omega}{k}\frac{\Delta\rho}{\rho_0} = c_s\frac{\Delta\rho}{\rho_0}$$

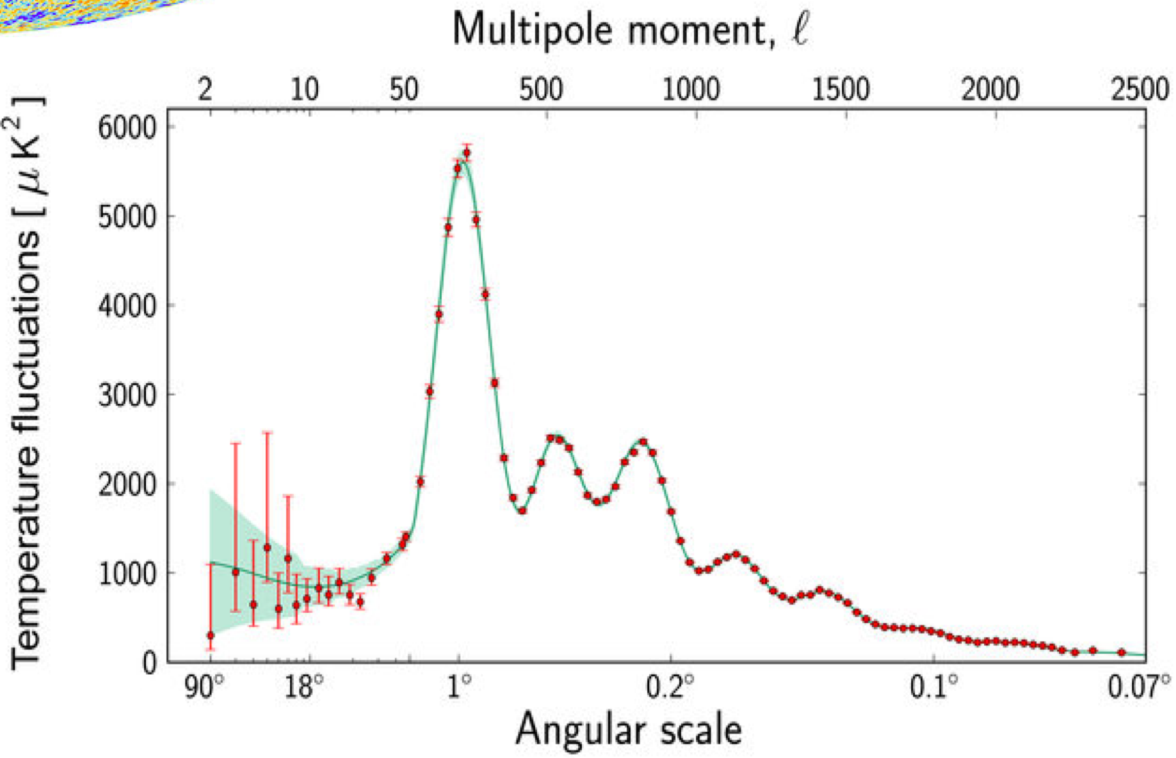
So we see that:

- Density and velocity perturbations are in phase since $\Delta u/\Delta\rho \in \mathbb{R}$
- The actual fluid velocity is much less than the sound speed:

$$\Delta u_0 = c_s\frac{\Delta\rho_0}{\rho_0} \ll c_s$$

CMB (Planck collaboration)





Sound waves in a stratified medium

Let’s now look at situation with (externally imposed) gravity; sound waves propagating in a background atmosphere which is in hydrostatic equilibrium.

Consider isothermal atmosphere with constant gravitational field, $\mathbf{g} = -g\hat{\mathbf{z}}$. Equilibrium is:

$$u_0 = 0$$

$$\rho_0(z) = \tilde{\rho} e^{-z/H}, \quad H \equiv \frac{\mathcal{R}_* T}{g\mu}$$

$$p_0(z) = \frac{\mathcal{R}_* T}{\mu}\rho_0(z) = \tilde{p} e^{-z/H}.$$

Nature of sound waves traveling in x- and y-direction is unaltered. So let’s just examine waves travelling in z-direction. Let $\mathbf{u} = u\hat{\mathbf{z}}$

So relevant fluid equations are

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\rho u) = 0$$
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g,$$

Introduce perturbations

$$u \to \Delta u$$
$$\rho_0 \to \rho_0 + \Delta\rho$$
$$p_0 \to p_0 + \Delta p.$$

And recall relation between Eulerian and Lagrangian pert^n, $\delta\rho = \Delta\rho - \boldsymbol{\xi} \cdot \boldsymbol{\nabla}\rho.$

$$\left.\begin{aligned} \delta\rho &= \Delta\rho - \xi_z \frac{\partial \rho_0}{\partial z} \\ \delta p &= \Delta p - \xi_z \frac{\partial p_0}{\partial z} \\ \delta u &= \Delta u, \end{aligned}\right\} \text{ Eulerian to Lagrangian perturbation relation}$$

$$\Delta \mathbf{u} = \frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} = \frac{\partial \boldsymbol{\xi}}{\partial t} + \cancel{\mathbf{u} \cdot \boldsymbol{\nabla}\boldsymbol{\xi}}^{\text{2nd order}} = \frac{\partial \boldsymbol{\xi}}{\partial t}$$

Now we introduce perturbations into fluid equations. Start with continuity eqn:

$$\frac{\partial}{\partial t}(\rho_0 + \delta\rho) + \frac{\partial}{\partial z}[(\rho_0 + \delta\rho)\delta u_z] = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\left(\rho_0 + \Delta\rho - \xi_z \frac{\partial \rho_0}{\partial z}\right) + \frac{\partial}{\partial z}(\rho_0 \Delta u_z) = 0 \quad \text{(ignoring 2nd order terms)}$$

$$\Rightarrow \quad \cancelto{0}{\frac{\partial \rho_0}{\partial t}} + \frac{\partial \Delta\rho}{\partial t} - \frac{\partial \xi_z}{\partial t}\frac{\partial \rho_0}{\partial z} - \cancelto{0}{\xi_z \frac{\partial}{\partial t}\frac{\partial \rho_0}{\partial z}} + \frac{\partial \rho_0}{\partial z}\Delta u_z + \rho_0 \frac{\partial \Delta u_z}{\partial z} = 0$$

$$\Rightarrow \quad \frac{\partial \Delta\rho}{\partial t} - \underbrace{\Delta u_z}_{\partial \xi_z/\partial t} \frac{\partial \rho_0}{\partial z} + \frac{\partial \rho_0}{\partial z}\Delta u_z + \rho_0 \frac{\partial \Delta u_z}{\partial z} = 0$$

$$\Rightarrow \quad \frac{\partial \Delta\rho}{\partial t} + \rho_0 \frac{\partial \Delta u_z}{\partial z} = 0. \qquad ③$$

Similarly, momentum equation gives (exercise for student!)

$$\frac{\partial \Delta u_z}{\partial t} = -\frac{1}{\rho_0}\frac{\partial \Delta p}{\partial z}$$

$$\Rightarrow \quad \frac{\partial \Delta u_z}{\partial t} = -\frac{c_u^2}{\rho_0}\frac{\partial \Delta\rho}{\partial z}, \quad c_u^2 \equiv \left.\frac{\partial p}{\partial \rho}\right|_{\rho_0}.$$

So, taking time-derivative of (3)

$$\frac{\partial^2 \Delta\rho}{\partial t^2} + \rho_0 \frac{\partial}{\partial z}\left(\frac{\partial \Delta u_z}{\partial t}\right) = 0$$

$$\Rightarrow \quad \frac{\partial^2 \Delta\rho}{\partial t^2} - \rho_0 \frac{\partial}{\partial z}\left(\frac{c_u^2}{\rho_0}\frac{\partial \Delta\rho}{\partial z}\right) = 0,$$

$$\frac{\partial^2 \Delta\rho}{\partial t^2} - \cancel{\rho_0}\frac{c_u^2}{\cancel{\rho_0}}\frac{\partial^2 \Delta\rho}{\partial z^2} + \cancel{\rho_0}\frac{c_u^2}{\rho_0^{\cancel{2}}}\frac{\partial \rho_0}{\partial z}\frac{\partial \Delta\rho}{\partial z} = 0$$

Assuming isothermal so c_u constant

$$\Rightarrow \quad \underbrace{\frac{\partial^2 \Delta\rho}{\partial t^2} - c_u^2 \frac{\partial^2 \Delta\rho}{\partial z^2}}_{\text{normal sound wave equation}} + \underbrace{\frac{c_u^2}{\rho_0}\boxed{\frac{\partial \rho_0}{\partial z}}\frac{\partial \Delta\rho}{\partial z}}_{\text{extra piece associated with stratification}} = 0$$

$$\frac{\partial \rho_0}{\partial z} = \frac{\partial}{\partial z}\left(\tilde{\rho} e^{-z/H}\right) = -\frac{1}{H}\tilde{\rho} e^{-z/H} = -\frac{\rho_0}{H}$$

$$\boxed{\frac{\partial^2 \Delta\rho}{\partial t^2} - c_u^2 \frac{\partial^2 \Delta\rho}{\partial z^2} - \frac{c_u^2}{H}\frac{\partial \Delta\rho}{\partial z} = 0}$$

Dispersion relation… put $\Delta\rho \propto e^{\mathrm{i}(kz-\omega t)}$

$$\Rightarrow \qquad -\omega^2 = -c_u^2 k^2 + c_u^2 \frac{\mathrm{i}k}{H}$$

$$\Rightarrow \qquad \boxed{\omega^2 = c_u^2\left(k^2 - \frac{\mathrm{i}k}{H}\right)} \qquad \text{Dispersion Relation}$$

$$\Rightarrow \qquad k^2 - \frac{\mathrm{i}k}{H} - \frac{\omega^2}{c_u^2} = 0$$

$$\Rightarrow \qquad k = \frac{\mathrm{i}}{2H} \pm \sqrt{\frac{\omega^2}{c_u^2} - \frac{1}{4H^2}}$$

$$k = \frac{\mathrm{i}}{2H} \pm \sqrt{\frac{\omega^2}{c_u^2} - \frac{1}{4H^2}}$$

Case I : $\omega > c_u/2H$

$$\mathrm{Im}\, k = \frac{1}{2H} \qquad \mathrm{Re}\, k = \pm\sqrt{\left(\frac{\omega}{c_u}\right)^2 - \left(\frac{1}{2H}\right)^2}$$

$$\Delta\rho \propto \underbrace{e^{-z/2H}}\, \underbrace{e^{\mathrm{i}\left(\pm\sqrt{(\omega/c_u)^2 - (1/2H)^2}\, z - \omega t\right)}}$$

Exp. decay, scaleheight 2H

Traveling wave with phase velocity

$$v_{\mathrm{ph}} = \frac{\omega}{\mathbb{K}}, \quad \mathbb{K} \equiv \pm\sqrt{\left(\frac{\omega}{c_u}\right)^2 - \left(\frac{1}{2H}\right)^2}$$

$$\left.\begin{array}{l} \Delta\rho \propto e^{-z/2H} \\ \rho_0 \propto e^{-z/H} \end{array}\right\} \quad \frac{\Delta\rho}{\rho_0} \propto e^{+z/2H}$$

$$\Delta u_z \propto e^{+z/2H}$$

Eventually, perturbations become large; 1st order pertn theory fails.

Case II : $\omega < c_u/2H$

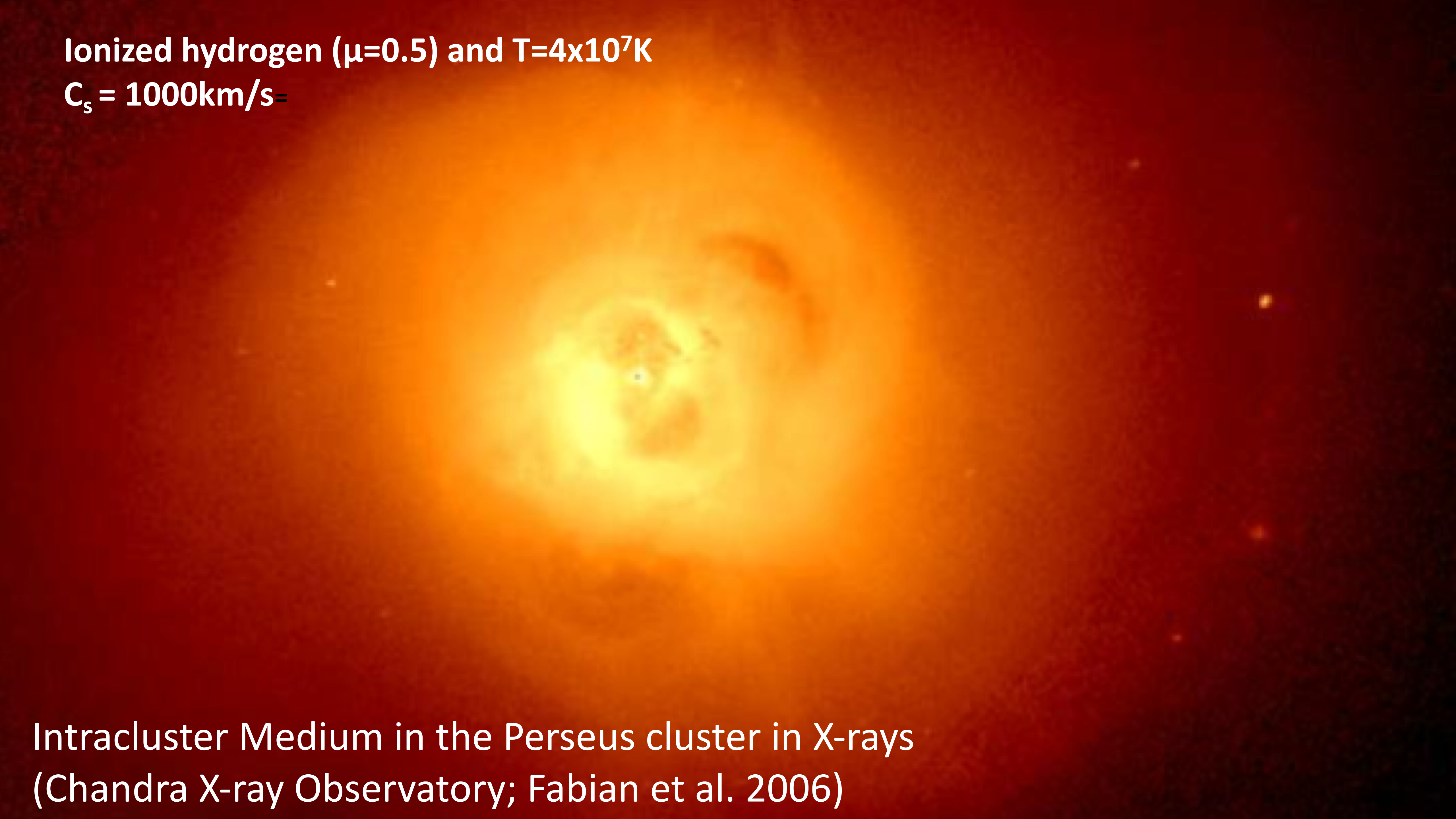
k purely imaginary

$$\Delta\rho \propto e^{-kz} e^{i\omega t}$$

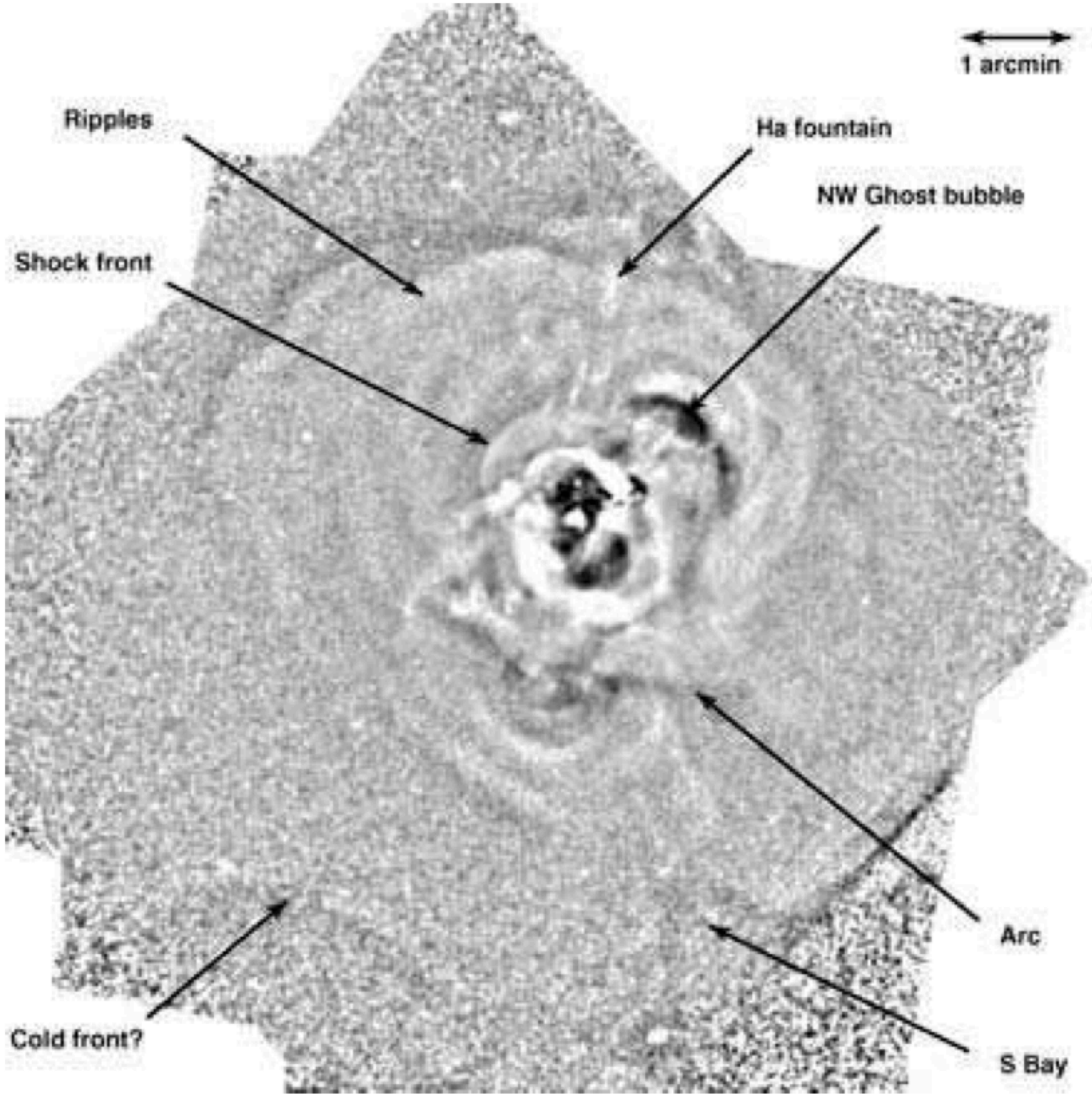
No propagation of energy

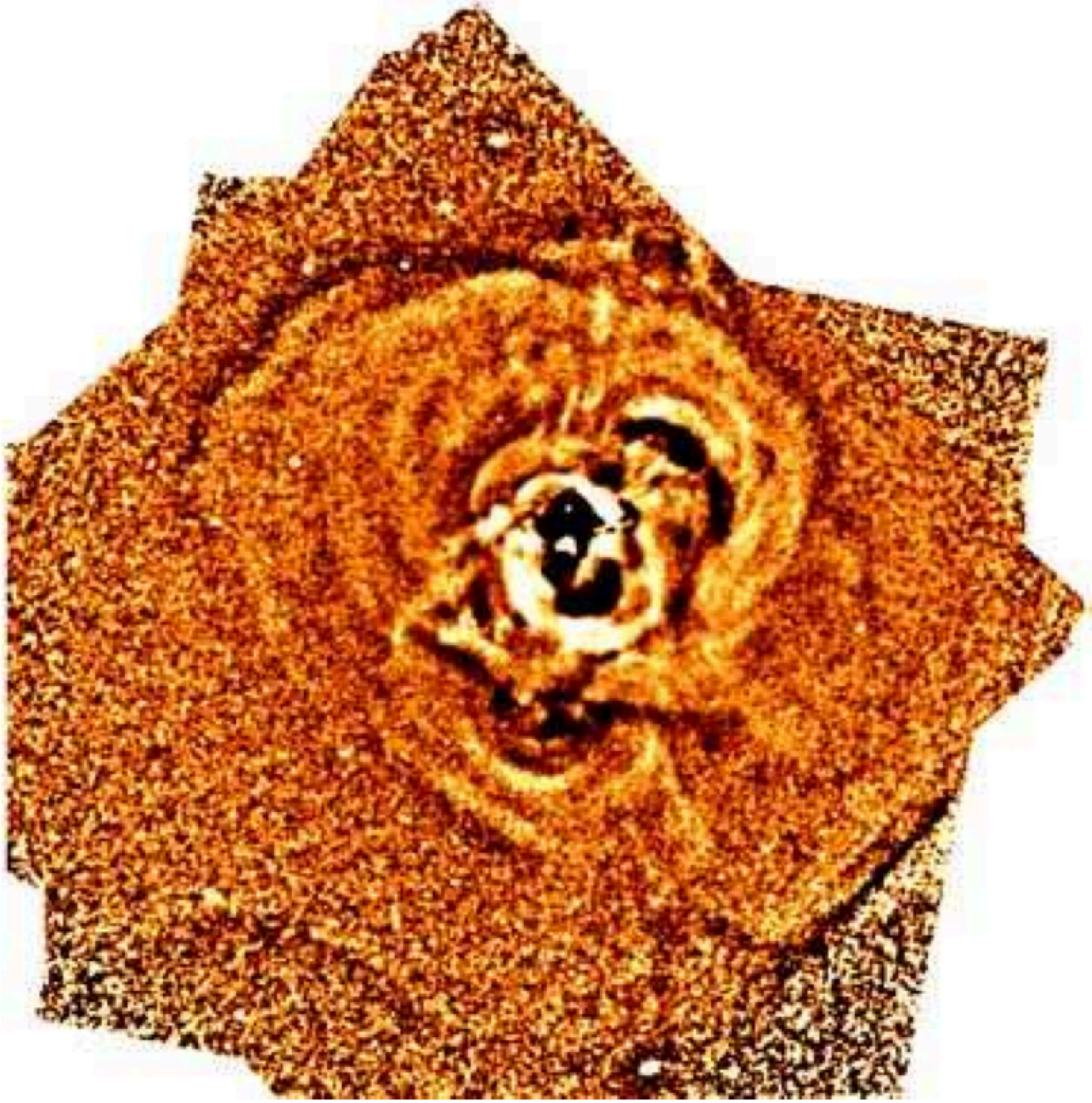
Ionized hydrogen (μ=0.5) and T=4x10^7K

C_s = 1000km/s

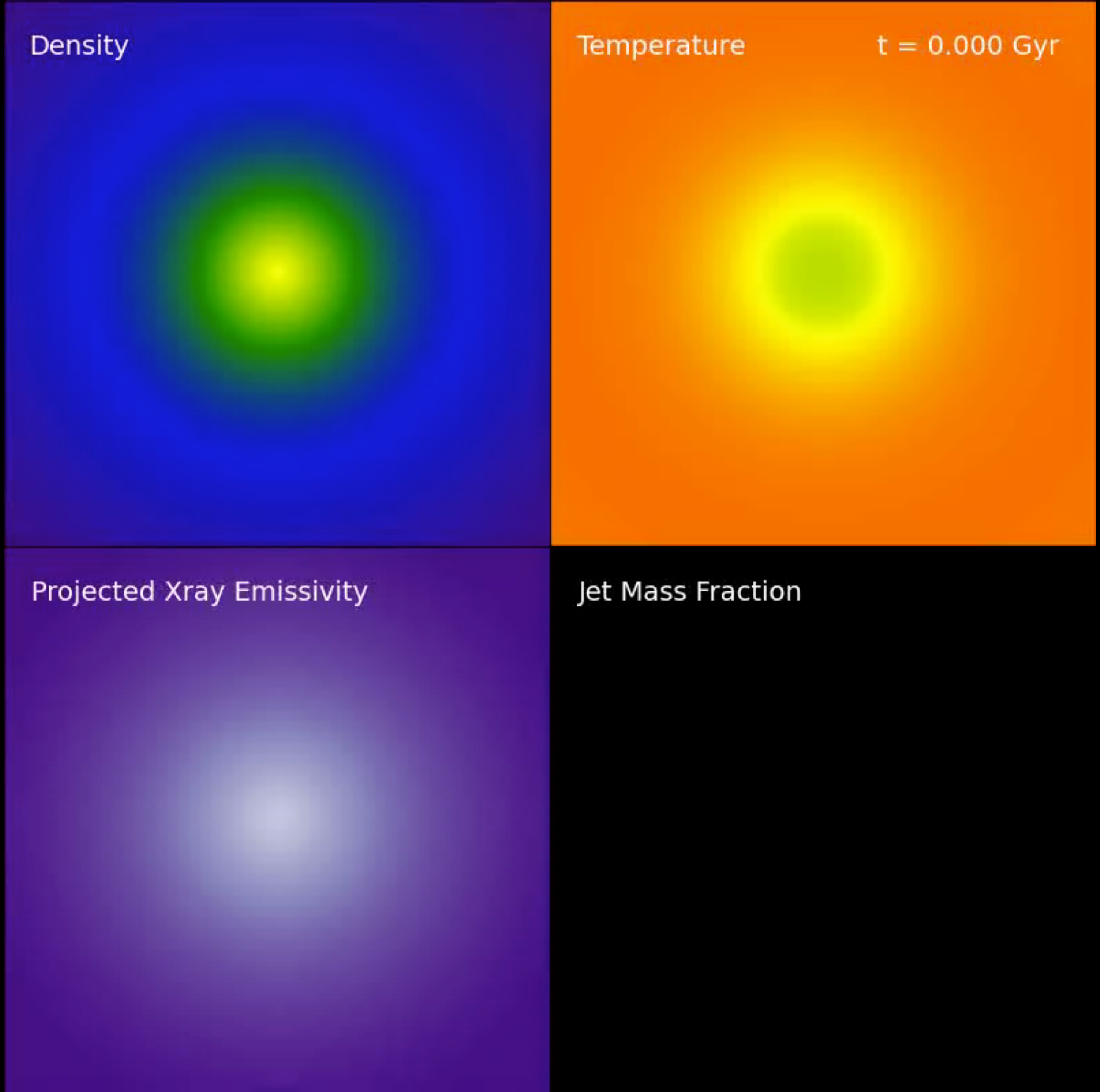


Intracluster Medium in the Perseus cluster in X-rays
(Chandra X-ray Observatory; Fabian et al. 2006)

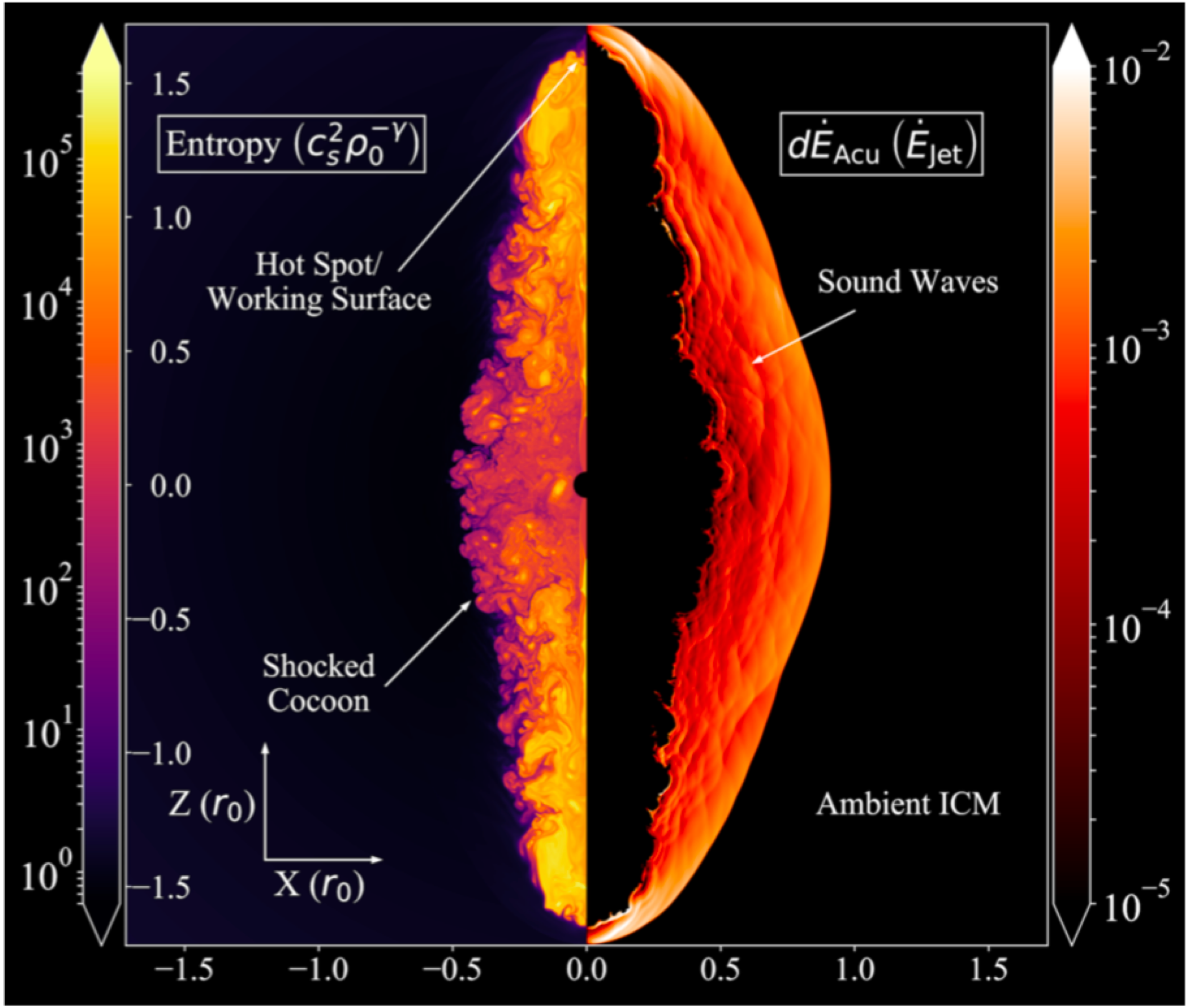




Unsharp-mask image of Perseus cluster of galaxies
Sanders et al. (2006)



Yang & Reynolds (2016)



Bambic & Reynolds (2019)