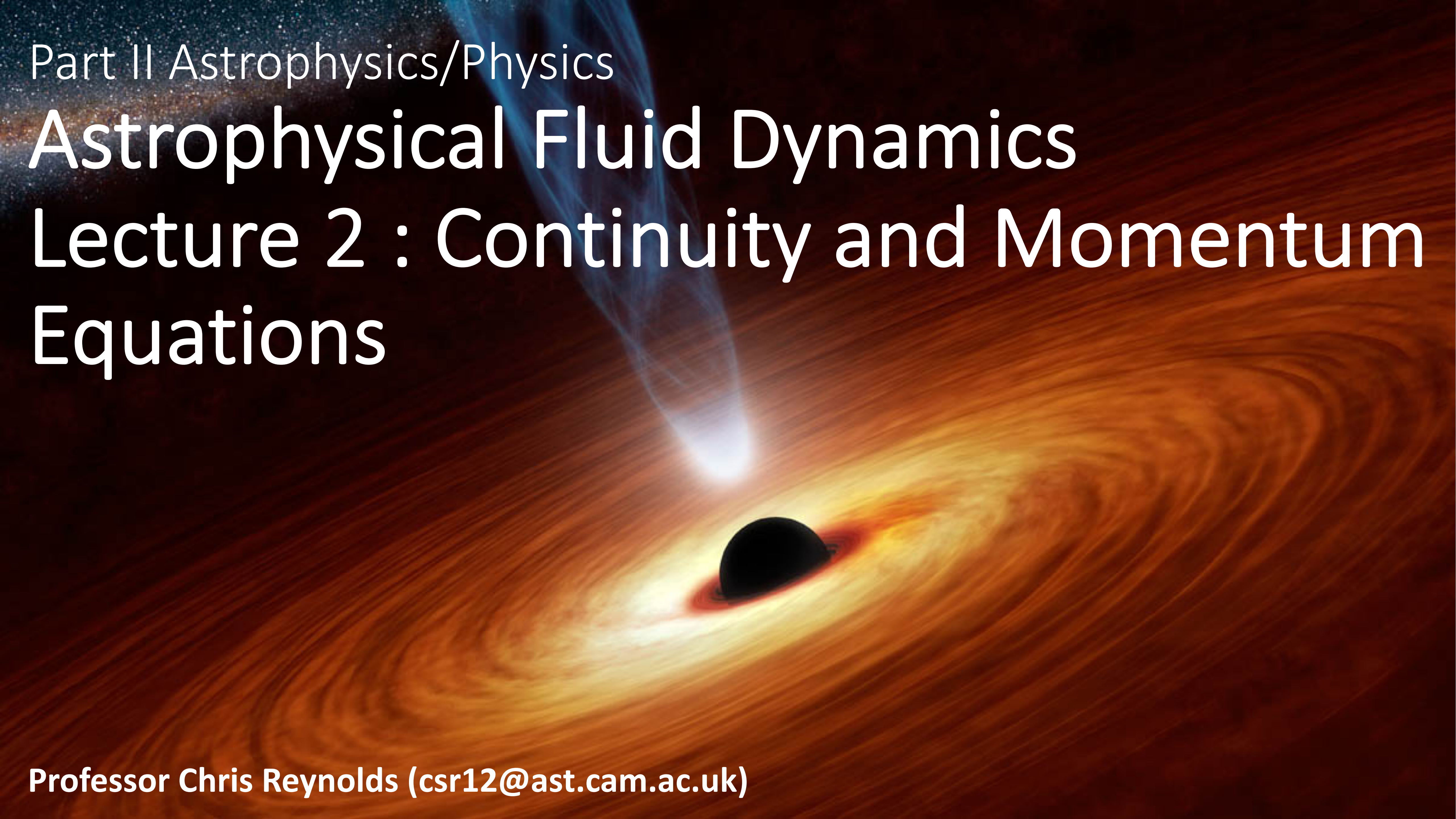

Part II Astrophysics/Physics

Astrophysical Fluid Dynamics

Lecture 2 : Continuity and Momentum Equations

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Recap

- Fluid as a continuous media that flows (fluid elements)
- Collisional vs collisionless fluids
- Eulerian and Lagrangian frameworks

$$\boxed{\underbrace{\frac{\mathrm{D}Q}{\mathrm{D}t}}_{\substack{\text{Lagrangian}\\ \text{time derivative}}} = \underbrace{\frac{\partial Q}{\partial t}}_{\substack{\text{Eulerian}\\ \text{time derivative}}} + \underbrace{\mathbf{u}\cdot\boldsymbol{\nabla} Q}_{\substack{\text{``convective''}\\ \text{derivative}}}}$$

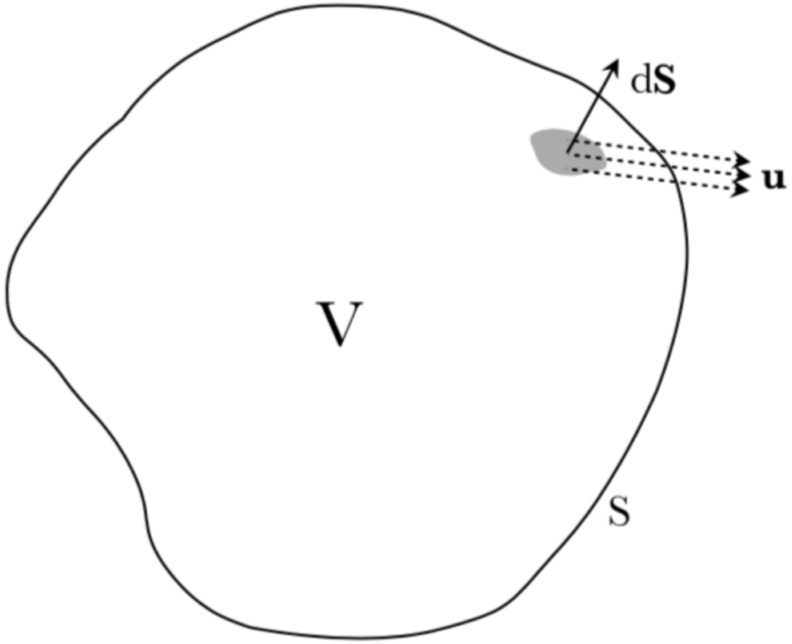
- Concept of streamlines, particle paths and streaklines

Today's lecture

- Continue with our formulation of the fluid equations…
  - Will establish a set of partial differential equations and constitutive relations that describe the time-changing properties of the fluid $\rho(\mathbf{r},t)$, $p(\mathbf{r},t)$, $\mathbf{u}(\mathbf{r},t)$…
  - Here, we will focus on non-relativistic fluids
  - Generalization to relativistic systems is straightforward in principle

- Conservation of mass (B.3)
  - Continuity equation for fluid
- Conservation of momentum (B.4)
  - Pressure and stress tensor
  - Momentum equation for fluid
  - Concept of ram-pressure

B.3 : Conservation of Mass



rate of change of mass in $V = -$ rate that mass is flowing out across S

$$\frac{\partial}{\partial t}\int_V \rho \,\mathrm{d}V = -\int_S \rho\mathbf{u}\cdot \mathrm{d}\mathbf{S}$$

$$\Rightarrow \qquad \int_V \frac{\partial \rho}{\partial t}\,\mathrm{d}V = -\int_V \boldsymbol{\nabla}\cdot(\rho\mathbf{u})\,\mathrm{d}V$$

$$\Rightarrow \qquad \int_V \left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u})\right)\mathrm{d}V = 0.$$

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0}$$

Eulerian Continuity Equation

Lagrangian view:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{\partial\rho}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\rho = \overbrace{-\boldsymbol{\nabla}\cdot\rho\mathbf{u}}^{-\nabla\cdot\rho\mathbf{u} = -\rho\nabla\cdot\mathbf{u} - \mathrm{u}\cdot\nabla\rho} + \mathbf{u}\cdot\boldsymbol{\nabla}\rho = -\rho\boldsymbol{\nabla}\cdot\mathbf{u}.$$

$$\boxed{\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\boldsymbol{\nabla}\cdot\mathbf{u} = 0} \qquad \text{LAGRANGIAN CONTINUITY EQUATION}$$

Important special case is an **incompressible fluid**:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0. \qquad \Leftrightarrow \qquad \boldsymbol{\nabla}\cdot\mathbf{u} = 0.$$

B.4 : Conservation of Momentum

- Recall elementary concept of pressure: force d**F** acting on one side of surface d**S** is

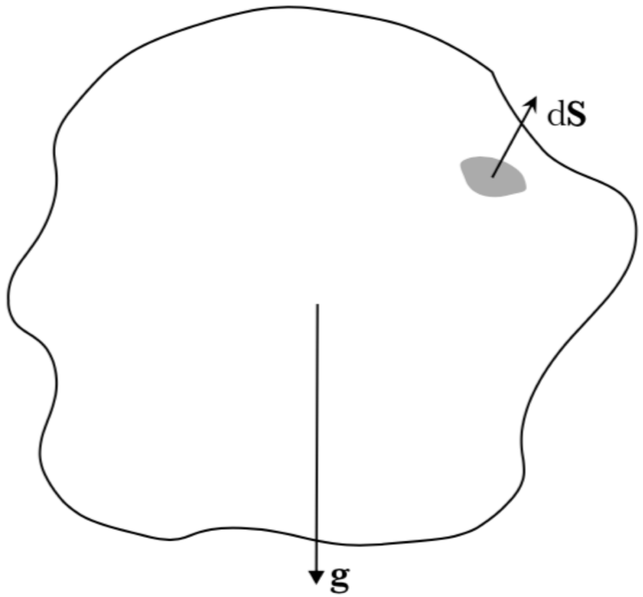
$$\mathrm{d}\mathbf{F} = p\,\mathrm{d}\mathbf{S}$$

- More generally, it is possible to have forces that are not-perpendicular to the surface (e.g. viscous stresses), so we will have a tensor relation

$$\mathrm{d}F_i = \sigma_{ij}\mathrm{d}S_j$$

- Simple isotropic fluid pressure corresponds to $\sigma_{ij} = p\delta_{ij}$

- Examine momentum conservation for fluid element subject to pressure forces and an external gravitational field **g**. Pick some arbitrary direction **n** in which to project forces



$$\mathbf{F}\cdot\hat{\mathbf{n}} = -\int_S p\hat{\mathbf{n}}\cdot \mathrm{d}\mathbf{S} = -\int_V \boldsymbol{\nabla}\cdot(p\hat{\mathbf{n}})\,\mathrm{d}V = -\int_V \hat{\mathbf{n}}\cdot\boldsymbol{\nabla}p\,\mathrm{d}V$$

So, equation of motion for fluid element is

$$\left(\frac{\mathrm{D}}{\mathrm{D}t}\int_V \rho\mathbf{u}\,\mathrm{d}V\right)\cdot\hat{\mathbf{n}} = -\int_V \hat{\mathbf{n}}\cdot\boldsymbol{\nabla}p\,\mathrm{d}V + \int_V \rho\mathbf{g}\cdot\hat{\mathbf{n}}\,\mathrm{d}V$$

$$\Rightarrow \quad \frac{\mathrm{D}}{\mathrm{D}t}(\rho\mathbf{u}\ \delta V)\cdot\hat{\mathbf{n}} = -\delta V\,\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}p + \delta V\,\rho\mathbf{g}\cdot\hat{\mathbf{n}}$$

$$\Rightarrow \quad \hat{\mathbf{n}}\cdot\mathbf{u}\ \underbrace{\frac{\mathrm{D}}{\mathrm{D}t}(\rho\delta V)}_{\substack{=0 \text{ by mass}\\ \text{conservation}}} + \rho\delta V\,\hat{\mathbf{n}}\cdot\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\delta V\,\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}p + \delta V\,\rho\mathbf{g}\cdot\hat{\mathbf{n}}$$

$$\Rightarrow \quad \delta V\,\hat{\mathbf{n}}\cdot\left(\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}+\boldsymbol{\nabla}p-\rho\mathbf{g}\right)=0 \qquad \forall \;\; \delta V, \hat{n}$$

$$\Rightarrow \quad \boxed{\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}=-\boldsymbol{\nabla}p+\rho\mathbf{g}}$$

Lagrangian Momentum Equation

$$\boxed{\rho\frac{\partial\mathbf{u}}{\partial t}+\rho(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}=-\boldsymbol{\nabla}p+\rho\mathbf{g}}$$

Eulerian Momentum Equation

This is just “F=ma” for the fluid element.

See importance of pressure gradients

Let's make explicit the conservation law:

- Notational convenience:

$$\frac{\partial}{\partial t} \equiv \partial_t \qquad \frac{\partial}{\partial x_i} \equiv \partial_i$$

- Then

$$\begin{aligned} \frac{\partial}{\partial t}(\rho u_i) &\equiv \partial_t(\rho u_i) \\ &= \boxed{\rho \partial_t u_i} + \boxed{u_i \partial_t \rho} \\ &= \boxed{-\rho u_j \partial_j u_i - \partial_j p \delta_{ij} + \rho g_i} \boxed{- u_i \partial_j (\rho u_j)} \end{aligned}$$

mtm equation

continuity equation

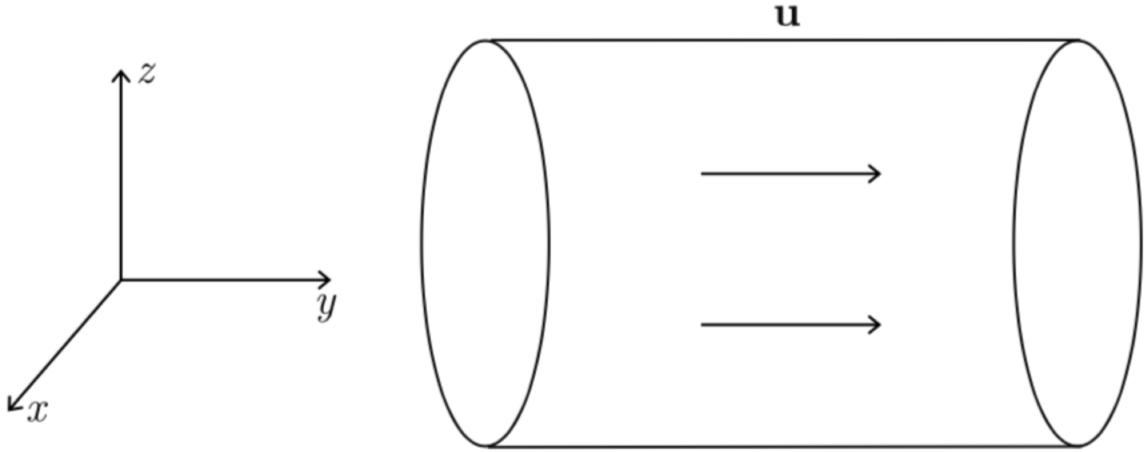
$$\Rightarrow \quad \partial_t(\rho u_i) = -\partial_j(\underbrace{\rho u_i u_j}_{\substack{\text{stress tensor}\\ \text{due to bulk flow}\\ \text{`Ram Pressure'}}} + \underbrace{p\delta_{ij}}_{\substack{\text{stress tensor}\\ \text{due to random}\\ \text{thermal motions}}}) + \rho g_i = -\partial_j \sigma_{ij} + \rho g_i$$

$$\sigma_{ij} = p\delta_{ij} + \rho u_i u_j$$

$$\partial_t(\rho\mathbf{u}) = -\boldsymbol{\nabla}\cdot\underbrace{(\rho\mathbf{u}\otimes\mathbf{u} + p\underline{\underline{\mathbf{I}}})}_{\substack{\text{flux of}\\ \text{momentum}\\ \text{density}}} + \rho\mathbf{g}$$

$$\partial_t\rho = -\nabla\cdot(\rho\mathbf{u})$$

- Example – flow in a pipe



$$\sigma_{ij} = \begin{pmatrix} p & 0 & 0 \\ 0 & p + \rho u^2 & 0 \\ 0 & 0 & p \end{pmatrix}$$

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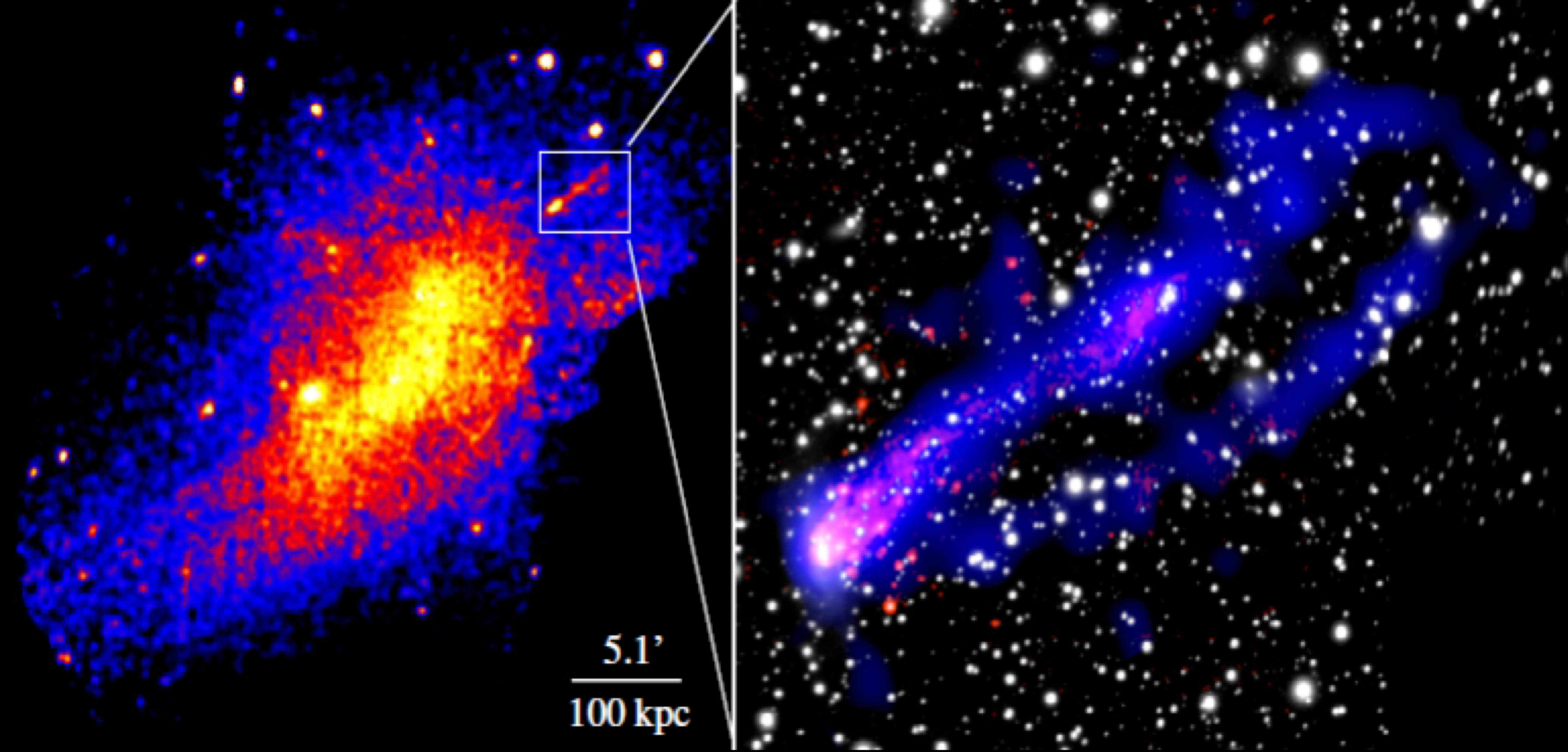
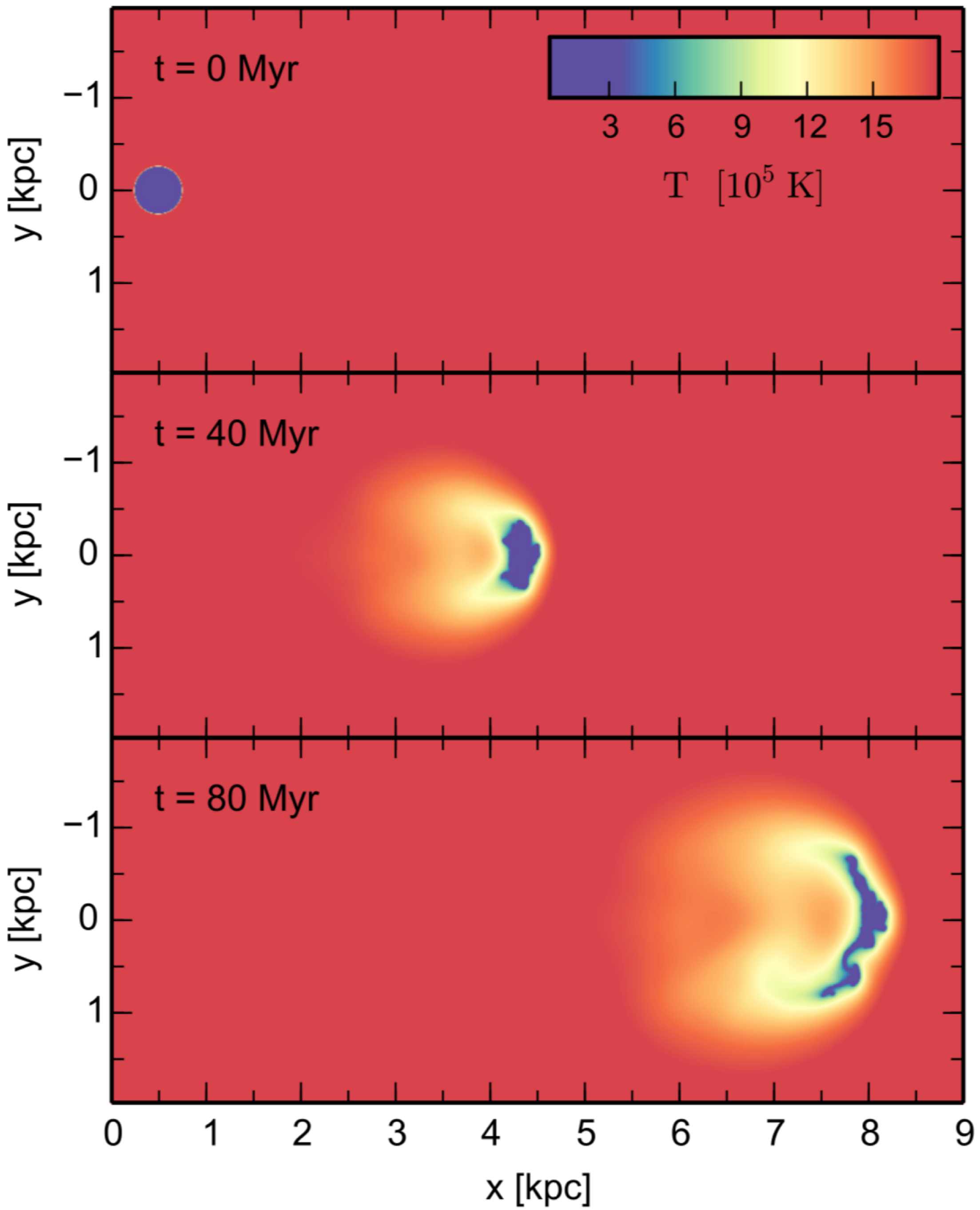


Galaxy ESO137-001 in the cluster Abell 3627 (Sun et al. 2007)



Armillotta et al. (2017)