Part II Astrophysics/Physics

Astrophysical Fluid Dynamics
Lecture 7 : Stellar Scaling Relations

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Recap

Notion of hydrostatic equilibrium

$$\boxed{\frac{1}{\rho}\boldsymbol{\nabla} p = -\boldsymbol{\nabla}\Psi}$$

EQUATION OF HYDROSTATIC EQM.

If system self-gravitating, also see to solve

$$\boxed{\nabla^2\Psi = 4\pi G\rho}$$

Spherically-symmetric self-gravitating polytropes ($p = K\rho^{1+1/n}$)

$$\boxed{\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n}$$

LANE-EMDEN EQN. OF INDEX n

$$\theta = \frac{\Psi_T - \Psi}{\Psi_T - \Psi_c} \qquad \xi = r\sqrt{\frac{4\pi G\rho_c}{\Psi_T - \Psi_c}} \qquad \rho = \rho_c\left(\frac{\Psi_T - \Psi}{\Psi_T - \Psi_c}\right)^n$$

This Lecture

- Hydrostatic equilibrium (cont)…
- Scaling relations (E.4)
  - Mass-radius relationship for polytropic stars
    - E.g. White Dwarfs
  - Reconciling with mass-radius relation of main sequence stars
  - Consideration of timescales
  - Mass transfer binaries and response of the stars

- After this lecture, you can do…
  - Example Sheet 1 (all)
  - Example Sheet 2 up to Q6

Mass-Radius Relation for Polytropic Stars

We can consider families of stars that can be treated as polytropes with a given polytropic index. Examples…

- Fully convective stars dominated by ideal gas pressure,

$$p = K\rho^{\gamma} \quad (\gamma = 5/3)$$

- White Dwarfs (well below the Chandrasekhar mass)

$$p = \frac{\pi^2 \hbar^2}{5 m_e m_{\rm ion}^{5/3}} \left(\frac{3}{\pi}\right)^{2/3} \rho^{5/3}$$

- Both cases correspond to n=3/2.

Key point is that the shape of the density distribution within each star in the family is identical, given by the appropriately scaled solution of Lane-Emden.

To be concrete, for now, consider a family of stars with $p = K\rho^{1+1/n}$ where both n and K are fixed across the family.

$$\rho = \left[\frac{\Psi_T - \Psi}{(n+1)K}\right]^n \qquad \Rightarrow \qquad \Psi_T - \Psi_c = K(n+1)\rho_c^{1/n}$$

$$\xi = \sqrt{\frac{4\pi G\rho_c}{\Psi_T - \Psi_c}}r \qquad \Rightarrow \qquad \xi = \sqrt{\frac{4\pi G\rho_c^{1-1/n}}{K(1+n)}}r$$

$$\rho = \rho_c\left[\frac{\Psi_T - \Psi}{\Psi_T - \Psi_c}\right]^n = \rho_c\theta^n$$

The surface of the star corresponds to the first zero of the solution of the Lane-Emden equation, i.e. $\xi_{\max}$ such that $\theta(\xi_{\max}) = 0$

Total mass of the polytrope is

$$M = \int_0^{r_{\max}} 4\pi r^2 \rho \, \mathrm{d}r$$

$$= 4\pi\rho_c \left[\frac{4\pi G \rho_c^{1-1/n}}{K(1+n)}\right]^{-3/2} \underbrace{\int_0^{\xi_{\max}} \theta^n \xi^2 \, \mathrm{d}\xi}_{\substack{\text{same for all} \\ \text{polytrope of index } n}}$$

$$\Rightarrow \qquad M \propto \rho_c^{\frac{1}{2}\left(\frac{3}{n}-1\right)}$$

Radius follows from unraveling definition of ξ,

$$\xi = \sqrt{\frac{4\pi G \rho_c^{1-1/n}}{K(1+n)}} r \qquad \Rightarrow \qquad r_{\max} \propto \rho_c^{\frac{1}{2}\left(\frac{1}{n}-1\right)}$$

Eliminating density

$$\boxed{M \propto R^{\frac{3-n}{1-n}}} \qquad \textsc{Mass-Radius Relation for Polytropic Stars}$$

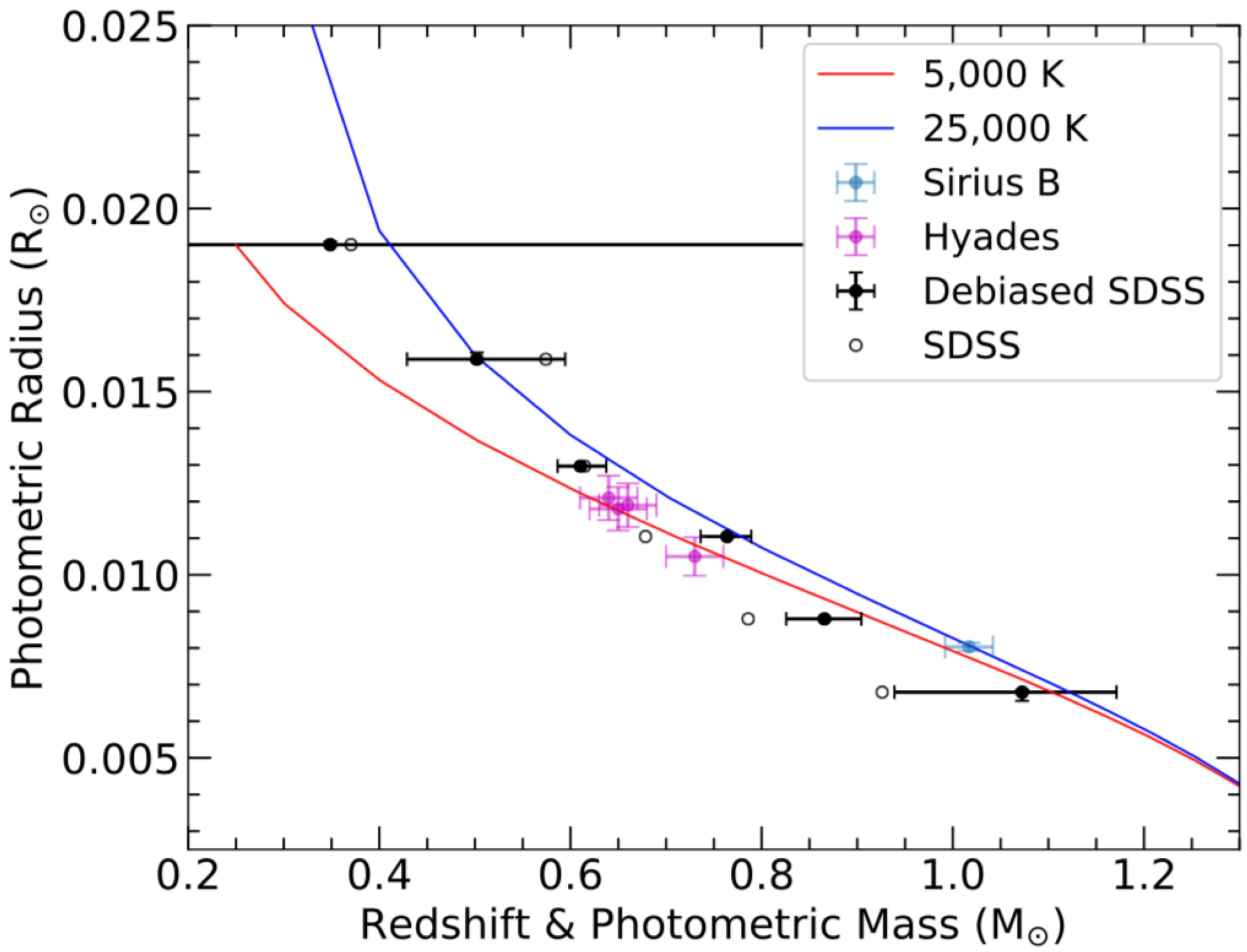
For n=3/2, this gives

$$M \propto R^{-3}$$

$$\text{or } R \propto M^{-1/3}$$

So more massive polytropes are smaller!

This is clearly seen in the white dwarf population.



Chandra et al. (2020; https://arxiv.org/abs/2007.14517)

As we consider progressively more massive white dwarfs, the bulk of the electrons need to be in high energy levels (Fermi surface is higher energy).

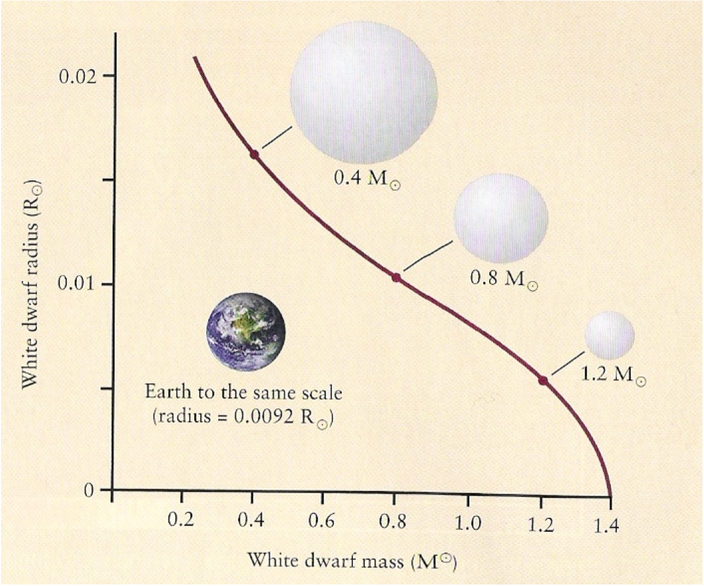
At some point, the electrons become relativistic… standard kinetic theory shows that the equation of state "softens" from $p = K\rho^{5/3}$ to $p = K'\rho^{4/3}$ (corresponding to n=3).

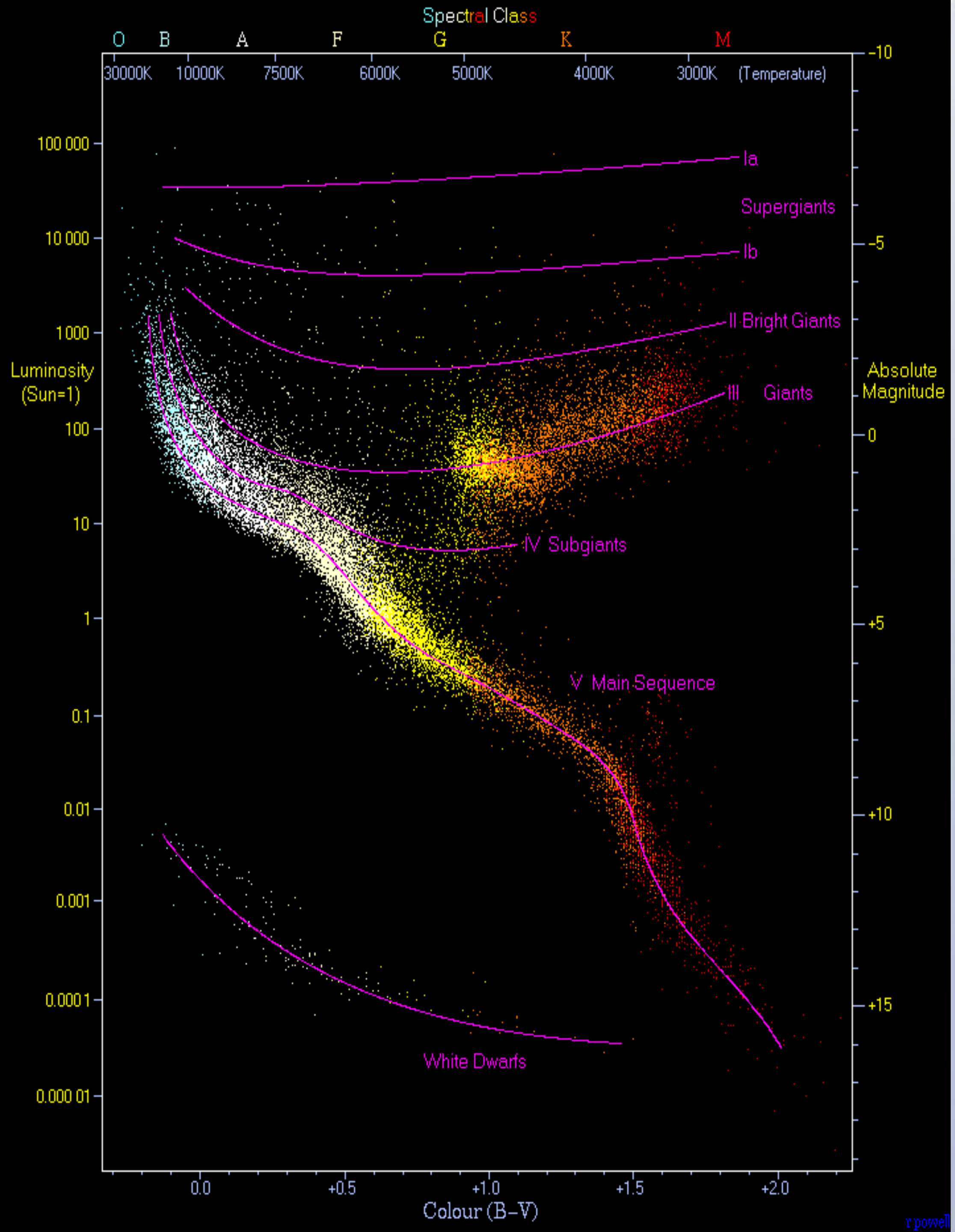
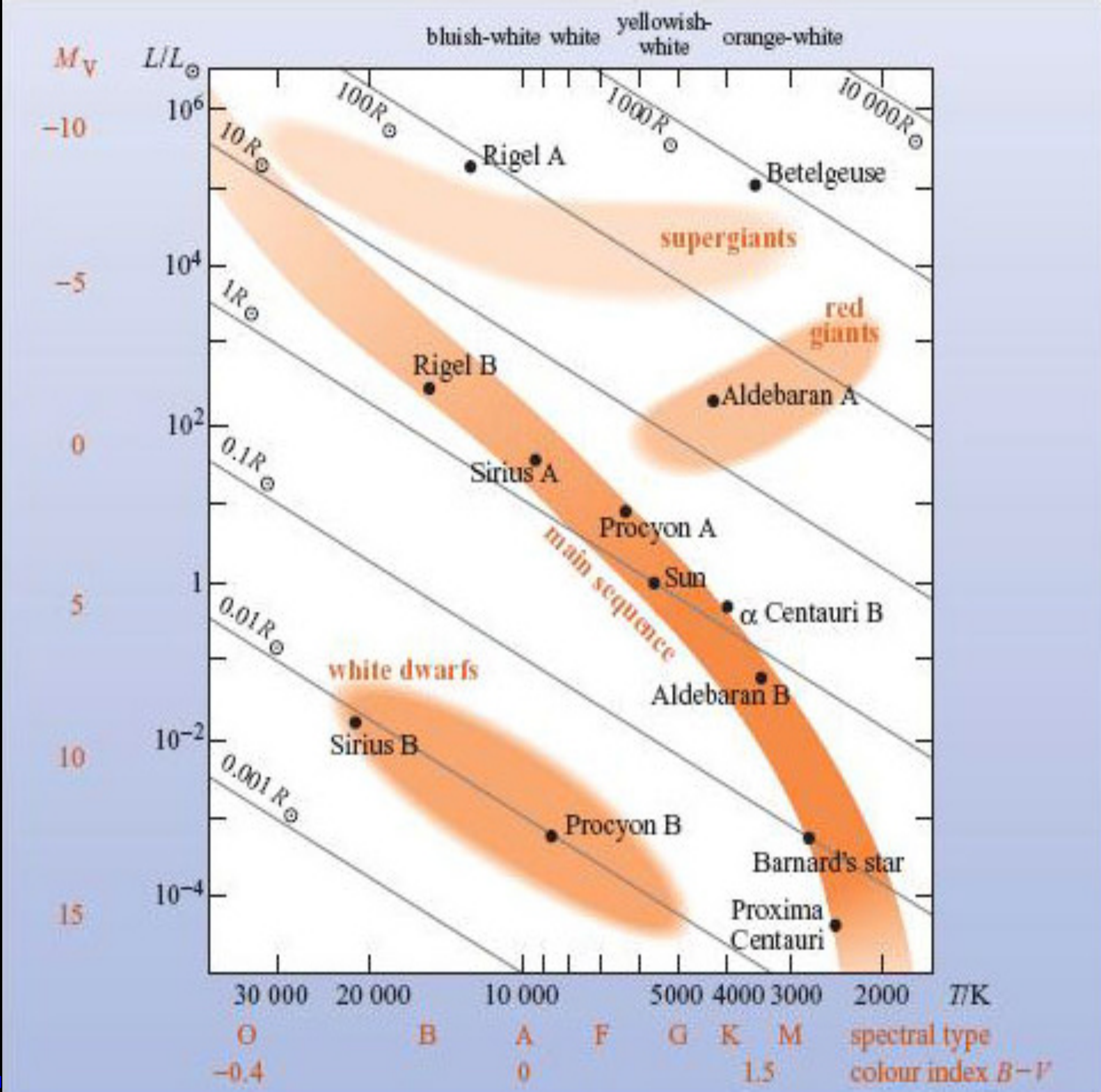
The scaling relation is $M \propto R^{\frac{3-n}{1-n}}$. So, for n=3, the mass is independent of radius… i.e., there is only one permitted mass for the configuration.

This is the Chandrasekar mass, about 1.4Msun

Plays a special part in Type-1a supernova

- WD gains mass (merger or accretion)
- Approaches Chandrasekhar mass
- Collapses
- Thermonuclear fusion of C/O to Fe
- Boom.




Spectral Class
O B A F G K M
30000K 10000K 7500K 6000K 5000K 4000K 3000K (Temperature)
Luminosity (Sun=1)
100 000
10 000
1000
100
10
1
0.1
0.01
0.001
0.0001
0.000 01
Absolute Magnitude
-10
-5
0
+5
+10
+15
Ia
Supergiants
Ib
II Bright Giants
III Giants
IV Subgiants
V Main Sequence
White Dwarfs
0.0 +0.5 +1.0 +1.5 +2.0
Colour (B−V)
r.powell

bluish-white white yellowish-white orange-white
M_V
$L/L_\odot$
10^6
10^4
10^2
1
10^{-2}
10^{-4}
−10
−5
0
5
10
15
100$R_\odot$
1000$R_\odot$
10 000$R_\odot$
10$R_\odot$
1$R_\odot$
0.1$R_\odot$
0.01$R_\odot$
0.001$R_\odot$
Rigel A
Betelgeuse
supergiants
red giants
Aldebaran A
Rigel B
Sirius A
Procyon A
main sequence
Sun
α Centauri B
white dwarfs
Aldebaran B
Sirius B
Procyon B
Barnard's star
Proxima Centauri
30 000 20 000 10 000 5000 4000 3000 2000 T/K
O B A F G K M spectral type
−0.4 0 1.5 colour index B−V

Mass Sequence stars to NOT show $R \propto M^{-1/3}$.

Instead, across much of the mass sequence we see $R \propto M$.

Our mistake was to assume constant K across the main sequence. Let's examine the temperature at the centre of the star:

$$\left.\begin{aligned} p &= K\rho^{1+1/n} \\ p &= \frac{\mathcal{R}_*}{\mu}\rho T \end{aligned}\right\} \quad \Rightarrow \quad T_c = \frac{\mu K}{\mathcal{R}_*}\rho_c^{1/n}$$

The thermonuclear reactions that power stars are extremely temperature sensitive… so across the main sequence the stars will adjust as to produce approximately the same temperature. Thus we need:

$$K \propto \rho_c^{-1/n}$$

Putting this into our analysis of mass and radius we get

$$M \propto \rho_c^{-1/2}, \quad R \propto \rho_c^{-1/2} \quad \Rightarrow \quad M \propto R$$

We can also use these techniques to examine the behaviour of an individual star that is gaining or losing mass.

In this case, should we use constant K or constant T_c? It depends on the timescale on which the star's mass is changing…

- Hydrostatic equilibrium is established on sound-crossing timescale…

$$t_{\rm h} \sim R/C_s \sim 1\ {\rm day}$$

- Energy/entropy of core can appreciably change on thermal timescale…

$$t_{\rm th} \sim E_{\rm tot}/L \sim \frac{GM^2}{RL} \sim 30\ {\rm Myr}$$

FOR SUN

Example – Spherical rotating polytropic star with angular velocity Ω gains non-rotating mass on less than the thermal timescale. How does Ω change?

- Conservation of angular momentum:

$$MR^2\Omega = \text{const.}$$

- So

$$MR^2\Delta\Omega + \Omega\Delta(MR^2) = 0$$

$$\Rightarrow \qquad \frac{\Delta\Omega}{\Omega} = -\frac{\Delta(MR^2)}{MR^2}$$

- But

$$R \propto M^{(1-n)/(3-n)}$$

$$\Rightarrow \qquad \frac{\Delta\Omega}{\Omega} \propto -\Delta\left(M^{(5-3n)/(3-n)}\right)$$

$$\Rightarrow \qquad \frac{\Delta\Omega}{\Omega} \propto -\left(\frac{5-3n}{3-n}\right)\Delta M$$

$$\Delta M > 0 \quad \Rightarrow \quad \begin{cases} \Delta\Omega < 0 & \text{if} \quad \frac{5-3n}{3-n} > 0 \qquad (\text{e.g. } n = \frac{3}{2}) \text{ Spin Down} \\ \Delta\Omega > 0 & \text{if} \quad \frac{5-3n}{3-n} < 0 \qquad (\text{e.g. } n = 2) \text{ Spin Up} \end{cases}$$

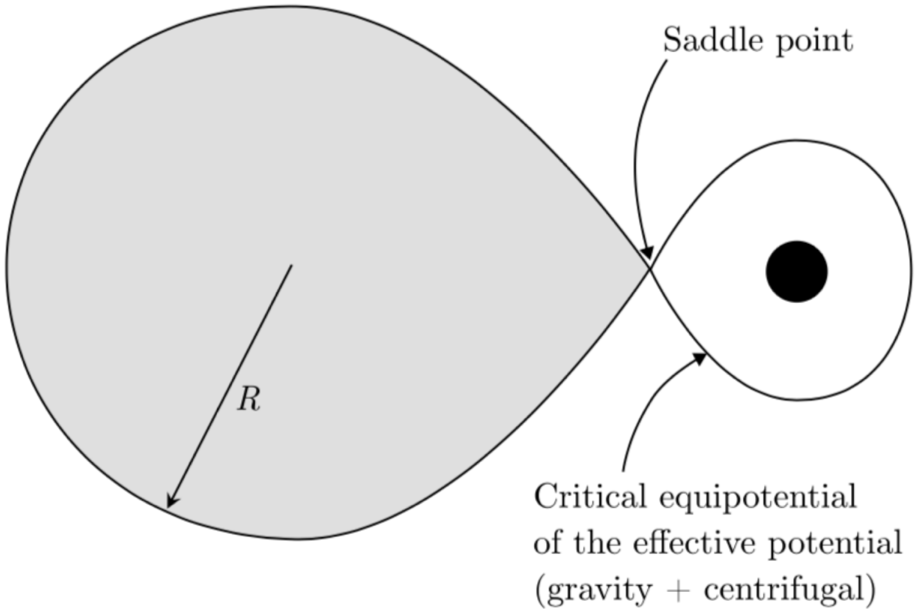
Example – Star in a close binary system loses mass to a companion.

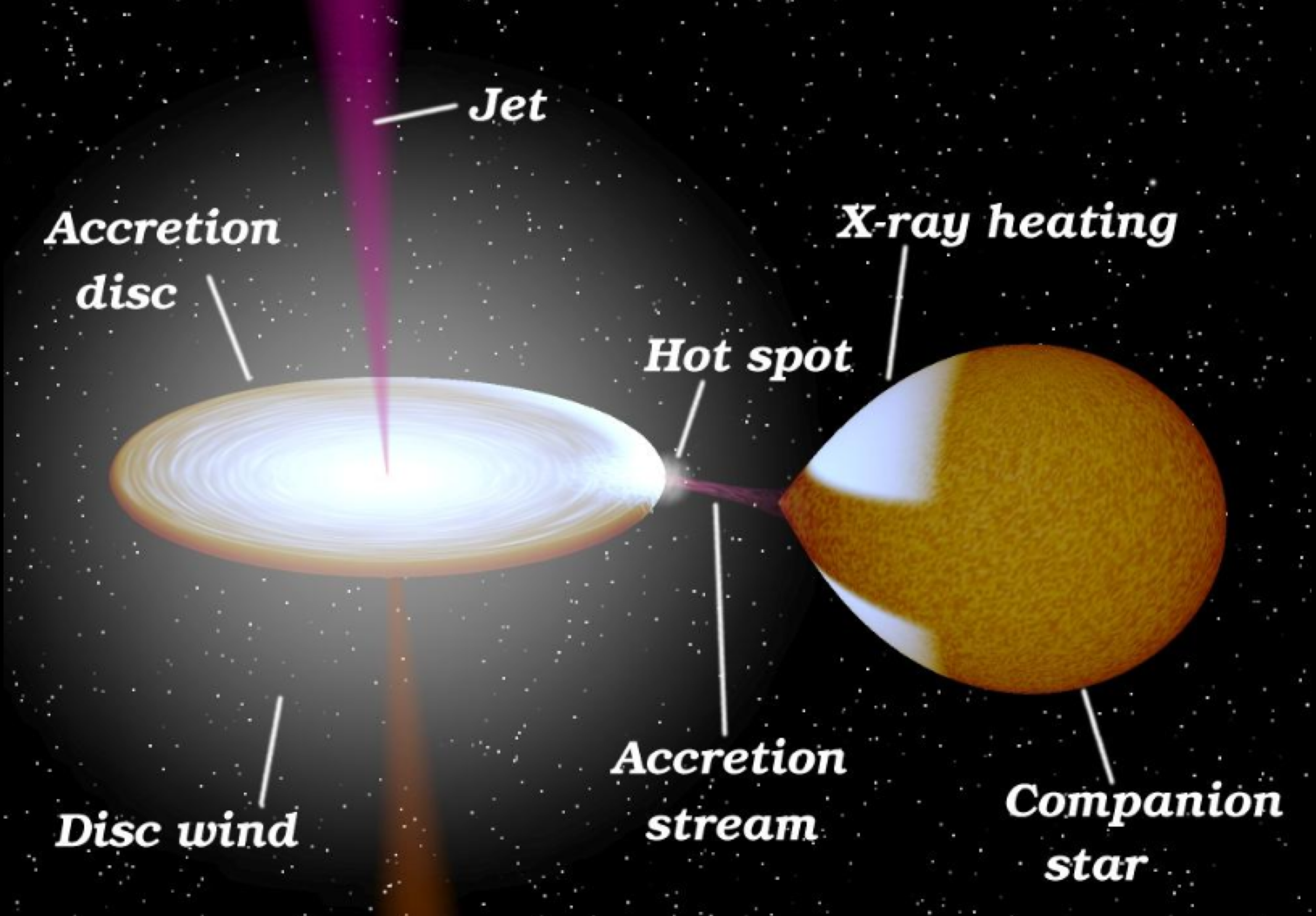
- Donor star loses mass

$$\Delta M < 0.$$

$$R \propto M^{(1-n)/(3-n)}$$

- So radius increases if 1<n<3.
- There is then the potential for runaway mass transfer




Jet
Accretion disc
X-ray heating
Hot spot
Disc wind
Accretion stream
Companion star