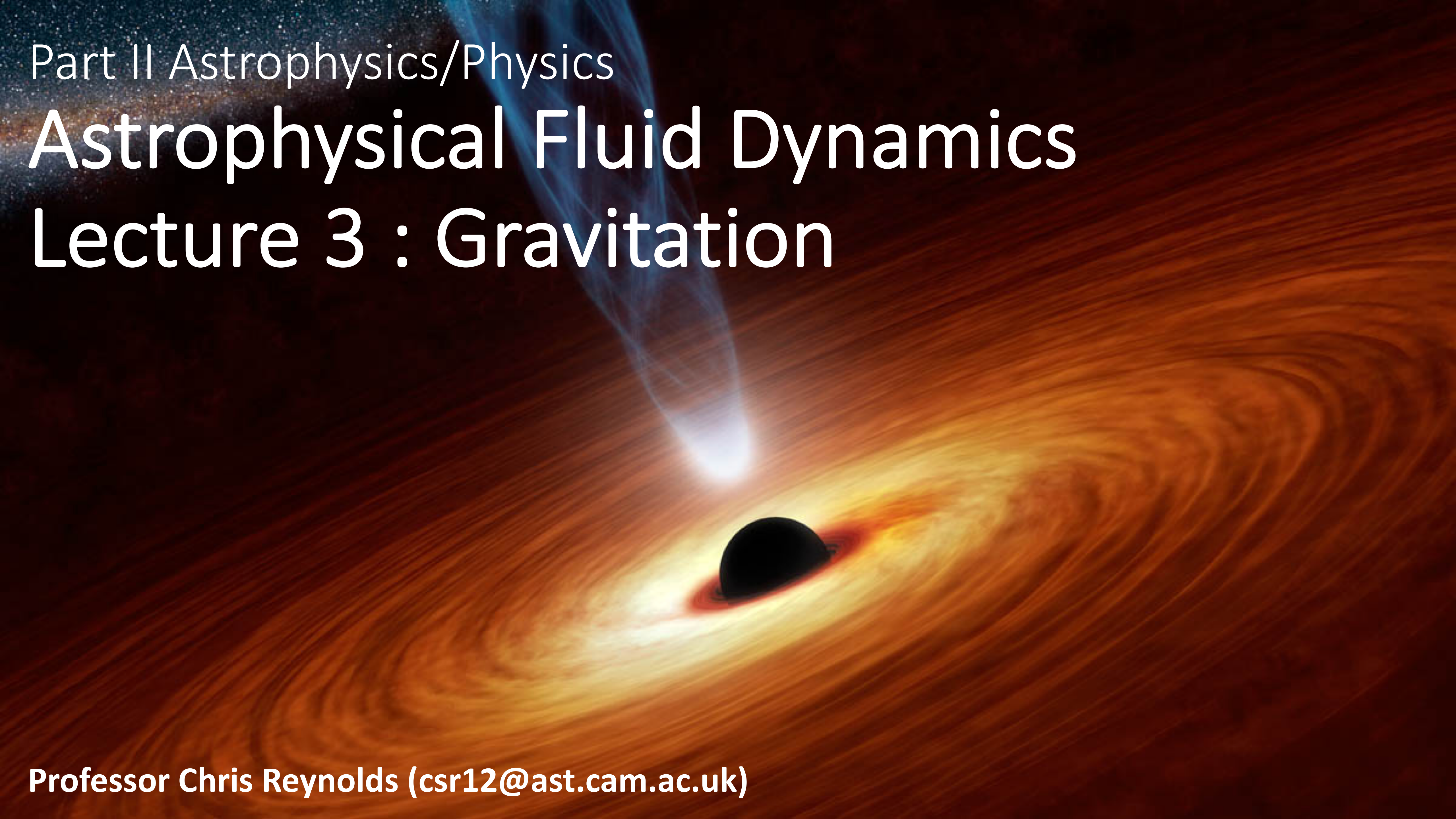


Part II Astrophysics/Physics

Astrophysical Fluid Dynamics
Lecture 3 : Gravitation

Professor Chris Reynolds (csr12@ast.cam.ac.uk)

Recap

- Continuity equation (conservation of mass)

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0}$$

EULERIAN CONTINUITY EQUATION

$$\boxed{\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \boldsymbol{\nabla} \cdot \mathbf{u} = 0}$$

LAGRANGIAN CONTINUITY EQUATION

- Momentum equation

$$\boxed{\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\boldsymbol{\nabla} p + \rho \mathbf{g}}$$

LAGRANGIAN MOMENTUM EQUATION

$$\boxed{\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \rho \mathbf{g}}$$

EULERIAN MOMENTUM EQUATION

$$\boxed{\partial_t(\rho \mathbf{u}) = -\boldsymbol{\nabla} \cdot (\rho \mathbf{u} \otimes \mathbf{u} + p\underline{\underline{\mathbf{I}}}) + \rho \mathbf{g}}$$

CONSERVATIVE FORM

Today’s lecture

- Gravitation (Chapter C)

- Basics (C.1)
  - Gravitational potential and Poisson’s Equation
  - Cases with special symmetries
- Gravitational potential energy (C.2+C.3)
- The Virial Theorem (C.4)

C.1 : Recap of basics

Define **gravitational potential** Ψ : force per unit mass g given by $\boxed{\mathbf{g} = -\boldsymbol{\nabla}\Psi}$

Conservative force… work done independent of path:

$$-\int_{\mathbf{r}}^{\infty} \mathbf{g} \cdot \mathrm{d}\mathbf{l} = \int_{\mathbf{r}}^{\infty} \boldsymbol{\nabla}\Psi \cdot \mathrm{d}\mathbf{l} = \Psi(\infty) - \Psi(\mathbf{r})$$

Newton's law for point mass

$$\Psi = -\frac{GM}{r} \qquad \text{if mass at origin}$$

For system of masses: $$\Psi = -\sum_i \frac{GM_i}{|\mathbf{r} - \mathbf{r}_i'|}$$

$$\Rightarrow \qquad \mathbf{g} = -\boldsymbol{\nabla}\Psi = -\sum_i \frac{GM_i(\mathbf{r} - \mathbf{r}_i')}{|\mathbf{r} - \mathbf{r}_i'|^3} \longrightarrow -G \int \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \mathrm{d}V'$$

Continuum limit

- So, we have

$$\boldsymbol{\nabla}_{\mathbf{r}} \cdot \mathbf{g} = -G \int \rho(\mathbf{r}') \underbrace{\boldsymbol{\nabla}_{\mathbf{r}} \cdot \left[\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right]}_{4\pi\delta(\mathbf{r}-\mathbf{r}')} \mathrm{d}V'$$

$$= -4\pi G \int \rho(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}')\, \mathrm{d}V'$$

$$= -4\pi G \rho(\mathbf{r})$$

$$\boxed{\boldsymbol{\nabla} \cdot \mathbf{g} = -\nabla^2 \Psi = -4\pi G \rho} \qquad \text{Poisson's Equation}$$

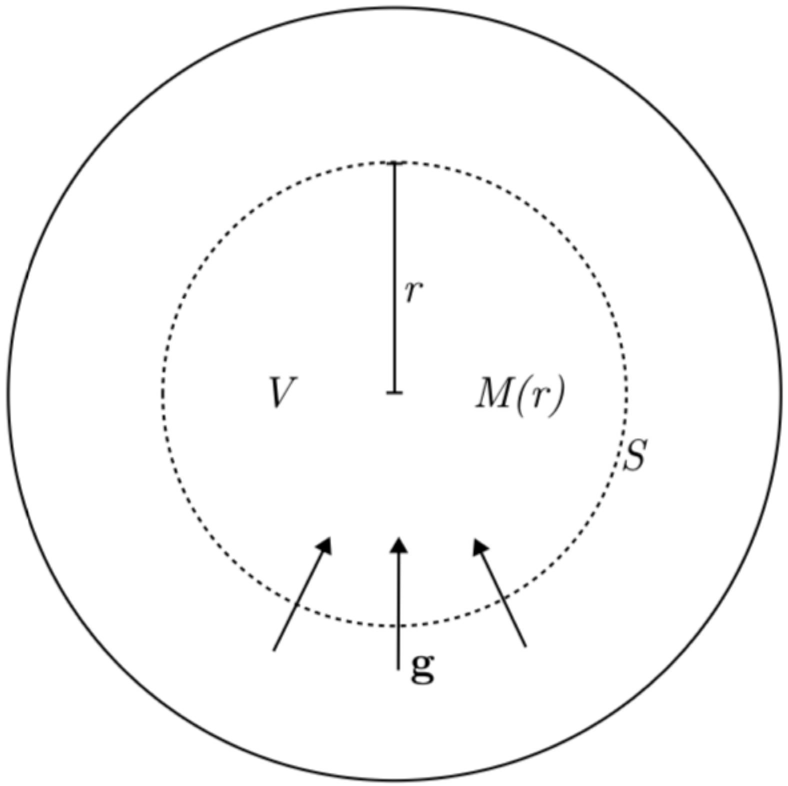
- Or in integral form:

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{g}\, \mathrm{d}V = -4\pi G \int_V \rho\, \mathrm{d}V$$

$$\Rightarrow \qquad \int_S \mathbf{g} \cdot \mathrm{d}\mathbf{S} = -4\pi G M$$

Integral form very useful for computing g when there are symmetries that permit trivial evaluation of surface integral…

- Example 1 : Spherically-symmetric system

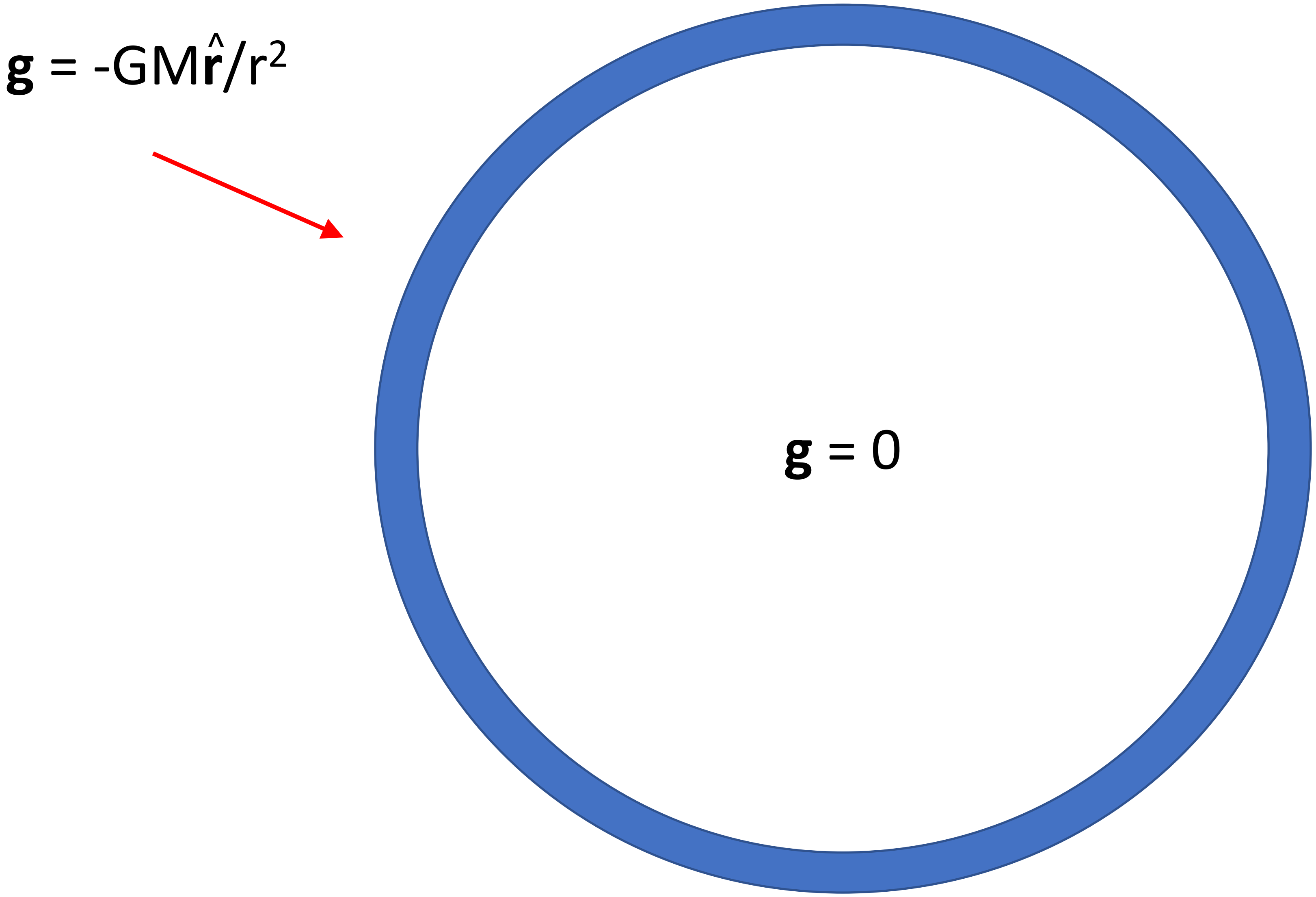


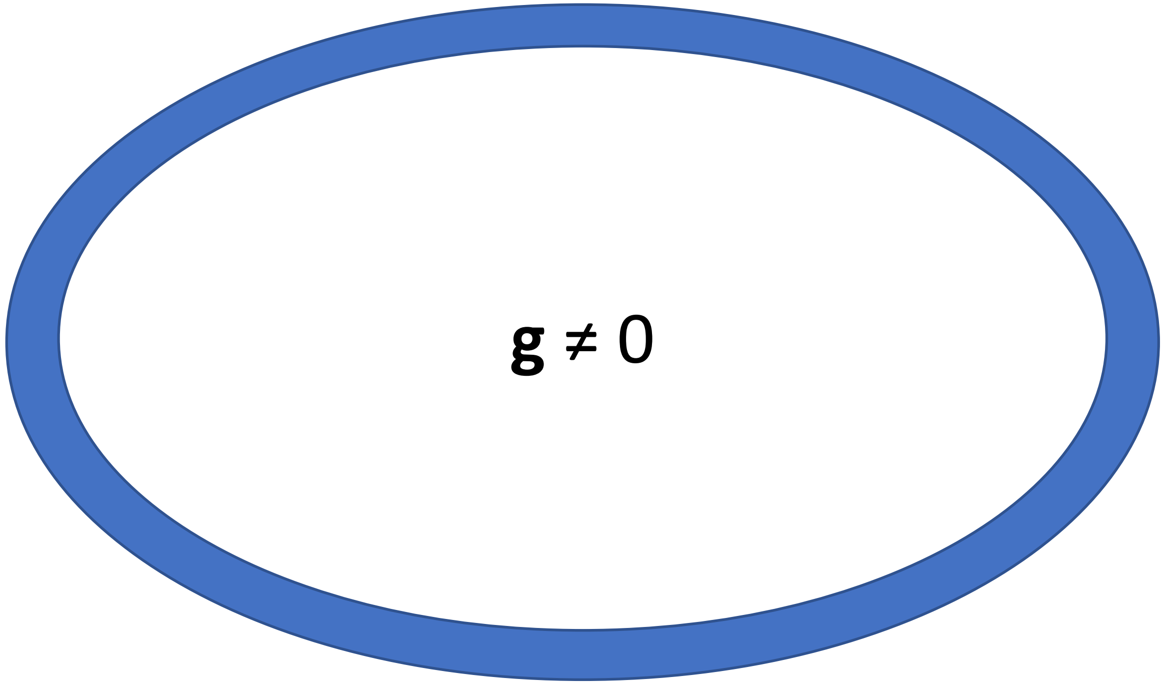
$$\int \mathbf{g} \cdot \mathrm{d}\mathbf{S} = -4\pi G \underbrace{M(r)}_{\substack{\text{mass} \\ \text{enclosed}}}$$

$$\Rightarrow \quad -4\pi r^2 |\mathbf{g}| = -4\pi G M(r)$$

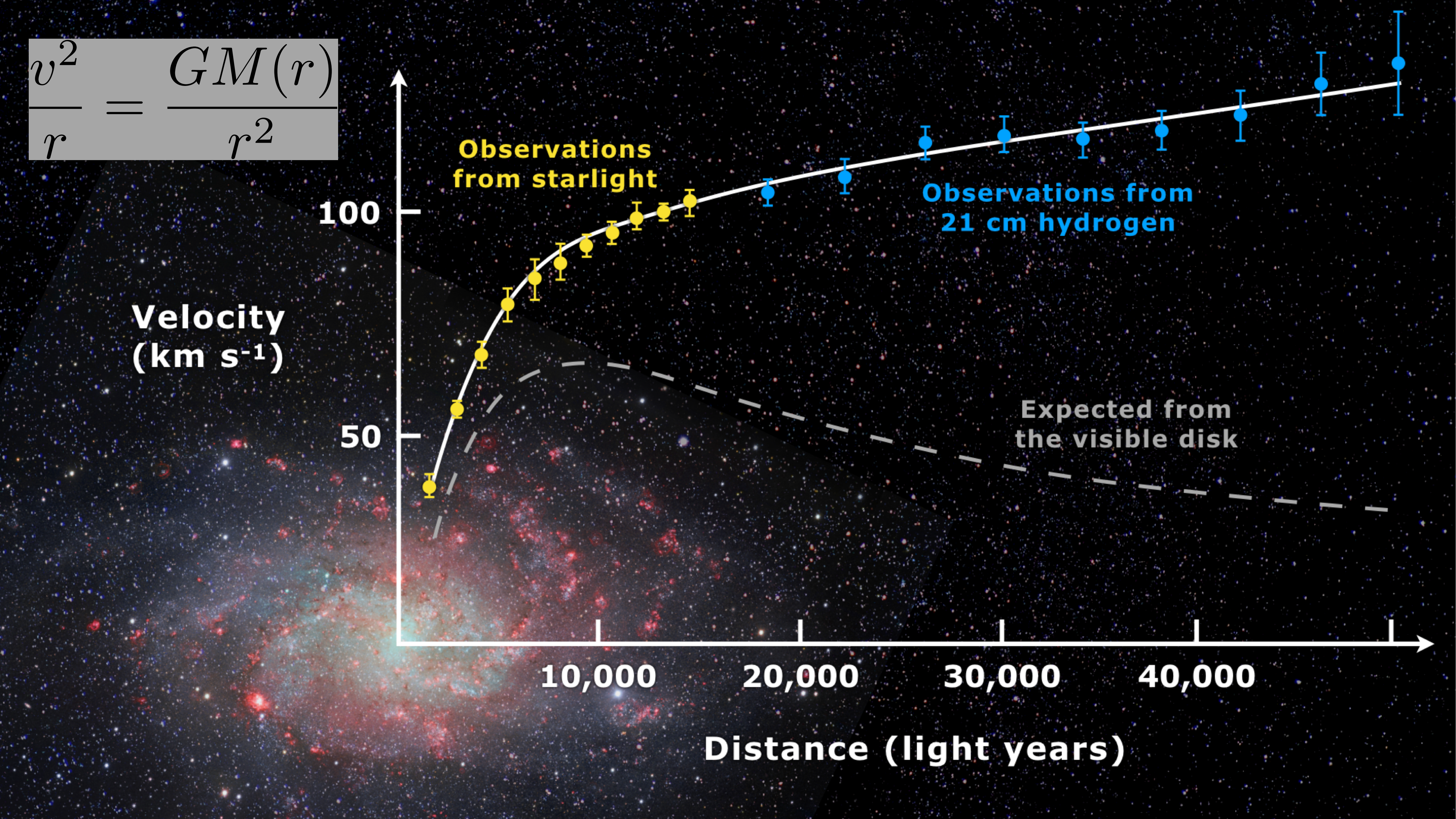
$$\Rightarrow \quad |\mathbf{g}| = \frac{GM(r)}{r^2}$$

$$\therefore \quad \mathbf{g} = -\frac{GM(r)}{r^2}\hat{\mathbf{r}}$$

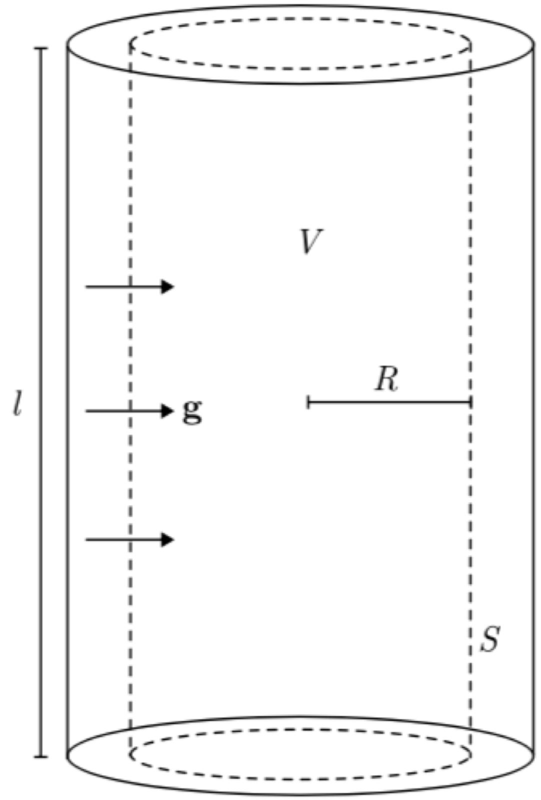

$\mathbf{g} = -GM\hat{\mathbf{r}}/r^2$
$\mathbf{g} = 0$


$\mathbf{g} \neq 0$

$$\frac{v^2}{r} = \frac{GM(r)}{r^2}$$



- Example 2 : Cylindrically-symmetric system

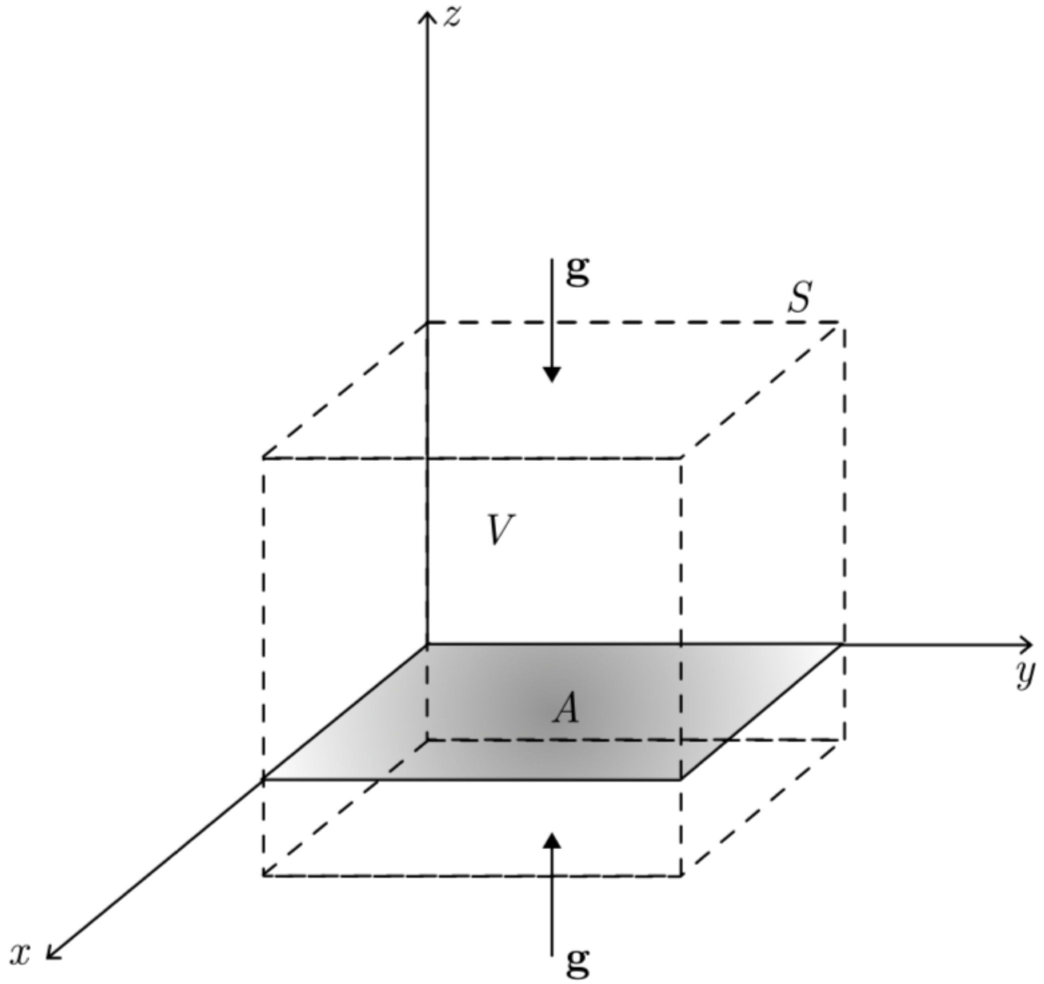


$$\int \mathbf{g} \cdot \mathrm{d}\mathbf{S} = -4\pi G \int_V \rho \, \mathrm{d}V$$

$$\Rightarrow \quad -2\pi R l |\mathbf{g}| = -4\pi G l \cdot \underbrace{M(r)}_{\substack{\text{enclosed mass} \\ \text{per unit length}}}$$

$$\Rightarrow \quad \mathbf{g} = -\frac{2GM(R)}{R}\hat{\mathbf{R}}$$

- Example 3 : Planar geometry with reflection symmetric in z=0



$$\int_S \mathbf{g} \cdot \mathrm{d}\mathbf{S} = -4\pi G \int_V \rho \, \mathrm{d}V$$

$$\Rightarrow \qquad -2|\mathbf{g}|A = -4\pi G A \int_{-z}^{z} \rho(z) \, \mathrm{d}z$$

$$\Rightarrow \qquad \mathbf{g} = -4\pi G \hat{\mathbf{z}} \int_0^z \rho(z) \, \mathrm{d}z$$

C.2 : Potential of spherically-symmetric system

- For spherically-symmetric system,

$$\mathbf{g} = -|\mathbf{g}|\hat{\mathbf{r}}, \qquad |\mathbf{g}| = \frac{G}{r^2}\underbrace{\int_0^r 4\pi\rho(r')r'^2\,\mathrm{d}r'}_{M(r)} = \frac{\mathrm{d}\Psi}{\mathrm{d}r}$$

- so

$$\Psi = \int_\infty^{r_0} \frac{G}{r^2}\left\{\int_0^r 4\pi\rho(r')r'^2\,\mathrm{d}r'\right\}\mathrm{d}r$$

Defining zero of potential at infinity

$$\Psi = -\underbrace{\left\{\frac{G}{r}\int_0^r 4\pi\rho(r')r'^2\,\mathrm{d}r'\right\}\Bigg|_{r=\infty}^{r_0}}_{} + \int_\infty^{r_0}\frac{G}{r}4\pi\rho(r)r^2\,\mathrm{d}r$$

Integrate by parts

Evaluate assuming
$M(r) \to 0$ as $r \to 0$

C.2 : Potential of spherically-symmetric system

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Defining zero of potential at infinity

$$\Psi = -\left\{\frac{G}{r}\int_0^r 4\pi\rho(r')r'^2\,\mathrm{d}r'\right\}\Bigg|_{r=\infty}^{r_0} + \int_\infty^{r_0}\frac{G}{r}4\pi\rho(r)r^2\,\mathrm{d}r$$

Integrate by parts

$$\Rightarrow \qquad \Psi = -\frac{GM(r_0)}{r_0} + \int_\infty^{r_0} 4\pi G\rho(r)r\,\mathrm{d}r$$

C.3 : Gravitational Potential Energy

Thought experiment to assess gravitational potential energy of system of particles : consider dismantling a system of point masses, taking them to infinity one-by-one. Can then see

$$\Omega = -\frac{1}{2}\sum_{j\neq i}\sum_{i}\frac{GM_iM_j}{|\mathbf{r}_j - \mathbf{r}_i|} = \frac{1}{2}\sum_j M_j\Psi_j \xrightarrow{\text{continuum limit}} \frac{1}{2}\int \rho(\mathbf{r})\Psi(\mathbf{r})\,\mathrm{d}V$$

Let us again evaluate this for spherical system. We have:

$$\Omega = \frac{1}{2}\int_0^\infty 4\pi\rho(r)r^2\Psi(r)\,\mathrm{d}r = \frac{1}{2}\left[M(r)\Psi(r)\Big|_0^\infty - \int_0^\infty M(r)\frac{\mathrm{d}\Psi}{\mathrm{d}r}\mathrm{d}r\right]$$

$$\frac{\mathrm{d}\Psi}{\mathrm{d}r} = \frac{GM(r)}{r^2}$$

- So…

$$\Omega = -\frac{1}{2}\int_0^\infty \frac{GM(r)^2}{r^2}\,\mathrm{d}r$$

- Integrate by parts again:

$$\Omega = \underbrace{\frac{1}{2}\,GM(r)^2\frac{1}{r}\Big|_0^\infty}_{=0} - \frac{1}{2}\int_0^\infty \frac{1}{r}2GM\frac{\mathrm{d}M}{\mathrm{d}r}\,\mathrm{d}r$$

$$\Rightarrow \qquad \Omega = -G\int_0^\infty \frac{M(r)}{r}\,\mathrm{d}M$$

- This last form has a nice interpretation : evaluate energy by “peeling” away shells.

C.4 : Virial Theorem

Virial Theorem is a powerful result relevant to isolated gravitating systems.

Here, we will examine the **scalar virial theorem** (∃ general tensor virial theorem).

Consider gravitating system of many particles of mass m_i and position $\mathbf{r}_i$. Start by considering the time derivative of the quantity $I_i = m_i r_i^2$

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}(m_i r_i^2) = m_i \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{r}_i \cdot \frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t}\right)$$

$$= m_i \mathbf{r}_i \cdot \frac{\mathrm{d}^2\mathbf{r}_i}{\mathrm{d}t^2} + m_i \left(\frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t}\right)^2$$

$$= \mathbf{r}_i \cdot \mathbf{F}_i + \underbrace{m_i \left(\frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t}\right)^2}_{2\times\text{Kinetic Energy } T_i}$$

where $\mathbf{F}_i = m_i \frac{d^2\mathbf{r}_i}{dt^2}$

Sum over particles

$$\frac{1}{2}\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = \underbrace{\sum_i (\mathbf{r}_i \cdot \mathbf{F}_i)}_{\substack{V\text{, the virial} \\ \text{(R. Clausius)}}} + 2T \qquad \text{where} \qquad I \equiv \textstyle\sum_i m_i r_i^2$$

System is <u>isolated</u>, so $\mathbf{F}_i = \sum_j \mathbf{F}_{ij}$ with $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$

Taking particles in pairs, we have

$$V = \sum_i \sum_{j>i} \mathbf{F}_{ij} \cdot (\mathbf{r}_i - \mathbf{r}_j)$$

Assume forces are local ($\mathbf{r}_i = \mathbf{r}_j$) or gravitational

$$\mathbf{F}_{ij} = -\frac{G m_i m_j}{r_{ij}^3}\mathbf{r}_{ij} \qquad \text{where} \qquad \mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$$

So:

$$V = -\sum_i \sum_{j>i} \frac{G m_i m_j}{r_{ij}}$$

i.e., the virial is simply the gravitational potential energy.

Putting into previous result:

$$\frac{1}{2}\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 2T + \Omega$$

Now assume that the system is in a state of dynamical equilibrium… LHS zero.

$$\boxed{2T + \Omega = 0} \qquad \text{THE VIRIAL THEOREM}$$

Implications of the Virial Theorem

1. Connects mass, velocity, and size of gravitating system:

$$T = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2} M \langle v^2 \rangle$$

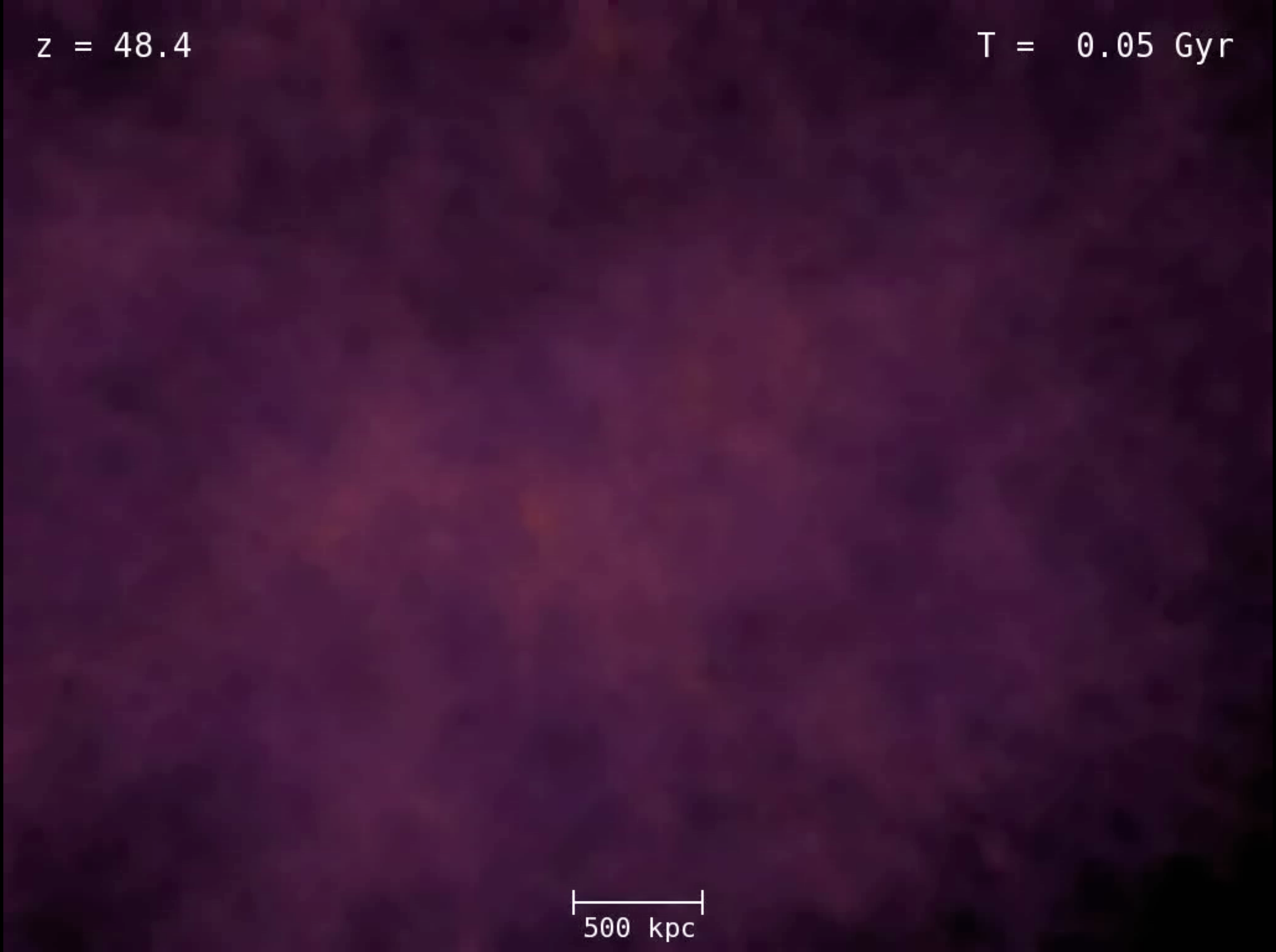
$$\Omega = -\frac{1}{2} \int_0^\infty \frac{GM(r)^2}{r^2} \, \mathrm{d}r = -\int_0^\infty \frac{GM(r)}{r} \, \mathrm{d}M = -\frac{GM^2}{\bar{r}}$$

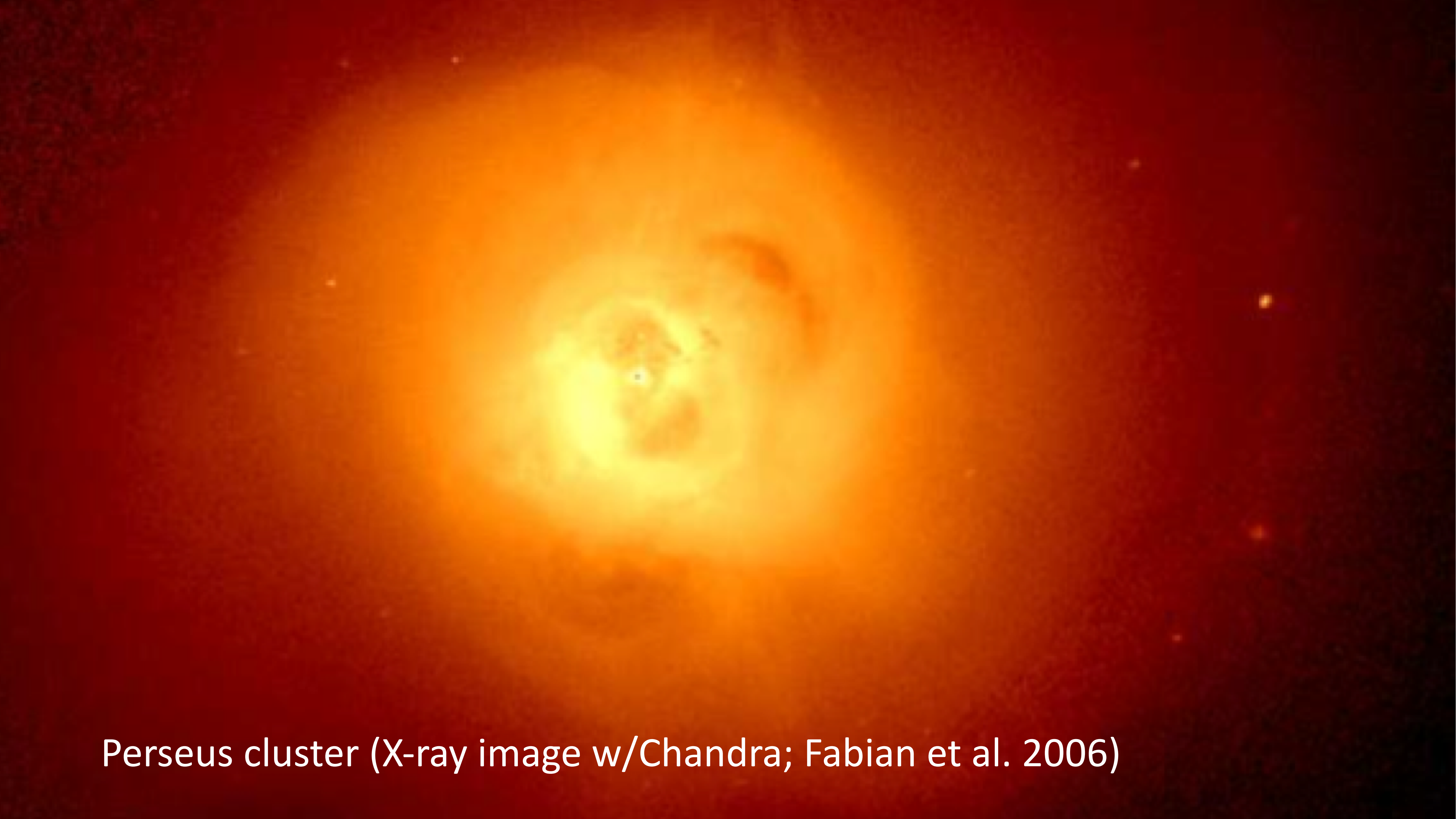
So

$$2T = -\Omega$$

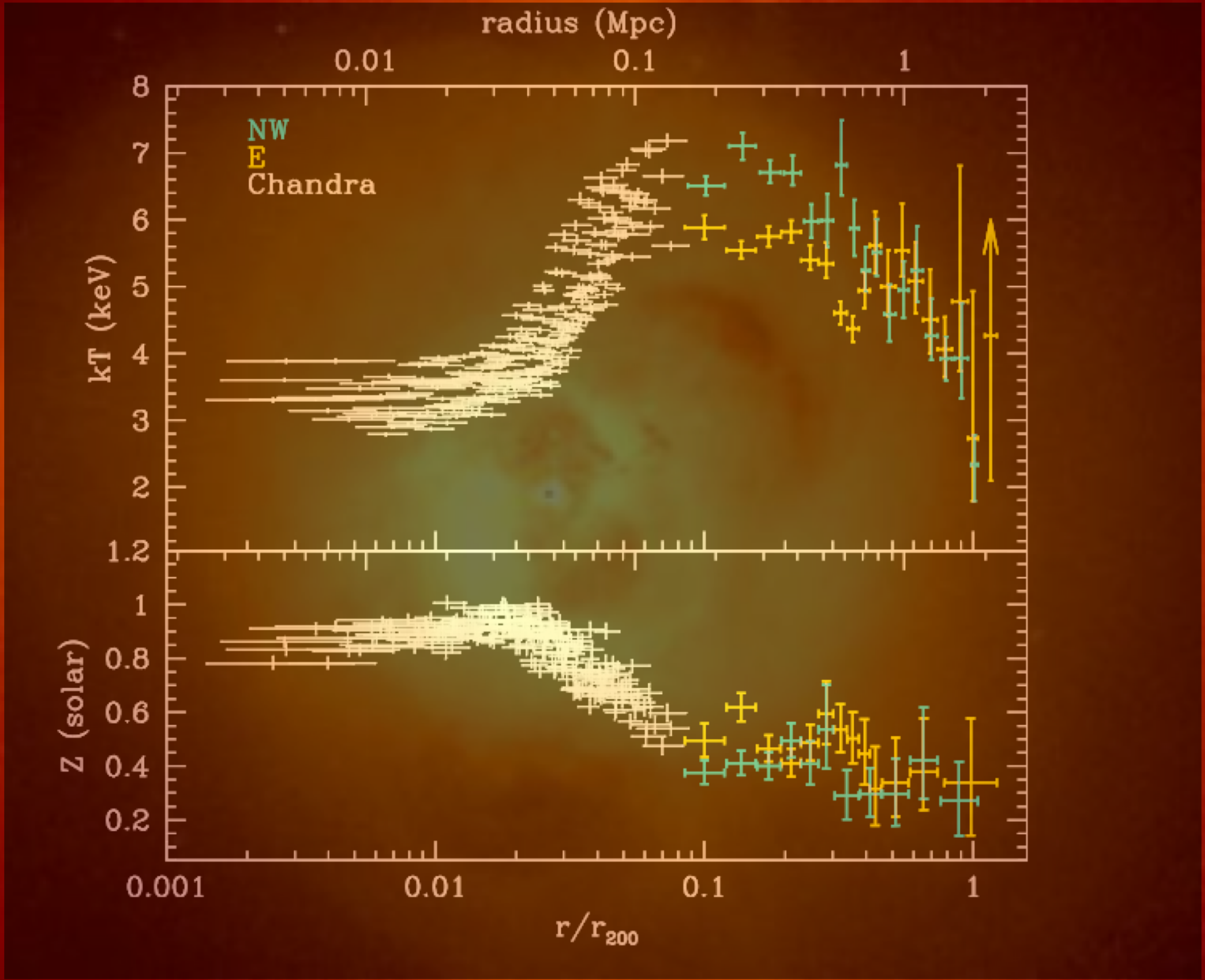
$$\Rightarrow M \langle v^2 \rangle = \frac{GM^2}{\bar{r}}$$

$$\Rightarrow \langle v^2 \rangle = \frac{GM}{\bar{r}}$$


z = 48.4
T = 0.05 Gyr
500 kpc



Perseus cluster (X-ray image w/Chandra; Fabian et al. 2006)



1keV=1.16x10^{7}K

Perseus cluster (X-ray image w/Chandra; Fabian et al. 2006)

2. Gravitating systems have negative specific "heat" capacity

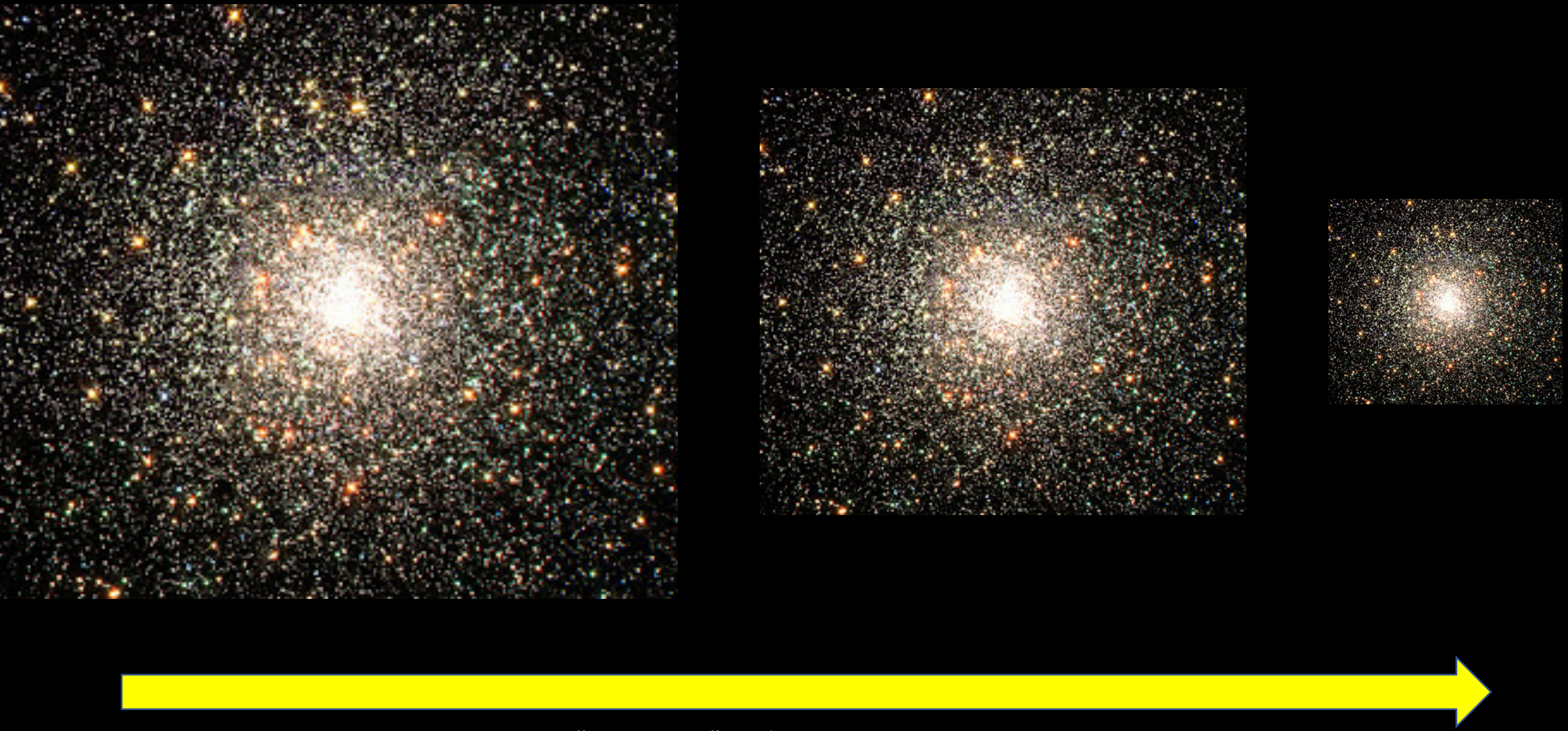
$$E_{\mathrm{total}} = T + \Omega$$

$$\Rightarrow E_{\mathrm{total}} = -T = -\frac{1}{2}M\langle v^2\rangle = -\frac{GM^2}{\bar{r}}$$

Broadly, this is why gravitation creates structure from initially smooth conditions

Dramatic manifestation is the gravothermal collapse (e.g. globular clusters).

Note that, for gravitating gas ball, *T* is directly related to gas temperature



Fastest stars “evaporate” and carry away energy
System shrinks and “heats” up
Runaway process (eventually stabilized by binary formation)