Is space haunted? Exorcising ghosts from the gravitational particle spectrum

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Abstract

In the context of finding a renormalisable theory of gravity, Poincaré Gauge Theory (PGT) of gravity offers a promising avenue. Recent works have shown that subsets of this theory linearised in the vicinity of Minkowski spacetime are power-counting renormalisable and have a healthy particle spectrum, free of ghosts and tachyons. If such theories were to describe our physical universe, their particle spectrum should also be healthy during periods of accelerated expansion such as during inflation and dark-energy domination. This work discusses the method used to analyse the propagator structure of PGT and advances the analysis to allow for the study of the particle spectrum of these theories around de Sitter spacetime. The results for Einstein–Cartan Theory and General Relativity are reviewed in detail while the particle spectrum of the Teleparallel Equivalent of General Relativity is studied for the first time using this method. A novel generalised Mathematica implementation of the method around both Minkowski and de Sitter spacetime is discussed.

1 Introduction

Einstein's theory of General Relativity (GR) has proven to be one of the most successful theories to date [1], with the detection of gravitational waves from a binary black hole merger being the latest prediction to be confirmed by observations [2]. Although its validity at intermediate length scales has been confirmed by experimental and observational tests [3], there is no direct evidence that GR is the correct theory of gravity on cosmological scales. Assuming Einstein's GR to be the correct description of gravity, the standard cosmological model, the ΛCDM model, demands that over 96% of the Universe's energy budget is 'dark' so as to explain astrophysical and cosmological observations. Cold dark matter (CDM) must be invoked to explain gravitational phenomena such as galaxy rotation curves [4] and gravitational lensing [5], whilst dark energy, mathematically expressed as a cosmological constant Λ, is what drives the acceleration of the Universe [6]. This leads to some conceptual problems, namely;

The dark energy and dark matter puzzles. Although the required properties for dark matter to explain observations are known, we have so far not been able to detect any particles that fit these criteria [7]. Moreover, the nature of dark energy is still an open area of research [8].

The cosmological constant problem. Quantum Field Theory (QFT) predicts a form of vacuum energy sourced by virtual particles, which is our best theoretical description of dark energy. However, the density of this energy is around fifty orders of magnitude larger than the value of Λ measured from observations [9].

The coincidence problem. The transition between a matter-dominated universe and a dark-energy-dominated universe happened at redshift 0.55 [10]. Is it a coincidence that the energy density of matter and dark energy are of the same order of magnitude at the present epoch [11]?

On top of these conceptual problems, the ΛCDM model is facing some growing tensions between early-time and late-time values of some key cosmological parameters. The most prominent of these tensions is the Hubble tension [12, 13]. A numerical value of the Hubble constant H_0, which measures the current rate of expansion of the Universe, can be obtained using two main methods. The first uses distance measurements from the local universe obtained from astronomical objects such as Cepheid stars and supernovae type Ia [14]. Together with spectroscopic measurements of redshift, this gives a model-independent value of $H_0 = 73.04 \pm 1.04 \mathrm{km\,s^{-1}\,Mpc^{-1}}$ [15]. The second way to get a value of H_0 uses measurements from the power spectrum of the cosmic microwave background (CMB) whose peaks are sensitive to joint constraints of density parameters and H_0. By making assumptions based on ΛCDM, this degeneracy can be broken to obtain an early-time measurement of $H_0 = 67.4 \pm 0.5 \mathrm{km\,s^{-1}\,Mpc^{-1}}$ [16]. The tension between these two measurements has now reached the statistically significant level of 5σ [15]. Future measurements of H_0 from gravitational waves [17] have the potential to determine whether this tension arises from systematic errors, or whether it is an indication towards new physics, potentially in the gravitational sector of the cosmological model [18].

Another tension is detected in the structure growth parameter $S_8 = \sigma_8\sqrt{\Omega_m/0.3}$, which gives a joint constraint on the matter density parameter Ω_m, and the amplitude of matter perturbations σ_8 [19]. Measurements of S_8 from late-time universe surveys such as the Kilo-Degree Survey are $8.3 \pm 2.6\%$ lower than Planck [20, 21]. Physically this means that matter in the low-redshift universe favours a smoother distribution than that predicted from CMB measurements. The tension currently lies at a $2.3 - 2.7\sigma$ level [22].

From a theoretical perspective, GR poses issues for the search of a consistent theory of quantum gravity. This is essential to fully describe regions of spacetime like the centre of black holes, where both gravity and quantum effects are expected to play a crucial role [23]. For a QFT to be predictive and give experimentally testable results it must be renormalisable, meaning that measurable quantities can be redefined to remove infinities. GR is perturbatively non-renormalisable; in order to remove the divergences that arise in the theory, an infinite number of parametrised counter-terms are needed [24, 25, 26]. A useful first step towards checking whether a theory is renormalisable is the power-counting renormalisability (PCR) criterion. This is the condition that the coupling constants within the theory do not have a negative mass dimension [27]. GR violates this condition and thus is not a PCR theory since the mass scale, Newton's constant, has the dimension of a negative power of mass [28]. It has been long known that adding higher derivative terms, such as quadratic products of the curvature tensor, to the Einstein–Hilbert action can lead to the theory becoming PCR [29]. These higher derivative terms have also shown potential in avoiding the formation of singularities, which in standard GR are unavoidable as a result of the Hawking–Penrose singularity theorem [30, 31].

Together with the challenges that ΛCDM is currently facing, the inability to express GR as a consistent QFT has motivated the investigation of modified theories of gravity in which the action that gives rise to the equations of motion is modified [32]. There are multiple ways in which this can be done, including $f(\mathcal{R})$ theories [33] and infinite derivative gravity [34]. The class of modified gravity theories relevant to this project is that of *Poincaré gauge theories* (PGT) which arises from the gauging of the Poincaré group and allows gravity to be described by both torsion and curvature [35, 36, 37] as will be fully discussed in Sec. 2.

Although GR is not renormalisable, its propagator, the graviton, is a healthy quantum particle. The theory predicts the graviton to be a spin-2 particle with two polarisations, as has indeed been observed through the two polarisations of gravitational waves [17, 38]. When modifying the theory of gravity we might be changing this particle spectrum by introducing new particles to the space-time and as a consequence, potentially introducing instabilities in the theory. To avoid this, the particle

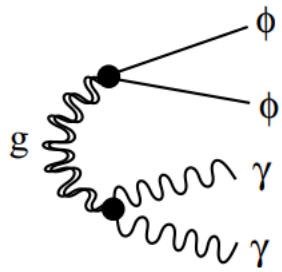

Figure 1: On-shell decay of vacuum into a ghost-pair and a photon-pair mediated by a graviton, highlighting the interaction of a ghost field with other fields via gravity. Figure by Cline, Jeon, and Moore [44].

spectrum of a healthy unitary theory should not contain ghosts, i.e. particles with negative energy, or tachyons, i.e. particles with imaginary mass [39, 40].

It is important to note that the ghosts that haunt modified theories of gravity are distinct from Faddeev–Popov ghosts which only represent internal lines in Feynman diagrams and thus cannot represent a physical degree of freedom in a theory [41]. Instead, the ghosts we are concerned with can be represented by an external line in Feynman diagrams and are thus physical particles themselves [42]. The propagator of a ghost can have one of two forms

$$\frac{-i}{p^2 - m^2 + i\epsilon} \quad \text{or} \quad \frac{-i}{p^2 - m^2 - i\epsilon}\,. \tag{1.1}$$

In the first case, the optical theorem will be violated making the theory a non-unitary one [43]. In the second case, particles with negative energy propagate forwards in time. These particles couple with matter fields and through scattering processes dump their energy onto this conventional sector of the theory. The consequence is that ghost-non-ghost pairs are created in the vacuum at a divergent rate. Although one might think that a solution to this is to decouple ghost fields from matter fields, this is impossible to do as the ghost field can still couple to other fields through gravity, an interaction which cannot be completely eliminated [44]. This is illustrated in Fig. 1.

The type of instability caused by tachyonic fields in a theory is best understood using the analogy of a one-dimensional lattice of pendulums in which each pendulum is connected to its neighbour by a spring. The system has two equilibrium points, one at $\theta = 0$, in which all pendulums are pointing down, and one at $\theta = \pi$, where all the pendulums are pointing up. The first equilibrium point is stable and any perturbation is described by the usual Klein–Gordon equation for particles with real mass [45]. The second equilibrium point is described by the Klein–Gordon equation for imaginary mass particles. This $\theta = \pi$ position is unstable, a perturbation in any one of the pendulums will set off an exponentially growing wave through the spring. In the limit of an infinite number of such pendulums that are infinitely close to each other this describes a tachyonic field [46]. It can be shown that the group velocity of such a tachyonic field is superluminal however, this is not equivalent to the physical speed of propagation [47]. Regardless of the fact that tachyonic fields do not violate causality, they introduce unwanted instabilities in the theory and so a healthy unitary theory should not contain any tachyonic particles.

The existence of these particles for a theory of gravity can be uncovered or excluded by investigating the propagator structure of such theories. There has been a significant effort in investigating the propagator structure of subsets of PGT in flat Minkowski spacetime [48, 49, 50]. Recently Lin, Hobson, and Lasenby [51] used a systematic approach to study the propagator of

PGT and identified a total of 58 theories which are free of ghosts and tachyons, and which are power counting renormalisable (PCR). Note that although finding a PCR theory is an improvement upon GR, it is still not a guarantee that the theory is renormalisable. Indeed PCR theories may turn out to be non-renormalisable because of issues such as anomalies, and non-PCR theories could potentially be renormalisable by non-perturbative means, for example by realising the asymptotic safety conjecture [52, 53].

Analysing the propagator structure in Minkowski space is an obvious first step in checking the validity of theories of gravity in weak gravitational fields. However, quantum field theory in curved spacetime predicts that the particle spectrum of theories, or the interpretation of these spectra, might change if we investigate the theories in backgrounds which deviate from Minkowski [54]. A theory of gravity that describes Nature should posses a healthy particle spectrum in backgrounds which we expect to encounter in our physical Universe, such as around black holes or in de Sitter spacetime, not only in flat spacetime. The study of de Sitter spacetime, which is a maximally symmetric Lorentzian manifold with constant positive curvature [55], is of particular importance when considering the early and late universe. In its early stages, the Universe is expected to have undergone a period of accelerated expansion described by a quasi-de Sitter metric. This period of inflation is commonly assumed to find a resolution to the flatness and horizon problems [56, 57]. Presently, we are observing a period of accelerated expansion which is thought to be driven by dark energy, the energy component that now dominates in the Universe [16, 6]. According to the ΛCDM model, the Universe is thus evolving towards a de Sitter one in which the only energy component is dark energy.

The potential that PGT has shown in describing the gravitational force whilst providing a renormalisable theory of gravity, together with the relevance of de Sitter spacetime, has motivated the key aim of this project; investigating the propagator structure of PGT in a de Sitter background.

The structure of the report is as follows. In Sec. 2 we describe the mathematical formulation of PGT. In Sec. 3 we first introduce the method used to analyse the particle spectrum by applying it to electromagnetism. We then describe the method used to analyse the propagator of PGT, in both Minkowski and de Sitter spacetimes. The use of the xAct package in Mathematica to carry out calculations is also discussed in this section. In Sec. 4 we discuss some interesting results in both spacetime backgrounds before concluding in Sec. 5.

The signature used throughout this report is $\eta_{AB} = (+,-,-,-)$.

2 Poincaré Gauge Theories of Gravity

PGTs are of specific interest as modified theories of gravity as they express gravity as a gauge theory, just like the electromagnetic, weak and strong interactions. In a gauge theory, a global symmetry of the Lagrangian can be transformed to a local symmetry, i.e. one that depends on the spacetime coordinates, via the introduction of compensating gauge fields, an idea which was extended to non-Abelian gauge groups, such as the Poincaré group, by Yang and Mills [58]. In the absence of a gravitational field, we know that the Poincaré group gives rise to the symmetries of fundamental interactions, namely under translations, rotations and boosts. To localise Poincaré symmetry we thus have to introduce compensating fields ${h_A}^{\mu}$ and ${A^{AB}}_{\mu}$ which represent gravitational interactions [59]. These fields are the tetrad and spin connection and correspond to translations and Lorentz transformations, respectively. Here, and throughout the rest of this report, upper and lower case Latin indices $(A, B, ..., i, j, ...)$ refer to local Lorentz frames while Greek indices $(\alpha, \beta, ...)$ refer to coordinate frames. Note that the spin connection ${A^{AB}}_{\mu}$ is symmetric in AB.

The gauging of the Poincaré group leads to a generalisation of GR in which the connection

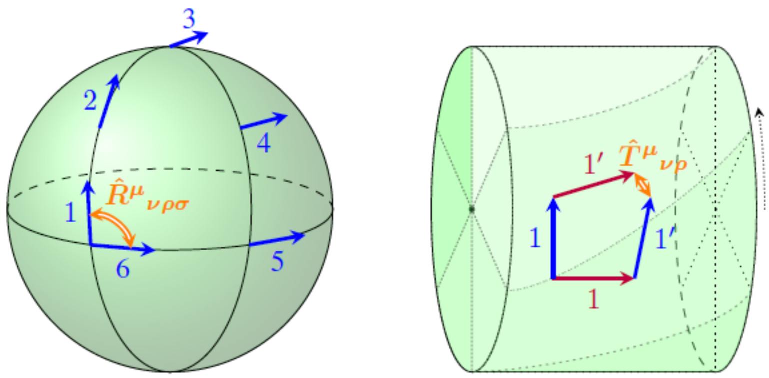

Figure 2: Geometrical difference between the curvature and torsion tensors. Curvature quantifies the rotation of a vector that is parallel transported along a closed curve on a manifold while torsion quantifies the non-closure of a parallelogram formed when two vectors are transported along each other. Figure taken from Ref. [60].

is no longer necessarily torsion-free. Geometrically, we are generalising the Riemann spacetime V_4 to Einstein–Cartan spacetime U_4. The physical implications of this is that alongside energy and momentum which source the gravitational field through curvature, spin can also source a gravitational field through torsion [35]. The geometrical difference between curvature and torsion is highlighted in Fig. 2. The field strengths associated to the gauge fields are indeed the curvature and torsion of spacetime and are given by $\mathcal{R}^{AB}{}_{CD} \equiv h_C{}^{\mu} h_D{}^{\nu} \mathcal{R}^{AB}{}_{\mu\nu}$ and $\mathcal{T}^{A}{}_{BC} \equiv h_B{}^{\mu} h_C{}^{\nu} \mathcal{T}^{A}{}_{\mu\nu}$ where

$$\mathcal{R}^{AB}{}_{\mu\nu} \equiv 2\left(\partial_{[\mu} A^{AB}{}_{\nu]} + A^{A}{}_{E[\mu} A^{EB}{}_{\nu]}\right), \tag{2.1a}$$

$$\mathcal{T}^{A}{}_{\mu\nu} \equiv 2\left(\partial_{[\mu} b^{A}{}_{\nu]} + A^{A}{}_{E[\mu} b^{E}{}_{\nu]}\right). \tag{2.1b}$$

Here $b^{A}{}_{\mu}$ is the inverse tetrad field such that $b^{A}{}_{\mu} h_B{}^{\mu} \equiv \delta^{A}_{B}$ and $b^{A}{}_{\mu} h_A{}^{\nu} \equiv \delta^{\nu}_{\mu}$.

A direct generalisation of GR in U_4 geometry gives rise to Einstein–Cartan Theory (ECT) [61] and is given by the gravitational action

$$S_{\mathrm{ECT}} \equiv \int \mathrm{d}^4 x b \mathcal{R}, \tag{2.2}$$

where b is the determinant of $b^{A}{}_{\mu}$ and $\mathcal{R} = h_A{}^{\mu} h_B{}^{\nu} \mathcal{R}^{AB}{}_{\mu\nu}$ is the Ricci scalar. The most general form of the gravitational Lagrangian for parity-preserving Poincaré Gauge Theories (PGT^{+}) that is at most quadratic in the gravitational gauge fields can be written as

$$\begin{aligned}
\frac{\mathcal{L}_G}{b} \equiv& -\lambda \mathcal{R} + (r_4 + r_5)\mathcal{R}^{AB}\mathcal{R}_{AB} + (r_4 - r_5)\mathcal{R}^{AB}\mathcal{R}_{BA} + \left(\frac{r_1}{3} + \frac{r_2}{6}\right)\mathcal{R}^{ABCD}\mathcal{R}_{ABCD} \\
&+ \left(\frac{2r_1}{3} - \frac{2r_2}{3}\right)\mathcal{R}^{ABCD}\mathcal{R}_{ACBD} + \left(\frac{r_1}{3} + \frac{r_2}{6} - r_3\right)\mathcal{R}^{ABCD}\mathcal{R}_{CDAB} \\
&+ \left(\frac{\lambda}{4} + \frac{t_1}{3} + \frac{t_2}{12}\right)\mathcal{T}^{ABC}\mathcal{T}_{ABC} + \left(-\frac{\lambda}{2} - \frac{t_1}{3} + \frac{t_2}{6}\right)\mathcal{T}^{ABC}\mathcal{T}_{BCA} \\
&+ \left(-\lambda - \frac{t_1}{3} + \frac{2t_3}{3}\right)\mathcal{T}_{B}{}^{AB}\mathcal{T}_{CA}{}^{C},
\end{aligned} \tag{2.3}$$

where λ, r_i and t_i are arbitrary constants [62, 49]. Alongside ECT, whose action is given in Eq. (2.2), there are other specific forms of the PGT^{+} Lagrangian that are of interest. Namely, GR itself is a special case of Eq. (2.3) with

$$\mathcal{L}_{\mathrm{GR}} \equiv b\left(\mathcal{R} - \mathcal{T}^2\right), \tag{2.4}$$

where $\mathcal{T}^2 \equiv \frac{1}{4}\mathcal{T}^{ABC}\mathcal{T}_{ABC} + \frac{1}{2}\mathcal{T}^{ABC}\mathcal{T}_{BCA} - \mathcal{T}_B{}^{AB}\mathcal{T}_{CA}{}^C$. It is clear from the action in Eq. (2.4) that GR sources gravity only through curvature, and the effects from torsion are subtracted off. Another theory of interest is the teleparallel equivalent of GR (TEGR), whose Lagrangian is simply given by

$$\mathcal{L}_{\mathrm{TEGR}} \equiv b\mathcal{T}^2. \tag{2.5}$$

This theory was initially investigated by Einstein himself, and in the Weitzenböck gauge, leads to the same dynamical equations as GR, as will be discussed in detail in Sec. 4.1. In this theory the gravitational field is sourced fully by torsion instead of curvature [63]. It is important to note that the terms quadratic in the curvature field-strength, denoted by $\mathcal{R}^2$ are essential for the spin-connection to achieve a full dynamical content [35] as will be highlighted in Sec. 4.

3 Propagator Analysis

3.1 Propagator Structure Analysis for Electromagnetism

As an introduction to the method that will be used to investigate the nature of particles in PGT, the particle spectrum of the electromagnetic (EM) field is analysed [64]. The analysis will follow the same steps for PGT with the specific differences described in Sec. 3.2.

The Lagrangian of the EM field is given by

$$\mathcal{L}_{\mathrm{EM}} \equiv -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{3.1}$$

where

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{3.2}$$

By noting that adding total derivatives to the Lagrangian does not result in a change in the equations of motion, we can write the Lagrangian as

$$\mathcal{L}_{\mathrm{EM}} = \frac{1}{2}A_\mu \mathcal{O}^{\mu\nu}\mathcal{A}_\nu, \tag{3.3}$$

where the symmetric differential operator is given by

$$\mathcal{O}^{\mu\nu} = \Box\eta^{\mu\nu} - \partial^\mu\partial^\nu. \tag{3.4}$$

We now introduce a complete set of orthogonal *spin projection operators* $\{\theta, \omega\}$ which decompose the vector A^μ into its spin-0 and spin-1 components;

$$\theta_{\mu\nu} \equiv \eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}, \quad \omega_{\mu\nu} \equiv \frac{\partial_\mu\partial_\nu}{\Box}. \tag{3.5}$$

This is also possible for higher order tensors which are decomposed into their scalar, vector and tensor components. Writing the operators in Eq. (3.5) in momentum space, it is easy to check that $k^\mu\theta_{\mu\nu} = 0$, and $k^\mu\omega_{\mu\nu} = k_\nu$, where k^μ is the four-momentum of the electromagnetic wave. Hence, θ and ω decompose A_μ into its transverse and longitudinal components, respectively.

Obtaining the field equations via the Euler–Lagrange equations yields

$$\Box\theta_{\mu\nu}A^\nu = 0\,. \tag{3.6}$$

Writing this in momentum space, with E being the energy and $\mathbf{k}$ the spatial momentum, one can see that this component is massless

$$k^2\theta_{\mu\nu}A^\nu = 0 \implies k^2 = 0 \implies E^2 = |\mathbf{k}|^2 \implies m = 0\,. \tag{3.7}$$

Since the field equations give no information about the longitudinal component, only the massless transverse component propagates.

For a Lagrangian written in terms of an operator $\mathcal{O}$ as in Eq. (3.3), we can define the *propagator* as $\mathcal{O}^{-1}$. Writing $\mathcal{O} = a\theta + b\omega$, it follows that

$$\mathcal{O}^{-1} = \frac{1}{a}\theta + \frac{1}{b}\omega\,. \tag{3.8}$$

For the photon, in momentum space, $\mathcal{O} = k^2\theta$, and so since $b = 0$, $\mathcal{O}$ is not invertible. This singularity can be overcome by adding a gauge fixing term to the Lagrangian. Note that this is also required to account for the extra degrees of freedom in the theory. In this way the propagator $\mathcal{O}^{-1}$ consists of two parts; a gauge-dependant part which is non-physical (as physical laws should not depend on the choice of gauge), and a gauge-independent part referred to as the *saturated propagator*. This is defined as [39]

$$\Pi(k) \equiv j^\dagger(k).\mathcal{O}^{-1}.j(k)\ , \tag{3.9}$$

with j being the *conserved* source current.

Notice that the gauge freedoms of the theory correspond to gauge constraints on the source current j. In the case of EM, this constraint is the condition that the positive parity scalar part of j must vanish. This is equivalent to the condition of a vanishing four-divergence of the source current, i.e. charge conservation. This is in perfect agreement with Noether's theorem which states that any continuous symmetry of the action of a physical system corresponds to a conservation law [65, 45].

Finally, the unitarity conditions of the theory are as follows; the saturated propagator of the Lagrangian should have only first order poles at $k^2 - m^2 = 0$, with real masses m to have no tachyons, and non-negative residues to have no ghosts [66].

3.2 Propagator Analysis for PGT in a Minkowski background

In this section we will describe the method used to analyse the particle spectrum around flat Minkowski spacetime for a general PGT. This is based on the analysis used in Ref. [39]. We will use ECT throughout this section as a concrete example to show the method in action and in Sec. 4.1 the results of this analysis for GR, ECT and TEGR will be discussed. In order to obtain the unitarity conditions, we need to express the Lagrangian in terms of coupling constants. Thus we write $\mathcal{L}_{\mathrm{ECT}} \equiv \lambda b\mathcal{R}$.

The first step towards analysing the particle spectrum of PGT is to linearise the Lagrangian in Eq. (2.3). If we choose to perturb around Minkowski background then we can write the tetrad field as

$$h_A{}^\mu = \delta_A{}^\mu + f_A{}^\mu\,, \tag{3.10}$$

where $f_A{}^\mu$ is the perturbation of the tetrad field. The inverse tetrad field is then

$$b^A{}_\mu = \delta^A{}_\mu - f^A{}_\mu + \mathcal{O}(f^2), \tag{3.11}$$

with determinant

$$b = 1 - f^i_{\ i} + \mathcal{O}(f^2)\,. \tag{3.12}$$

The perturbation f can be decomposed into its symmetric and antisymmetric parts,

$$f_{AB} = s_{AB} + a_{AB}, \tag{3.13}$$

where s and a are treated as separate fields for the analysis. In a Minkowski background, the spin connection is perturbative in nature as a consequence of a constant background tetrad field and is considered to be $\mathcal{O}(f)$. Note that changing between Roman and Greek indices will be $\mathcal{O}(f^2)$ and so in the linearised theory we can exchange between these two freely. This will no longer be the case for a de Sitter background as will be seen in Sec. 3.3.

Next we substitute Eqs. (3.10), (3.11) and (3.12) into the specific form of the PGT Lagrangian in Eq. (2.3) under investigation and keep terms up to $\mathcal{O}(f^2)$. For ECT the Lagrangian up to $\mathcal{O}(f^2)$ becomes

$$\mathcal{L}_{\text{ECT}} = \lambda(A_{acb}A^{abc} + A^{ab}{}_{a}A_{b}{}^{c}{}_{c} + 2f^{ab}\partial_b A_{a}{}^{c}{}_{c} - 2\partial_b A^{ab}{}_{a} - 2f^{ab}\partial_c A_{a}{}^{c}{}_{b} + 2f^{a}{}_{a}\partial_c A^{bc}{}_{b})\,. \tag{3.14}$$

Note that the $\mathcal{O}(f)$ term, $-2\partial_b A^{ab}{}_{a}$, is simply a divergence term and thus confirms that the Lagrangian satisfies the background field equations. This will be true for any general Lagrangian; the terms which are of first order in the perturbation should satisfy the background field equations. Making sure that these first order terms result in the field equations expected when applying the Euler–Lagrange equations, especially for theories such as GR and ECT, is a useful validity check for the linearising scheme being used. The second order terms then contain information about the particle spectrum of the theory.

We can now express the linearised Lagrangian in terms of a differential operator in a similar way as in Eq. (3.3) for the EM Lagrangian. However, in the PGT case there are now three fields at play rather than the singular field in EM. These are the spin connection and the symmetric and anti-symmetric part of the perturbation of the tetrad field. We can write the linearised Lagrangian in compact notation as

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_F + \mathcal{L}_I \\
&= \frac{1}{2}\sum_{a,b} \xi^{(a)}_{\acute{\alpha}}(x)\mathcal{O}^{(ab)}(\partial)^{\acute{\alpha}\acute{\beta}}\xi^{(b)}_{\acute{\beta}}(x) - \sum_{a} \xi^{(a)}_{\acute{\alpha}}(x) j^{(a)\acute{\alpha}}(x) \\
&= \frac{1}{2}\hat{\xi}^{\mathrm{T}}(x)\cdot\hat{\mathcal{O}}(\partial)\cdot\hat{\xi}(x) - \hat{\xi}^{\mathrm{T}}(x)\cdot\hat{j}(x)\,.
\end{aligned} \tag{3.15}$$

Here $\xi^{(1)}_{\acute{\alpha}_1}...\xi^{(n)}_{\acute{\alpha}_n}$ are the fields in the Lagrangian under investigation with corresponding source currents $j^{(a)}_{\acute{\alpha}}(x)$. The Greek indices with an acute accent, for example $\acute{\alpha}$, represent the collection of local Lorentz indices of the fields. To obtain Eq. (3.15), we are defining the generalised field vector

$$\hat{\xi} \equiv \sum_{a=1}^{n} \xi^{(a)}_{\acute{\alpha}_a}\mathbf{e}_a\,, \tag{3.16}$$

where $\mathbf{e}_a$ is a coloumn vector with the ath element equal to one and zeros in the other entries, and the generalised operator

$$\hat{\mathcal{O}}(\partial) \equiv \mathcal{O}^{(ab)}(\partial)^{\acute{\alpha}\acute{\beta}}\mathbf{e}_a\mathbf{e}_b^{\dagger}\,. \tag{3.17}$$

Thus we can write the free-field part of the Lagrangian in Fourier space as

$$\mathcal{L}_{\mathrm{F}} = \frac{1}{2}\hat{\xi}^{\mathrm{T}}(-k)\cdot\hat{\mathcal{O}}(k)\cdot\hat{\xi}(k)\,. \tag{3.18}$$

Schematically in PGT this is equivalent to expressing the Lagrangian in the following form,

$$\mathcal{L} = A_{abc}\mathcal{I}^{abcdef}A_{def} + f_{ab}\mathcal{D}^{abcde}A_{cde} + f_{ab}\mathcal{S}^{abcd}f_{cd} \tag{3.19}$$

where $\mathcal{I}$, $\mathcal{D}$ and $\mathcal{S}$ are differential operators like that in Eq. (3.3). For ECT, these operators in Fourier space were found to be

$$\mathcal{I}^{abcdef} = \eta^{ac}\eta^{bd}\eta^{ef} - \eta^{af}\eta^{bd}\eta^{ce}, \tag{3.20a}$$

$$\mathcal{D}^{abcde} = -ik^d\eta^{ac}\eta^{eb} + ik^c\eta^{ad}\eta^{eb} + ik^d\eta^{ab}\eta^{ec} - ik^b\eta^{ad}\eta^{ec} - ik^c\eta^{ab}\eta^{ed} + ik^b\eta^{ac}\eta^{ed}\,, \tag{3.20b}$$

$$\mathcal{S}^{abcd} = 0. \tag{3.20c}$$

$\mathcal{S}$ is the zero tensor since there are no terms which are quadratic in the tetrad perturbation f in $\mathcal{L}_{\mathrm{ECT}}$.

As was the case for EM, the saturated propagator $\Pi(k)$ as defined in Eq. (3.9) will include all the required information about the particle spectrum of the theory. In order to calculate the inverse of the operator $\hat{\mathcal{O}}(k)$ so we can obtain $\Pi(k)$, it is useful to use the spin projection operator (SPO) formalism [67]. SPOs $P_{ii}(J^P,k)_{\acute{\alpha}}{}^{\acute{\beta}}$ are used to decompose a field into its separate spin J and parity P components;

$$\xi(k)_{\acute{\alpha}} = \sum_{J,P,i}\xi_i(J^P,k)_{\acute{\alpha}} \quad \text{with} \quad \xi_i(J^P,k)_{\acute{\alpha}} \equiv P_{ii}(J^P,k)_{\acute{\alpha}}{}^{\acute{\beta}}\xi(k)_{\acute{\beta}}\,. \tag{3.21}$$

In order to define a Hermitian, complete and orthonormal basis for parity-conserving operators acting on $\xi_{\acute{\alpha}}$, we can introduce off-diagonal SPOs $P_{ij}(J^P,k)_{\acute{\alpha}}{}^{\acute{\beta}}$ where $i \neq j$. For a theory which has n distinct fields, the tensor quantities $P^{(ab)}_{ij}(J^P)_{\acute{\alpha}\acute{\beta}}$ are elements of a matrix $P(J^P)$. The indices (a,b) label which $n \times n$ block the element lies in, while the (i,j) indices label the element within this block. We then redefine $P_{ij}(J^P)_{\acute{\alpha}\acute{\beta}}$ to refer to the (i,j)th element of the matrix $P(J^P)$.

We can use these SPOs to decompose our operator $\hat{\mathcal{O}}(k)$

$$\hat{\mathcal{O}}(k) = \sum_{i,j,J,P} a_{ij}(J^P,k)\hat{P}_{ij}(J^P,k)\,. \tag{3.22}$$

Pre and post multiplying Eq. (3.22) by SPOs and using their orthonormality conditions we obtain

$$\hat{P}_{ii}(J^P)_{\acute{\mu}\acute{\alpha}} \cdot \hat{\mathcal{O}}^{\acute{\alpha}\acute{\beta}} \cdot \hat{P}_{jj}(J^P)_{\acute{\beta}\acute{\rho}} = a_{ij}\hat{P}_{ij}(J^P)_{\acute{\mu}\acute{\rho}}\,, \tag{3.23}$$

which we can use to solve for the coefficients a_{ij}. Here we are explicitly writing the Lorentz indices for clarity. The derivation of Eq. (3.23) can be found in Appendix A.

As an example, consider the SPO matrix corresponding to the $J^P = 2^+$ sector which is given by

$$P(2^+) = \begin{array}{c} \\ A^*_{ijk} \\ s^*_{ij} \end{array} \begin{array}{c} \begin{array}{cc} \qquad\qquad A_{abc} \qquad\qquad\qquad & \qquad\qquad s_{ab} \end{array} \\ \begin{pmatrix} -\frac{2}{3}\Theta_{cb}\Theta_{kj}\Omega_{ia} + \Theta_{ic}\Theta_{ka}\Omega_{jb} + \Theta_{ia}\Theta_{kc}\Omega_{jb} & \sqrt{2}\tilde{k}_j(\Theta_{ia}\Theta_{kb} - \frac{1}{3}\Theta_{ab}\Theta_{ki}) \\ \sqrt{2}\tilde{k}_b(\Theta_{cj}\Theta_{ia} - \frac{1}{3}\Theta_{ca}\Theta_{ij}) & -\frac{1}{3}\Theta_{ab}\Theta_{ij} + \Theta_{ia}\Theta_{jb} \end{pmatrix} \end{array} \tag{3.24}$$

where $\tilde{k}_a = k_a/\sqrt{k^2}$, $\Omega^{ab} = k^a k^b/k^2$ and $\Theta^{ab} = \eta^{ab} - k^a k^b/k^2$. The elements of the matrix in Eq. (3.24) are correctly symmetrised according to the symmetry properties of the A, a and s fields. Then Eq. (3.23) defines a system of four equations;

$$P_{11}(2^+)_{ijkabc}\mathcal{I}^{abcdef}P_{11}(2^+)_{deflmn} - a_{11}P_{11}(2^+)_{ijklmn} = 0\,, \tag{3.25a}$$

$$P_{11}(2^+)_{ijkabc}\mathcal{D}^{abcde}P_{22}(2^+)_{delm} - a_{12}P_{12}(2^+)_{ijklm} = 0\,, \tag{3.25b}$$

$$P_{22}(2^+)_{ijab}\mathcal{D}^{abcde}P_{11}(2^+)_{cdeklm} - a_{21}P_{21}(2^+)_{ijklm} = 0\,, \tag{3.25c}$$

$$P_{22}(2^+)_{ijab}\mathcal{S}^{abcd}P_{22}(2^+)_{cdkl} - a_{22}P_{22}(2^+)_{ijkl} = 0\,. \tag{3.25d}$$

Solving this system of equations for ECT using the operators in Eqs. (3.20a) - (3.20c) gives the a matrix for the 2+ sector of the theory

$$a(2^+) = \begin{pmatrix} \frac{\lambda}{2} & \frac{-ik\lambda}{\sqrt{2}} \\ \frac{ik\lambda}{\sqrt{2}} & 0 \end{pmatrix}. \tag{3.26}$$

The a matrices for the other spin parity components can be obtained in a similar way.

Because of the orthonormality properties of SPOs, we are now in a good position to obtain $\hat{\mathcal{O}}^{-1}(k)$ as we simply need to invert the a matrices. However, some of these a matrices might be singular. The singularities of these a matrices corresponds to gauge freedoms within the theory, as was the case for EM. In particular if $a(J^P)$ is an $s \times s$ matrix with rank r then it has $(s-r)$ null eigenvectors, i.e. eigenvectors which correspond to a zero eigenvalue. The equations of motion $\hat{\mathcal{O}} \cdot \hat{\xi} = \hat{\jmath}$ are invariant under a transformation of each of these $(s-r)$ eigenvectors. We can thus use these gauge freedoms to fix the gauge by setting the corresponding parts of the fields to zero, making the a matrices non-singular. In previous works this was done by deleting rows/coloumns which upon deletion do not change the rank of the matrix, until an $r \times r$ invertible matrix is obtained [39]. In this work, we obtain generalised inverse matrices $b^{-1}(J^P)$ using the Moore–Penrose method [68, 69, 70] which has a stronger mathematical foundation. This is highlighted in the following example with $y \in \mathbb{R}$

$$a(J^P) = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} \implies b^{-1}(J^P) = \begin{pmatrix} \frac{1}{y} & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.27}$$

The $(s-r)$ null eigenvectors also correspond to gauge constraints on the source current $\hat{\jmath}$ which can then be interpreted as physical conservation laws in accordance with Noether's theorem, as was seen for the EM case. Each PGT will contain the source constraints which arise from the Poincaré symmetry. Defining τ_{AB} as the source current corresponding to the graviton field f_{AB}, and σ_{ABC} as the source current for the tordion field A_{ABC}, these are given by,

$$\partial^B \tau_{AB} = 0 \quad \text{and} \quad \tau_{[AB]} - \partial^C \sigma_{ABC} = 0\,. \tag{3.28}$$

Note that the graviton is a quantum of curvature while the tordion is a quantum of torsion. The first of the conditions in Eq. (3.28) is familiar from GR as it is a consequence of the diffeomorphism invariance of the theory. τ_{AB} can be interpreted as the stress energy-momentum tensor and this first expression is a set of four equations. $A = 0$ corresponds to the conservation of energy while the spatial components of this expression reflect conservation of momentum [71]. The second expression in Eq. (3.28) defines six extra source constraints which arise from the extra symmetries in PGT, three rotational symmetries and three from symmetries under boosts. In a similar way as for τ, these correspond to the conservation of angular-momentum current [35]. Specific forms of PGT might contain additional gauge freedoms corresponding to additional source constraints.

After we obtain the matrices $b^{-1}(J^P)$ we can then write the inverse of $\hat{\mathcal{O}}(J^P)$ as

$$\hat{\mathcal{O}}^{-1}(J^P) = \sum_{i,j} b^{-1}_{ij}(J^P)\hat{P}_{ij}(J^P), \tag{3.29}$$

with the saturated propagator being

$$\Pi = \sum_{a,b,J,P} b^{-1}_{ab}(J^P)\hat{\jmath}^{\dagger} \cdot \hat{P}_{ab}(J^P) \cdot \hat{\jmath}. \tag{3.30}$$

Focusing on the poles which arise from taking the inverse of the b matrices, these are a consequence of the $1/\det[b(J^P)]$ factor. The determinant can be written as

$$\det[b(J^P)] = \alpha k^{2q}(k^2 - m_1^2)...(k^2 - m_r^2) \tag{3.31}$$

where α and the m_is are functions of the Lagrangian parameters but independent of k by construction. Thus the propagator will have poles at $k^2 = 0$ and $k^2 = m_i^2$. The no ghost and no tachyon conditions remain the same as for the EM case; the masses m_i should be real for no tachyons, and the poles should have non-negative residues for no ghosts.

Going back to ECT, we can calculate the inverse of the $a(2^+)$ matrix in Eq. (3.26) in the usual way as this is not singular. We then find that

$$\det[b(2^+)] = -\frac{2}{k^2\lambda^2} \tag{3.32}$$

which is indeed in the same form as in Eq. (2.3). This already tells us that there will be massless modes with $k^2 = 0$ propagating in the theory in the $J^P = 2^+$ sector. We find, as expected, that there are no massive modes from the other spin-parity sectors and that the residues of the massless poles are

$$\left\{-\frac{9p^2}{\lambda}, -\frac{9p^2}{\lambda}\right\}. \tag{3.33}$$

Notice that since we have massless modes, $k^2 = p^2$, where p is the momentum of the particle. The unitarity condition for ECT is $\lambda < 0$, as a consequence of the requirement of non-negative residues.

A hidden power of this method is that it also allows for investigating whether a theory is PCR. This is only possible for theories for which the mixing terms in the b matrices vanish. An example of such a mixing term is the top right entry of Eq. (3.26) which mixes the s and the A field. If this condition is satisfied then for a PCR theory, the propagator of the graviton, the particle associated with the f field, should go as k^{-4} while that of the tordion, the particle associated with the A field, should go as k^{-2}. An example of how this analysis is applied will be given in Sec. 4.

3.3 Perturbing About dS Spacetime

As discussed in the introduction, a crucial step in advancing the propagator analysis of PGT is to perform the same analaysis in a dS spacetime, which is important for cosmology. Linearising around this background is significantly more complex than linearising around a trivial Minkowski background. In this section the details of this process are described.

The metric in dS spacetime is given by

$$g_{\mu\nu} = a^2(\tau)(d\tau^2 - dx^2 - dy^2 - dz^2), \tag{3.34}$$

where τ is conformal time and a is the scale factor [10]. When implementing this linearising scheme using a Mathematica code, which will be described in detail in Sec. 3.4, we want to do so in a coordinate free manner. Thus we express the dS metric in terms of the Minkowski one $\eta_{\mu\nu}$ as $g_{\mu\nu} = a^2(\tau)\eta_{\mu\nu}$. The inverse metric is given by $g^{\mu\nu} = \frac{1}{a^2(\tau)}\eta^{\mu\nu}$.

Using the relations between the tetrad field $e_A{}^\mu$, the inverse tetrad field $E^A{}_\mu$ and the metric

$$g_{\mu\nu} = \eta_{AB}E^A{}_\mu E^B{}_\nu, \quad g^{\mu\nu} = \eta^{AB}e_A{}^\mu e_B{}^\nu, \tag{3.35}$$

we can obtain an expression for the dS tetrad field and its inverse

$$e_A{}^\mu = \frac{1}{a}\delta_A{}^\mu, \quad E^A{}_\mu = a\delta^A{}_\mu. \tag{3.36}$$

Here we are using $e_A{}^\mu$ and $E^A{}_\mu$ to refer to the background tetrad and inverse tetrad field respectively. This allows us to distinguish between the full tetrad field $h_A{}^\mu$ and inverse tetrad field $b^A{}_\mu$ which also include the perturbations of these fields.

In the linearising process we will also need to introduce the timelike vector v^μ which will be used to define derivatives of $a(\tau)$. In (τ, x, y, z) coordinates this vector is given by $v^\mu = (1/a, 0, 0, 0)$ or equivalently $v_\mu = (a, 0, 0, 0)$. This timelike vector is defined in this way so that the norm in the dS metric is equal to one; $g_{\mu\nu}v^\mu v^\nu = a^2(\tau)/a^2(\tau) = 1$. Since the scale factor $a(\tau)$ is a function of conformal time only, and using the definition of the Hubble parameter $H = a'/a^2$, where prime refers to differentiation wrt τ, the derivatives of a are

$$\partial_\tau a = a^2 H \quad \text{and} \quad \partial_i a = 0, \tag{3.37}$$

for $i \in \{1, 2, 3\}$. We can thus express any partial derivative of a as

$$\partial_\mu a = v_\mu a H \quad \text{and} \quad \partial^\mu a = v^\mu a^3 H\,. \tag{3.38}$$

Since the components of v_μ are functions of a, we also need to consider the derivatives of this timelike vector. Namely,

$$\partial_\mu v_\nu = v_\mu v_\nu H, \quad \partial_\mu v^\nu = -v_\mu v^\nu H, \quad \partial^\mu v_\nu = v^\mu v_\nu a^2 H, \quad \partial^\mu v^\nu = -v^\mu v^\nu a^2 H. \tag{3.39}$$

Higher order derivatives will not be necessary in the calculation at this point.

We now want to perturb around a dS background by introducing a perturbation in both the tetrad field and the spin connection. The full tetrad field will be defined as,

$$h_A{}^\mu = e_A{}^\mu + \epsilon f_A{}^\mu\,, \tag{3.40}$$

with inverse,

$$b^A{}_\mu = E^A{}_\mu - \epsilon f^A{}_\mu\,. \tag{3.41}$$

The parameter ϵ is introduced to keep track of the order of the perturbations. $e_A{}^\mu$ and $E^A{}_\mu$ are as defined in Eq. (3.36). Note that in full,

$$f^A{}_\mu = \eta^{AB} g_{\mu\nu} f_B{}^\nu = a^2 \eta^{AB} \eta_{\mu\nu} f_B{}^\nu\,. \tag{3.42}$$

This is useful to keep in mind as in our Mathematica implementation we will define the metric to be the Minkowski one, as will be discussed in Sec. 3.4, and thus keeping track of the a factors is crucial. Note also that $h_A{}^\mu b^A{}_\nu = \delta^\mu{}_\nu + \mathcal{O}(\epsilon^2)$ and $h_A{}^\mu b^B{}_\mu = \delta_A{}^B + \mathcal{O}(\epsilon^2)$ as required for the tetrad and inverse tetrad fields.

In the dS case, the spin connection is no longer perturbative in nature as it was for the Minkowski case discussed in Sec. 3.2 and we now need to explicitly calculate the background spin connection. The full spin connection $\omega^{ij}{}_\mu$ is given by

$$\omega_{ij\mu} = \Delta_{ij\mu} + \epsilon A_{ij\mu} \tag{3.43}$$

with Δ, the Ricci rotation coefficients, defined as [35]

$$\Delta_{ij\mu} \equiv \frac{1}{2}(c_{ijm} - c_{mij} + c_{jmi}) e^m{}_\mu \tag{3.44}$$

and

$$c^i{}_{\mu\nu} \equiv \partial_\mu e^i{}_\nu - \partial_\nu e^i{}_\mu\,. \tag{3.45}$$

Since the c tensor in Eqs. (3.44) and (3.45) appears with different indices, suitable contractions with the relevant tetrad field and metric need to be done. Namely, $c_{ijm} = e_j{}^\mu e_m{}^\nu \eta_{ik} c^k{}_{\mu\nu}$.

Next we define the curvature and torsion field-strength tensors as in Eqs. (2.1a) and (2.1b) but using the full spin connection ${\omega^{ij}}_\mu$

$$\mathcal{R}^{AB}{}_{\mu\nu} = 2\left(\partial_{[\mu}\omega^{AB}{}_{\nu]} + \omega^{A}{}_{E[\mu}\omega^{EB}{}_{\nu]}\right) , \tag{3.46a}$$

$$\mathcal{T}^{A}{}_{\mu\nu} = 2\left(\partial_{[\mu}b^{A}{}_{\nu]} + \omega^{A}{}_{E[\mu}b^{E}{}_{\nu]}\right) . \tag{3.46b}$$

We then expand Eqs. (3.46a) and (3.46b) fully to obtain an expression in terms of a, H, v^μ, ${f_A}^\mu$ and ${A^{AB}}_\mu$ and the derivatives of the last two terms. We keep terms up to second order in the perturbation, i.e. to $\mathcal{O}(\epsilon^2)$ to obtain the linearised field-strength tensors.

In order to obtain any PGT Lagrangian, we need to include the determinant of the inverse tetrad field defined in Eq. (3.41). Using Jacobi's formula for the identity matrix I and a square matrix M

$$\det(I + \epsilon M) = 1 + \epsilon \mathrm{Tr}(M) + \mathcal{O}(\epsilon^2), \tag{3.47}$$

where $\mathrm{Tr}(M)$ denotes the trace. Since

$$b^{A}{}_{\mu} = a\delta^{A}{}_{\mu} - a^2 f^{A}{}_{\mu} = a(\delta^{A}{}_{\mu} - af^{A}{}_{\mu}), \tag{3.48}$$

the determinant of the inverse tetrad field according to Eq. (3.47) is given by

$$\det(b) = a^4(1 - af^i_{\ i}). \tag{3.49}$$

Any linearised PGT Lagrangian is then obtained by substituting into Eq. (2.3) for the linearised field-strength tensors in Eqs. (3.46a) and (3.46b), tetrad in Eq. (3.40), spin-connection in Eq. (3.43), and inverse tetrad determinant in Eq. (3.49), and truncating again to $\mathcal{O}(\epsilon^2)$.

3.4 Implementation using the xAct package in Mathematica

Although in theory, the analysis discussed in Secs. 3.2 and 3.3 can be performed analytically using pen and paper, for complex specific cases of the PGT Lagrangian in Eq. (2.3), the calculations quickly become impractical. This is also the case for a de Sitter background. Thus, the use of computer algebra is necessary to allow for the study of a large subset of PGT in an efficient way. The previous Mathematica implementation which performs the propagator analysis for PGT, developed by Lin, Hobson, and Lasenby [39], uses nested lists, making the code susceptible to errors. For this reason a novel Mathematica code was developed using the xAct package [72] which allows for tensor calculations in the Wolfram Language. This allows for a covariant representation of the tensor fields and provides a more elegant implementation of the method as will be discussed in this section.

We first define the manifold and the tensors on this manifold. For example, the inverse tetrad field in a Minkowski background is defined as follows;

```
DefManifold[M, 4, IndexRange[a, n]];
DefConstantSymbol[ϵ];
DefTensor[B[a, -b], M]; (*Inverse tetrad*)
DefTensor[F[a, -b], M];(*The perturbation of the tetrad field*)
ExpandB = With[{lhs = B[a, -b], rhs = delta[a, -b] - ϵ F[a, -b]},
  MakeRule[{lhs, rhs}, MetricOn -> All, ContractMetrics -> True]]
```

In both Minkowski and de Sitter backgrounds, we set the metric on the manifold to be equal to the Minkowski metric.

```
DefMetric[-1, η[-a, -b], pd, {",", "∂"}, FlatMetric -> True]
```

The reason for this is that in the xAct package it is difficult to define two seperate metrics which act on different sets of indices. In this tetrad formulation, we know that the de Sitter metric should be used to manipulate Greek indices while the Minkowski metric should be used to manipulate Latin indices. However, it is difficult to make the distinction between different indices and so everything is defined in terms of the Minkowski metric. Since for de Sitter space, the metric is the Minkowski metric multiplied by the relevant factors of a, as defined in Sec. 3.3, we have to ensure that these factors of a are dealt with independently of the metric. For example, in de Sitter space, the perturbation of the inverse tetrad field is related to the perturbation of the tetrad field as follows. We first define the background inverse tetrad field in terms of the Minkwoski tetrad and the scale factor `as[]` as seen in Eq. (3.36);

```
DefTensor[invtetrad[a, -b], M, PrintAs -> "E"](*the background de Sitter inverse
    tetrad field*)
AutomaticRules[invtetrad, MakeRule[{invtetrad[a, -b], as[] delta[a, -b]},
    ContractMetrics -> On, MetricOn -> All]]
```

We can then calculate the inverse of the tetrad perturbation;

```
AutomaticRules[invtetrad, MakeRule[{invtetrad[a, -b], as[] delta[a, -b]},
  ContractMetrics -> On, MetricOn -> All]]
DefTensor[F[-a, b], M](*The perturbation of the tetrad*)
DefTensor[Finv[a, -b], M](*Corresponding perturbation of the inverse tetrad*)
AutomaticRules[Finv, MakeRule[{Finv[a, -b], invtetrad[c, -b] invtetrad[a, -d] F[-c,
    d]}]]
```

This gives the full inverse tetrad to first order to be $b^A{}_\mu = a\delta^A_\mu - a^2 f^A{}_\mu$. Simply using `F[-a,b]` as the perturbation of the inverse would not introduce the correct a^2 factor multiplying $f^A{}_\mu$ since xAct would raise and lower indices only using the Minkowski metric which is the only metric defined in the code.

In Sec. 3.3, a timelike unit vector v^μ was introduced in order to correctly calculate derivatives of a as in Eq. (3.38). Since tensors and vectors are defined in a coordinate independent way in our xAct implementation, any properties must be defined as automatic rules at the start of the notebook. This means that xAct will automatically apply the rule for the relevant tensor whenever it encounters an instance of it in the code. For example, the inner product of the vector v_μ and the partial derivative of the scale factor a are set as follows in the script;

```
AutomaticRules[v, MakeRule[{v[a] v[-a], as[]^2}, ContractMetrics -> True, MetricOn
    -> All]]
AutomaticRules[as, MakeRule[{pd[-i][as[]], as[] v[-i] H[]}, ContractMetrics -> True,
     MetricOn -> All]] (*as[] is the scale factor*)
AutomaticRules[v, MakeRule[{pd[-j][v[-i]], v[-i] v[-j] H[]}, ContractMetrics ->
    False, MetricOn -> None]]
AutomaticRules[v, MakeRule[{pd[j][v[-i]], v[j] v[-i] as[]^2 H[]}, ContractMetrics ->
     False, MetricOn -> None]]
```

This means that any instance of $v^\mu v_\mu$ that will be encountered in later calculations will be replaced by a^2 and similarly $\partial_i a$ is replaced by $v_i H$ etc. Notice that the inner product $v_\mu v^\mu$ was defined to be equal to one in Sec. 3.3. However, since contraction in xAct is done using η_{ab}, an extra factor of a^2 needs to be added to the norm. Here a and H are being defined as scalar fields, denoted by the square bracket. We can then set up rules which replace these scalar fields with constant symbols so that they are no longer treated as tensors in xAct. For example a `Replacea` rule was defined that does this for the scale factor.

In de Sitter space we also had to define the Ricci rotation coefficients as given in Eq. (3.44);

```
DefTensor[ct[a, -b, -c], M, Antisymmetric[{-b, -c}], PrintAs -> "c"]
AutomaticRules[ct, MakeRule[{ct[a, -b, -c], pd[-b][invtetrad[a, -c]] - pd[-c][
    invtetrad[a, -b]]}]]
DefTensor[rrc[-a, -b, -c], M, Antisymmetric[{-a, -b}], PrintAs -> "Δ"];
AutomaticRules[rrc, MakeRule[{rrc[-i, -j, -k], 1/2 (tetrad[-j, a] tetrad[-m, b] η[-i
    , -l] - tetrad[-i, a] tetrad[-j, b] η[-m, -l] + tetrad[-m, a] tetrad[-i, b] η[-j,
    -l]) ct[l,-a,b] invtetrad[m, -k]}]]
```

and setup the spin-connection as defined in Eq. (3.43);

```
DefTensor[A[-a, -b, -c], M, Antisymmetric[{-a, -b}]](*Spin connection perturbation*)
DefTensor[ω[-a, -b, -c], M, Antisymmetric[{-a, -b}]](*The full spin connection*)
AutomaticRules[ω, MakeRule[{ω[-a, -b, -c], rrc[-a, -b, -c] + ϵ A[-a, -b, -c]}]]
```

The Riemann and torsion tensors are defined according to the definitions in Eqs. (2.1a) and (2.1b).

```
DefTensor[Ri[a, b, -c, -d], M]; (*Riemann tensor*)
ExpandR = With[{lhs = Ri[a, b, -e, -f], rhs = 2 (Antisymmetrize[pd[-e][ω[a, b, -f
    ]], {-e, -f}] + Antisymmetrize[ω[a, -g, -e] ω[g, b, -f], {-e, -f}])}, MakeRule[{
    lhs, rhs}, MetricOn -> All, ContractMetrics -> True]];
DefTensor[Tor[a, -b, -c], M, PrintAs -> "T"]
ExpandTor = With[{lhs = Tor[a, -b, -c], rhs = 2 Ht[-b, d] Ht[-c, e] (
    Antisymmetrize[pd[-d][B[a, -e]], {-d, -e}] + Antisymmetrize[ω[a, -f, -d] B[f, -e
    ], {-d, -e}])},
MakeRule[{lhs, rhs}, ContractMetrics -> True, MetricOn -> All]];
```

Here Ht[−c,e] is the full tetrad defined in Eq. (3.40).

We obtain the linearised Riemann tensor by keeping terms to $\mathcal{O}(\epsilon^2)$

```
ExpandLR = With[{lhs = LR[a, b, -c, -d], rhs = CoefficientList[Ri[a, b, -c, -d] /.
    ExpandR, ϵ[[1]] + ϵ CoefficientList[Ri[a, b, -c, -d] /. ExpandR, ϵ][[2]] + ϵ
    CoefficientList[Ri[a, b, -c, -d] /. ExpandR, ϵ][[3]] },
MakeRule[{lhs, rhs}, MetricOn -> All, ContractMetrics -> True]];
```

and similarly for the torsion tensor.

We can then setup any PGT Lagrangian using the relevant tensors and index contractions as seen in Eq. (2.3), and multiplying by b. For example the Lagrangian for ECT in dS is setup as follows;

```
detb = as[]^4 (1 - as[] delta[i, -j] ϵ F[j, -i])
R = ToCanonical[ScreenDollarIndices[Ht[-a, c] Ht[-b, d] LR[a, b, -c, -d] /. ExpandLR
    ]]
LECT = ToCanonical[detb R]
LinearECT = CollectTensors[CoefficientList[LECT, ϵ][[1]] + CoefficientList[LECT, ϵ
    ][[2]] + CoefficientList[LECT, ϵ][[3]] /. Replacea]
```

Notice the use of the Replacea rule so that we can then collect the tensors in the expression.

Linearisation in Minkowski background follows the exact same steps as presented here but was significantly simpler than the linearisation in de Sitter spacetime. This can clearly be seen in the increased complexity even when simply obtaining the perturbation of the inverse tetrad field. In Minkowski, the introduction of the vector v^μ was not necessary, no Ricci rotation coefficients needed to be calculated, as these were known a priori to be equal to zero, and the background tetrad field was simply δ^μ_A.

The next challenge was to obtain the $\mathcal{I}$, $\mathcal{D}$ and $\mathcal{S}$ operators which together constitute the operator $\hat{\mathcal{O}}$ as defined in Eq. (3.19). The steps for obtaining these operators in a Minkowski background are presented below. In order to do this, an ansatz needed to be used as it was not possible to solve for a full tensor expression without specifying the form in xAct. The ansatz was obtained by considering a simple theory, such as ECT, and deriving the operators by hand to deduce the structure of their general form. Since PGTs all have the same structure but with increasing complexity, it was expected that these operators will have the same structure which is why the ansatz approach is appropriate. As a first example, consider the $\mathcal{I}$ operator for ECT; the terms which mix the A field with itself in the Lagrangian, are of the form $A_{abc}A^{abc}$ and terms with different contractions on the indices, see Eq. (3.14). Thus the following ansatz was used;

$$\begin{aligned}\mathcal{I}^{abcdef} &= C_1\eta^{af}\eta^{be}\eta^{cd} + C_{10}\eta^{ae}\eta^{bc}\eta^{df} + C_{11}\eta^{ac}\eta^{be}\eta^{df} + C_{12}\eta^{ab}\eta^{ce}\eta^{df} + C_{13}\eta^{ad}\eta^{bc}\eta^{ef} \\ &+ C_{14}\eta^{ac}\eta^{bd}\eta^{ef} + C_{15}\eta^{ab}\eta^{cd}\eta^{ef} + C_2\eta^{ae}\eta^{bf}\eta^{cd} + C_3\eta^{af}\eta^{bd}\eta^{cd} + C_4\eta^{ad}\eta^{bf}\eta^{ce} \\ &+ C_5\eta^{ae}\eta^{bd}\eta^{cd} + C_6\eta^{ad}\eta^{be}\eta^{cf} + C_7\eta^{af}\eta^{bc}\eta^{de} + C_8\eta^{ac}\eta^{bf}\eta^{de} + C_9\eta^{ab}\eta^{cf}\eta^{de}.\end{aligned} \tag{3.50}$$

In xAct this was implemented as follows;

```
DefTensor[Io[a, b, c, d, e, f], M];
ExpandIo = With[{lhs = Io[a, b, c, d, e, f], rhs = MakeAnsatz@IndexConfigurations[η[
    a, b] η[c, d] η[e, f]]}, MakeRule[{lhs, rhs}, ContractMetrics -> True, MetricOn
    -> All, UseSymmetries -> True]]
```

We can then easily solve for the coefficients C_i for the relevant part of the Lagrangian. For example, by picking the A^2 terms from the ECT Lagrangian we can solve as follows;

```
IocoeefsECT = With[{lhs = - A[a, b, c]  A[-b, -c, -a] + A[a, b, -a]   A[-b, c, -c],
    rhs = A[-a, -b, -c] Io[a, b, c, d, e, f] A[-d, -e, -f] /. ExpandIo},
    SolveConstants[lhs == rhs]][[1]].
```

This gives the C_i coefficients up to some degree of freedom. For instance, for ECT we get $C_7 = -C_{15}$. In this case we can simply set these extra coefficients to zero to obtain the simplest linear combination that gives the correct operator. After doing this manually we can then define a rule that sets the operator for the relevant theory;

```
FinalIoECT = With[{lhs = IoECT[a, b, c, d, e, f] , rhs = - η[a, f] η[b, d]  η[c, e]
    + η[a, c] η[b, d]  η[e, f]}, MakeRule[{lhs, rhs}, ContractMetrics -> On, MetricOn
     -> All]]
```

In order to obtain $\mathcal{D}$, the Lagrangian in Fourier space needed to be obtained. This was done using the following rules;

```
ftv = With[{lhs = pd[i][F[a, b]], rhs = I v[i] F[a, b]}, MakeRule[{lhs, rhs},
    MetricOn -> All, ContractMetrics -> On]]
ftA = With[{lhs = pd[i][A[a, b, c]], rhs = I v[i] A[a, b, c]}, MakeRule[{lhs, rhs},
    MetricOn -> All, ContractMetrics -> On]]
```

Note that `v[i]` refers to the Fourier space wave vector which was denoted by k^a in Sec. 3.2.

The relevant terms for the $\mathcal{D}$ operator take on the form $f\partial A$ thus in Fourier space an ansatz of linear combinations of terms such as $\eta^{ab}\eta^{cd}v^e$ and different index contractions was used;

```
AnsatzDECT = With[{lhs = DECT[a, b, c, d, e], rhs = MakeAnsatz@IndexConfigurations[η
    [a, b] η[c, d] v[e]] }, MakeRule[{lhs, rhs}, MetricOn -> All, ContractMetrics ->
    On]];
```

The operator $\mathcal{S}$ was trickier to obtain as it contains terms such as

$$\partial_d f^{di}\partial_i f^c{}_c \tag{3.51}$$

and so the Fourier transform is more complicated to perform. In order to do so efficiently, we used the fact that adding boundary terms to the Lagrangian does not change the equations of motion. By noticing that

$$\partial_d(f^{di}\partial_i f^c{}_c) = \partial_d f^{di}\partial_i f^c{}_c + f^{di}\partial_d\partial_i f^c{}_c, \tag{3.52}$$

we can replace the term in Eq. (3.51) by $-f^{di}\partial_d\partial_i f^c{}_c$ in our Lagrangian since the two terms will result in the same equations of motion. We can then perform the Fourier transform by applying the ftv and ftA rules defined above. The ansatz used for the $\mathcal{S}$ operator is given by a linear combination of two ansatz of the form $v^a v^b \eta^{cd}$ and $v^2\eta^{ab}\eta^{cd}$.

```
DefTensor[S[a, b, c, d], M]
ExpandSAnsatz = With[{lhs = S[a, b, c, d], rhs = MakeAnsatz@IndexConfigurations[v[a]
    v[b] η[c, d]] + MakeAnsatz[IndexConfigurations[vabs^2 η[a, b] η[c, d]],
    ConstantPrefix -> "A"]}, MakeRule[{lhs, rhs}, ContractMetrics -> True, MetricOn
    -> All]];
```

The next step is to obtain the a matrices. This was done element by element for the individual spin-parity components. First the SPO matrices as given by Lin, Hobson, and Lasenby [39] were defined in the code. As an example, the $P(2^+)$ matrix, given in Eq. (3.26), is defined as;

```
DefTensor[P2p11[-i, -j, -k, -a, -b, -c], M];
ExpandP2p11 = With[{lhs = P2p11[-i, -j, -k, -a, -b, -c], rhs = Simplify[
    Antisymmetrize[Antisymmetrize[-2/3 θ[-c, -b] θ[-k, -j] Ω[-i, -a] + θ[-i, -c] θ[-k
    , -a] Ω[-j, -b] + θ[-i, -a] θ[-k, -c] Ω[-j, -b], {-a, -b}], {-i, -j}]]},
    MakeRule[{lhs, rhs}, MetricOn -> All, ContractMetrics -> On]];
    ...
P2p = {{P2p11, P2p12}, {P2p21, P2p22}};
```

We can then solve for the relevant $a(2^+)$ element using Eq. (3.25a);

```
DefConstantSymbol[a2p11ECT]
P2p[[1, 1]][-i, -j, -k, -a, -b, -c] IoECT[a, b, c, d, e, f] P2p[[1, 1]][-d, -e, -f,
    -l, -m, -n] - a2p11ECT P2p[[1, 1]][-i, -j, -k, -l, -m, -n] == 0 /.
     ExpandP2p11 /. FinalIoECT /.
  Expandθ /. ExpandΩ
SolveConstants[%]
```

The exact same process was used to obtain the other elements of the $a(2^+)$ matrix and subsequently the other a matrices corresponding to the remaining spin-parity components.

In addition to the massless poles arising from the determinant of the b matrices, the SPOs also contain singularities of the form k^{-2n} with n being a positive integer. To deal with these poles, it is convenient to go over to an explicit coordinate system and then take the limit of the null cone [39]. To carry out calculations in a coordinate basis in xAct, the xCoba package is required [73]. For this reason, pre-existing code as part of the PSALTer package was used from this point onward[1].

4 Results and Discussion

4.1 GR, ECT & TEGR in Minkowski Space

The method described in Sec. 3.2 was first applied to ECT, GR and TEGR so as to compare the results with theories that are well understood and calibrate the code. Starting from GR, the

[1]The PSALTer package is a code that is currently being developed by the group to perform this particle spectrum analysis for various gravitational theories. It is not yet published and the dS linearising scheme that will be discussed in an upcoming section has been added to it to extend upon its functionality.

Lagrangian we would like to investigate is as defined in Eq. (2.4), multiplied by the coupling constant $-\lambda$ on which the unitarity conditions will depend

$$\frac{\mathcal{L}_{\mathrm{GR}}}{b} = -\lambda(\mathcal{R} - \mathcal{T}^2). \tag{4.1}$$

The a matrices which define the operator $\mathcal{O}$ as in Eq. (3.22) are given by

$$\begin{pmatrix} 0 & \frac{3ik\lambda}{\sqrt{2}} & i\sqrt{\frac{3}{2}}k\lambda \\ -\frac{3ik\lambda}{\sqrt{2}} & -2k^2\lambda & 0 \\ -i\sqrt{\frac{3}{2}}k\lambda & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & i\sqrt{2}k\lambda \\ 0 & 0 & 0 \\ -i\sqrt{2}k\lambda & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 & ik\lambda & -ik\lambda \\ 0 & 0 & 0 & 0 \\ -ik\lambda & 0 & 0 & 0 \\ ik\lambda & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & k^2\lambda \end{pmatrix}, \quad \begin{pmatrix} 0 \end{pmatrix} \tag{4.2}$$

These correspond to the 0^+, 0^1, 1^+, 1^-, 2^+ and 2^- spin-parity sectors respectively. The corresponding b^{-1} matrices which then define the propagator $\mathcal{O}^{-1}$ via Eq. (3.29) are

$$\begin{pmatrix} 0 & 0 & \frac{i\sqrt{\frac{2}{3}}}{k\lambda} \\ 0 & --\frac{1}{2k^2\lambda} & \frac{\sqrt{3}}{2k^2\lambda} \\ -\frac{i\sqrt{\frac{2}{3}}}{k\lambda} & \frac{\sqrt{3}}{2k^2\lambda} & -\frac{3}{2k^2\lambda} \end{pmatrix}, \quad \begin{pmatrix} 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & \frac{i}{\sqrt{2}k\lambda} \\ 0 & 0 & 0 \\ \frac{i}{\sqrt{2}k\lambda} & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 & \frac{i}{2k\lambda} & -\frac{i}{2k\lambda} \\ 0 & 0 & 0 & 0 \\ -\frac{i}{2k\lambda} & 0 & 0 & 0 \\ \frac{i}{2k\lambda} & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{k^2\lambda} \end{pmatrix}, \quad \begin{pmatrix} 0 \end{pmatrix}. \tag{4.3}$$

Using our analysis we find that GR has no massive modes and two massless eigenvalues

$$\left\{ \frac{p^2}{\lambda}, \frac{p^2}{\lambda} \right\}, \tag{4.4}$$

corresponding to two massive modes propagating in the 2^+ sector. This is in perfect agreement with the predictions from the linearised theory of GR in which the number of independent components of the symmetric polarisation tensor $H_{\mu\nu}$ is reduced to two after imposing gauge constraints. These independent modes correspond to a spin-2 particle since displacement under gravitational waves is invariant under rotations by π [74]. The unitarity condition for no tachyons or ghosts is given by $\lambda > 0$. This is in accordance with the Einstein–Hilbert action for which $\lambda = 1/2\kappa$ where $\kappa > 0$ [35].

The gauge constraints on the source currents for GR, as output by the code can be seen in Fig. 3. In order to understand the physical meaning of these constraints, we would need to go from the spin-parity decomposition seen in the output, to position space. Since the main aim of the project was to extend the analysis to dS spacetime, this was not explicitly done, however for GR in this

Gauge constraints on source currents:

$$\left\{ \overset{0^-}{\bullet}\sigma^{\flat\parallel} == 0,\ \overset{1^+}{\bullet}\sigma^{\flat\perp}{}^{ab} == 0,\ \overset{1^-}{\bullet}\sigma^{\flat\perp}{}^{a} == 0,\ \overset{1^-}{\bullet}\tau^{\flat\parallel}{}^{a} + \overset{1^-}{\bullet}\tau^{\flat\perp}{}^{a} == 0,\ \overset{2^+}{\bullet}\sigma^{\flat\parallel}{}^{ab} == 0,\ \overset{2^-}{\bullet}\sigma^{\flat\parallel}{}^{abc} == 0 \right\}$$

Figure 3: Gauge constraints on the source currents for GR as output by the Mathematica code.

PGT formulation we expect the source constraints to correspond to those arising from the Poincaré symmetry as given in Eq. (3.28).

In the same order as before, the a matrices for ECT are found to be

$$\begin{pmatrix} -\lambda & -\frac{ik\lambda}{\sqrt{2}} & -i\sqrt{\frac{3}{2}}k\lambda \\ \frac{ik\lambda}{\sqrt{2}} & 0 & 0 \\ i\sqrt{\frac{3}{2}}k\lambda & 0 & 0 \end{pmatrix}, \begin{pmatrix} -\lambda \end{pmatrix}, \begin{pmatrix} -\frac{\lambda}{2} & -\frac{\lambda}{\sqrt{2}} & -\frac{ik\lambda}{\sqrt{2}} \\ \frac{\lambda}{\sqrt{2}} & 0 & 0 \\ \frac{ik\lambda}{\sqrt{2}} & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} -\frac{\lambda}{2} & \frac{\lambda}{\sqrt{2}} & 0 & ik\lambda \\ \frac{\lambda}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -ik\lambda & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{\lambda}{2} & -\frac{ik\lambda}{\sqrt{2}} \\ \frac{ik\lambda}{\sqrt{2}} & 0 \end{pmatrix}, \begin{pmatrix} \frac{\lambda}{2} \end{pmatrix} \tag{4.5}$$

with corresponding b^{-1} matrices

$$\begin{pmatrix} 0 & -\frac{i}{2\sqrt{2}k\lambda} & -\frac{i\sqrt{\frac{3}{2}}}{2k\lambda} \\ \frac{i}{2\sqrt{2}k\lambda} & \frac{1}{8k^2\lambda} & \frac{\sqrt{3}}{8k^2\lambda} \\ \frac{i\sqrt{\frac{3}{2}}}{2k\lambda} & \frac{\sqrt{3}}{8k^2\lambda} & \frac{3}{8k^2\lambda} \end{pmatrix}, \quad \begin{pmatrix} -\frac{1}{\lambda} \end{pmatrix}, \quad \begin{pmatrix} 0 & -\frac{\sqrt{2}}{\lambda+k^2\lambda} & -\frac{i\sqrt{2}k}{\lambda+k^2\lambda} \\ -\frac{\sqrt{2}}{\lambda+k^2\lambda} & \frac{1}{(1+k^2)^2\lambda} & \frac{ik}{(1+k^2)^2\lambda} \\ \frac{i\sqrt{2}k}{\lambda+k^2\lambda} & -\frac{ik}{(1+k^2)^2\lambda} & \frac{k^2}{(1+k^2)^2\lambda} \end{pmatrix},$$
$$\begin{pmatrix} 0 & \frac{\sqrt{2}}{\lambda+2k^2\lambda} & 0 & \frac{2ik}{\lambda+2k^2\lambda} \\ \frac{\sqrt{2}}{\lambda+2k^2\lambda} & \frac{1}{(1+2k^2)^2\lambda} & 0 & \frac{i\sqrt{2}k}{(1+2k^2)^2\lambda} \\ 0 & 0 & 0 & 0 \\ -\frac{2ik}{\lambda_2 k^2\lambda} & -\frac{i\sqrt{2}k}{(1+2k^2)^2\lambda} & 0 & \frac{2k^2}{(1+2k^2)^2\lambda} \end{pmatrix}, \quad \begin{pmatrix} 0 & -\frac{i\sqrt{2}}{k\lambda} \\ \frac{i\sqrt{2}}{k\lambda} & -\frac{1}{k^2\lambda} \end{pmatrix}, \quad \begin{pmatrix} \frac{2}{\lambda} \end{pmatrix}. \tag{4.6}$$

These matrices result in no massive propagating modes and two massless eigenvalues as given in Eq. (3.33). This implies that *ECT and GR have an identical particle spectrum*, as expected. The unitarity condition is $\lambda < 0$. Notice that in the pure gravity case of ECT one can solve for the spin connection from the equations of motion to obtain the Levi–Civita connection and thus torsion does not play a dynamical role in the theory. Since ECT and GR give the same dynamics when not coupled to other fields, the particle spectrum should also be identical as is indeed observed [75].

A similar result is expected for TEGR since this theory is also expected to be dynamically equivalent to GR. Performing the analysis by treating both the tetrad field and the spin connection as independent fields in the theory (as is done for both GR and ECT) results in the theory having no massive or massless propagating modes. This is a result of not using a gauge condition which is necessary for TEGR to be dynamically equivalent to GR. One such gauge is that of Weitzenböck in which the spin connection vanishes i.e. ${A^{ij}}_\mu \equiv 0$ [60]. This gives rise to the Weitzenböck spacetime in which the curvature of the spin-connection vanishes. As a result, the Lagrangian of TEGR as in Eq. (2.5) is equal to the Einstein–Hilbert Lagrangian up to a divergence term which does not influence the dynamics of the theory [35]. The a matrices in this gauge are found to be

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -2k^2\lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & k^2\lambda \end{pmatrix}, \quad \begin{pmatrix} 0 \end{pmatrix} \tag{4.7}$$

with corresponding b^{-1} matrices

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2k^2\lambda} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{k^2\lambda} \end{pmatrix}, \quad \begin{pmatrix} 0 \end{pmatrix}. \tag{4.8}$$

These matrices give no massive propagating modes and two massless eigenvalues

$$\left\{ \frac{2p^2}{\lambda}, \frac{p^2}{\lambda} \right\} \tag{4.9}$$

corresponding to two massless 2^+ particles, in accordance with the analysis for GR and ECT. The unitarity condition for this theory is $\lambda > 0$.

Since there are no off-diagonal elements in the b^{-1} matrices in Eq. (4.8) we can investigate whether the theory is PCR as discussed in Sec. 3.2. In both the 0^+ and the 2^+ matrices, the non-zero terms that go as $\propto k^{-2}$ correspond to the s field, i.e. the symmetric part of the f field. Thus, TEGR is not PCR. This is because the PCR criterion requires the propagator of this s field to drop off as at least k^{-4}. Since the b^{-1} matrices for ECT and GR have off-diagonal terms it is not possible to apply the PCR criterion employed by Lin, Hobson, and Lasenby [39] although we know a priori that these theories are not PCR.

4.2 Linearised ECT in dS Spacetime

In this section, the general method developed in Sec. 3.3 to linearise a PGT Lagrangian around dS spacetime is applied to ECT. This simple theory provides an opportunity to calibrate the method and sheds light on the type of linearised PGT Lagrangians we expect to see for more complex theories.

The linearised Ricci scalar obtained using the xAct code discussed in Sec. 3.4 for ECT is

$$\begin{aligned} \mathcal{R} = 12H^2 + \frac{A_{acb}A^{abc}}{a^2} + \frac{{A^{ab}}_a {A_b}^c{}_c}{a^2} + 6a{f^a}_a H^2 - a^2 f^{ab} f_{ba} H^2 + a^2 {f^a}_a {f^b}_b H^2 \\ - \frac{2A_{abc}f^{ab}Hv^c}{a} - \frac{2A_{acb}f^{ab}Hv^c}{a} + \frac{2f^{ab}\partial_b {A_a}^c{}_c}{a} - \frac{2\partial_b {A^{ab}}_a}{a^2} - \frac{2f^{ab}\partial_c {A_a}^c{}_b}{a}. \end{aligned} \tag{4.10}$$

The only way to get a full guarantee that this is indeed the correct linearised Ricci scalar in de Sitter is to perform the calculation by hand. However, performing this calculation analytically can be very complex so instead some consistency checks can be performed. Firstly, the Ricci scalar to zeroth order in the perturbations can be read off from Eq. (4.10) as $\mathcal{R} = 12H^2$, this is exactly the value expected from the literature for a dS spacetime [71].

A second consistency check can be performed by analysing the factors of a appearing in Eq. (4.10). Expanding the definition of the Ricci scalar in terms of the perturbed tetrad and spin-connection gives

$$\begin{aligned}\mathcal{R} = &(\frac{1}{a}\delta_A{}^\mu + \epsilon f_A{}^\mu)(\frac{1}{a}\delta_B{}^\nu + \epsilon f_B{}^\nu)[\partial_{[\mu}\Delta^{AB}{}_{\nu]} + \epsilon\partial_{[\mu}A^{AB}{}_{\nu]} + \Delta^A{}_{E[\mu}\Delta^{EB}{}_{\nu]} \\ &+ \epsilon\Delta^A{}_{E[\mu}A^{EB}{}_{\nu]} + \epsilon A^A{}_{E[\mu}\Delta^{EB}{}_{\nu]} + \epsilon^2 A^A{}_{E[\mu}A^{EB}{}_{\nu]}].\end{aligned} \tag{4.11}$$

It is straightforward to analytically obtain an expression for the Ricci rotation coefficients

$$\Delta_{ij\mu} = aH\delta^m{}_\mu(\delta_j{}^0\eta_{im} - \delta_i{}^0\eta_{jm}). \tag{4.12}$$

Thus noticing that $\Delta \sim a$ and $\partial\Delta \sim a^2$ we can then calculate the a-dependence of the terms in Eq. (4.11). For example $\frac{1}{a}f\partial\Delta \sim af$. Note that we still have the vector v^c in Eq. (4.10) which is hiding a factor of a. We can thus replace this by a vector $\tilde{v}^c$ such that $v^c = a\tilde{v}^c$ where $\tilde{v}^c = (1, 0, 0, 0)$. After performing this substitution we find that the factors of a that appear in the linearised Ricci scalar as obtained using xAct precisely match those as expected from the analytical analysis discussed above.

The particle spectrum of a theory of gravity should not depend on the scale factor a. This is because we don't expect the particle content of a theory of gravity to change with time. Because of this, we expect each term of the Lagrangian of ECT to have the same scale factor dependence. In order to achieve this we make use of the principle of reparametrization invariance i.e. diffeomorphism invariance. Specifically, a physical system should remain invariant under a reparametrization of the time coordinate [71]. We can thus rescale the tetrad and spin connection as $f \to \tilde{f} = af$ and $A \to \tilde{A} = \frac{A}{a}$.

To perform the Fourier transform of the Lagrangian, we can set the coordinate frame in such a way that $\tilde{v}^c = k^c/|k| = \tilde{k}^c$. This is a valid choice of k^c due to diffeomorphism invariance. Performing a usual Fourier transform then gives the Ricci scalar in Fourier space with the re-scaled fields as

$$\begin{aligned}\mathcal{R} = &12H^2 + \tilde{A}_{acb}\tilde{A}^{abc} + \tilde{A}^{ab}{}_a\tilde{A}_b{}^c{}_c + 6\tilde{f}^a{}_a H^2 - \tilde{f}^{ab}\tilde{f}_{ba}H^2 + \tilde{f}^a{}_a\tilde{f}^b{}_b H^2 - 2\tilde{A}_b{}^c{}_c\tilde{f}^{ab}H\tilde{k}_a \\ &+ 4\tilde{A}_a{}^b{}_b H\tilde{k}^a + 2\tilde{A}_b{}^c{}_c\tilde{f}^a{}_a H\tilde{k}^b - 2\tilde{A}_{abc}\tilde{f}^{ab}H\tilde{k}^c - 2\tilde{A}_{acb}\tilde{f}^{ab}H\tilde{k}^c - \frac{2i\tilde{A}^{ab}{}_a k_b}{a} \\ &+ \frac{2i\tilde{A}_a{}^c{}_c\tilde{f}^{ab}k_b}{a} - \frac{2i\tilde{A}_a{}^c{}_b\tilde{f}^{ab}k_c}{a},\end{aligned} \tag{4.13}$$

where we are treating a as constant. Notice that the only factors of a that remain in the expression are accompanying the wave vector k^b. If we replace k^b by the wave vector in proper coordinates ak^b the scale factor dependence is completely eliminated. Note that this does not affect terms containing $\tilde{k}^a = \frac{k^a}{|k|} = \frac{ak^a}{a|k|}$.

In order to obtain the final ECT Lagrangian we need to multiply by b, which in terms of the rescaled fields becomes,

$$b = a^4(1 - af^i{}_i) = a^4(1 - \tilde{f}^i{}_i). \tag{4.14}$$

This will only introduce an over-all factor of a to our expression and gives the final ECT Lagrangian, linearised around dS spacetime in Fourier space to be,

$$\begin{aligned}\mathcal{L}_{\text{ECT}} = &a^4(12H^2 + \tilde{A}_{acb}\tilde{A}^{abc} + \tilde{A}^{ab}{}_a\tilde{A}_b{}^c{}_c - 6\tilde{f}^a{}_a H^2 - \tilde{f}^{ab}\tilde{f}_{ba}H^2 - 5\tilde{f}^a{}_a\tilde{f}^b{}_b H^2 \\ &+ 4\tilde{A}^{ab}{}_b H\tilde{k}_a - 2\tilde{A}^{ab}{}_b\tilde{f}^c{}_c H\tilde{k}_a - 2\tilde{A}^{abc}\tilde{f}_{ac}H\tilde{k}_b - 2\tilde{A}^{abc}\tilde{f}_{ab}H\tilde{k}_c - 2^{a\tilde{b}}{}_b\tilde{f}^c{}_a H\tilde{k}_c \\ &- 2i\tilde{A}^{ab}{}_a k_b - 2i\tilde{A}^{abc}\tilde{f}_{ac}k_b + 2i\tilde{A}^{ab}{}_a\tilde{f}^c{}_c k_b + 2i\tilde{A}^{ab}{}_b\tilde{f}_a{}^c k_c)\end{aligned} \tag{4.15}$$

In future work, the particle spectrum of ECT can be analysed by making use of this final form of the linearised Lagrangian in Eq. (4.15).

4.3 Particle spectrum of scalar field & linearised GR in de Sitter space

In this section, the particle spectra of a free scalar field and linearised GR in dS spacetime are analysed. These simpler theories are useful to understand the type of behaviour expected for the full PGT case. The following analysis is also useful to highlight any modifications needed for the code developed to be able to perform the analysis in this new background.

The simplest field we can consider in dS spacetime is that of a massless scalar field ϕ, whose Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi, \tag{4.16}$$

where $g_{\mu\nu}$ is the dS metric and g is its determinant. By using the relation between the dS metric and the Minkowski metric $g_{\mu\nu} = a^2\eta_{\mu\nu}$ we can rewrite this as

$$\mathcal{L} = \frac{1}{2}a^2\eta^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi \tag{4.17}$$

For Eq. (4.17) to have the same form of the Lagrangian of a free field in Minkowski space we can use reparametrisation invariance and make the substitution $\phi \to \tilde{\phi} = \frac{\phi}{a}$. Keeping track of the derivatives of a and making use of a unit timelike vector on Minkowski space n^μ, similar to what is done in Sec. 3.3, we obtain the Lagrangian

$$\mathcal{L} = \frac{1}{2}H^2\phi^2a^2 - Ha\phi^\mu\nabla_\mu\phi + \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi, \tag{4.18}$$

where contractions are all done using the Minkowski metric. Finally, noticing that adding boundary terms does not affect the equations of motion we get

$$\mathcal{L} = \frac{1}{2}\nabla_a\phi\nabla^a\phi + \mathcal{H}^2\phi^2, \tag{4.19}$$

where $\mathcal{H} = aH$. We can now compare this Lagrangian to that of a massive scalar field with mass m

$$\mathcal{L} = \frac{1}{2}\nabla_a\phi\nabla^a\phi - \frac{1}{2}m^2\phi^2. \tag{4.20}$$

By direct comparison of Eqs. (4.19) and (4.20) we can observe that by moving to a dS background, in Minkowski spacetime it *seems like* the scalar field is obtaining an effective tachyonic mass $m^2 = -2\mathcal{H}^2 < 0$. Indeed, when running the code with the linearised Lagrangian in Eq. (4.16) around dS, we find that there is a massive pole with a tachyonic square mass equal to $-2\mathcal{H}^2$. This is always negative since $\mathcal{H}$ is real. The code identifies no massless propagating modes. It is important to note that this does not imply that a massless scalar field gives rise to a tachyonic particle in dS but rather reflects the fact that the method for performing the spectrum analysis was designed around the case of perturbing around Minkowski spacetime and that this pole should be identified with a healthy dS mode function [76].

In a similar way as was done for PGT in Sec. 3.3, we can linearise GR in its metric formulation whose Lagrangian is given by

$$\mathcal{L}_{\mathrm{GR}} = -\frac{\sqrt{-g}}{2\kappa}(R - 2\Lambda), \tag{4.21}$$

with Λ being the cosmological constant. We do this by introducing a perturbation $h_{\mu\nu}$ of the dS metric. Excluding the details of the calculation, the linearised Lagrangian is found to be

$$\begin{aligned}\mathcal{L} = &\frac{\mathcal{H}^2 h_{ab}h^{ab}}{\kappa} - \frac{\mathcal{H}^2 h^a{}_a h^b{}_b}{\kappa} - \frac{2\mathcal{H}^2 h_a{}^c h_{bc} n^a n^b}{\kappa} - \frac{\mathcal{H}^2 h_{ab} h^c{}_c n^a n^b}{\kappa} + \frac{\mathcal{H} h_a{}^b n^a \nabla_b h^c{}_c}{\kappa} \\ &- \frac{h^{ab}\nabla_b \nabla_a h^c{}_c}{2\kappa} + \frac{\mathcal{H} h^{bc} n^a \nabla_c h_{ab}}{\kappa} - \frac{\mathcal{H} h^b{}_b n^a \nabla_c h_a{}^c}{\kappa} - \frac{\mathcal{H} h_a{}^b n^a \nabla_c h_b{}^c}{\kappa} + \frac{h^{ab}\nabla_c \nabla_b h_a{}^c}{\kappa} \\ &- \frac{h^a{}_a \nabla_c \nabla_b h^{bc}}{2\kappa} - \frac{h^{ab}\nabla_c \nabla^c h_{ab}}{2\kappa} + \frac{h^a{}_a \nabla_c \nabla^c h^b{}_b}{2\kappa}.\end{aligned} \tag{4.22}$$

For this linearised Lagrangian with coupling constant α we find the a-matrices to be

$$\begin{pmatrix} -3\alpha\mathcal{H}^2 & \frac{1}{2}\sqrt{3}\alpha(2ik - 3\mathcal{H})\mathcal{H} \\ \frac{1}{2}\sqrt{3}\alpha(-2ik - 3\mathcal{H})\mathcal{H} & -\alpha(k^2 + 2\mathcal{H}^2) \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \end{pmatrix}, \begin{pmatrix} 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}\alpha(k^2 + 2\mathcal{H}^2) \end{pmatrix}, \begin{pmatrix} 0 \end{pmatrix}. \tag{4.23}$$

Note that since the fundamental fields in the metric formulation of GR are different from those in PGT, these a matrices correspond to slightly different SPOs to those presented in Sec. 4 but serve the exact same function of decomposing the operator $\mathcal{O}$ of the theory as a linear combination in terms of the SPO orthonormal basis. The corresponding inverse b matrices are given by,

$$\begin{pmatrix} \frac{4(k^2+2\mathcal{H}^2)}{3\alpha\mathcal{H}^4} & \frac{4ik-6\mathcal{H}}{\sqrt{3}\alpha\mathcal{H}^3} \\ \frac{-4ik-6\mathcal{H}}{\sqrt{3}\alpha\mathcal{H}^3} & \frac{4}{\alpha\mathcal{H}^2} \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \end{pmatrix}, \quad \begin{pmatrix} \frac{2}{\alpha(k^2+2\mathcal{H}^2)} \end{pmatrix}, \quad \begin{pmatrix} 0 \end{pmatrix}. \tag{4.24}$$

We find that there are four gauge constraints on the source current which correspond to the conservation of the stress-energy tensor i.e. $\partial^b \tau_{ab} = 0$. This is exactly what we observe when linearising in Minkowski spacetime and is a good consistency check to make sure that important physical laws still hold in de Sitter space. Retaining important symmetries when moving away from flat Minkowski background is a stringent criterion that a modified theory of gravity should meet. Studying the gauge constraints when performing this analysis for different PGTs is thus critical in future works.

The propagator analysis finds that there is one massive propagating spin-parity sector with $m^2 = -2\mathcal{H}^2$ which according to the code is classified as a tachyonic massive mode. The residue of this pole is equal to $2/\alpha$. Notice how in the case of GR, the exact same tachyonic mass as that of the free field appears. This is in accordance with the literature in which the mode functions of the graviton polarisations and the free field are considered to be the same [77]. The code fails to identify the massless modes of the theory however, we can still conclude that the spectrum cannot have more than two massless eigenvalues. Conservation of the stress-energy tensor imposes four gauge constraints. This removes eight degrees of freedom from the ten symmetries of the symmetric rank two tensor, leaving two degrees of freedom [78].

The reason behind the failure of the code to identify the massless polarisations is related to the requirement to move to a coordinate basis in order to calculate the poles arising from the SPOs, as discussed at the end of Sec. 3.4. For the dS case, rather than taking the limit of the null cone, we would need to take the limit for the tachyonic one-sheet hyperbola lying outside the light cone as highlighted in Fig. 4. This is yet to be implemented in the code. Despite this limitation, the above analysis highlights the potential that the code developed has in analysing the particle spectrum of theories around a de Sitter spacetime.

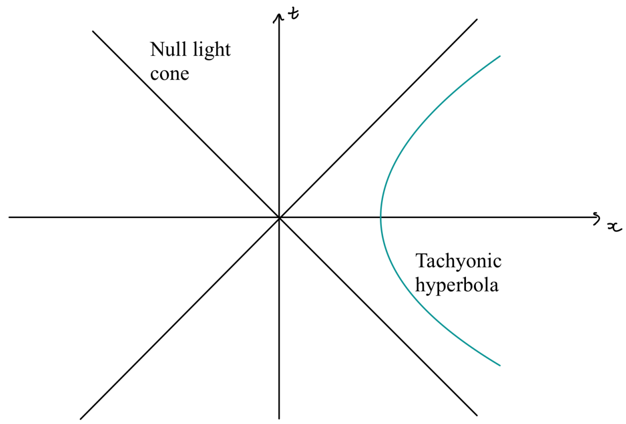

Figure 4: Sketch highlighting a tachyonic one-sheet hyperbola which lies outside the future light cone. In a dS spacetime, massless poles should be investigated in the limit of this hyperbola rather than that of the null cone.

5 Conclusion

In this project, the method for analysing the particle spectrum of PGT was studied in detail, implemented as a general novel Mathematica code and extended to dS spacetime. The main findings and conclusions are summarised below.

First, a systematic method for analysing the propagator structure of PGT around Minkowski spacetime was studied and a novel Mathematica code was developed to allow for the implementation of this method to any general PGT Lagrangian. The particle spectra of GR and ECT were investigated in detail. The analysis was also applied for the first time for TEGR. It was found that all three theories contain two massless propagating modes, corresponding to a spin-2 particle as expected. The method confirmed that for TEGR to be dynamically equivalent to GR, the Weitzenböck gauge is necessary as otherwise the theory contains no propagating modes.

Given the relevance of a dS cosmology, especially in the context of inflation and late-times acceleration, the propagator analysis was then extended to a dS background. In this non-trivial background, linearising a PGT Lagrangian proved to be significantly more complex than doing so around a Minkowski background. The mathematical details of this linearisation process, alongside a Mathematica implementation were presented in this work. It was seen that for ECT, the scale factor dependence of the Lagrangian can be eliminated by utilising the concept of reparametrisation invariance. This ensures that the particle spectrum of the theory is not time dependant.

The particle spectra of a free scalar field, and linearised GR were analysed around dS spacetime to provide an indication of the expected results for PGT. It was found that using the standard particle analysis developed for Minkowski space, switching to a dS background had the same effect as introducing a time-dependant tachyonic massive mode to the theory which should be interpreted as a healthy dS mode function.

The main focus after this project should be to modify the code developed to allow for a full analysis of the particle spectra of PGT in de Sitter spacetime. Specifically, the code should be able to distinguish between the effective tachyonic mass that a theory picks up from the dS background, and an actual tachyon that would spoil the stability of the theory. Furthermore, the inability to calculate the number of massless polarisations should be further investigated and corrected for so that the full particle spectrum can be analysed. ECT, GR and TEGR in the PGT formalism should then be used as calibration cases as was done for the Minkowski analysis.

Since the dS linearising method developed is a general one, this analysis can be easily extended to other PGTs, so that their particle spectra can be analysed using the code developed. In particular, future work should focus on investigating the PCR theories free of ghosts and tachyons found by Lin, Hobson, and Lasenby [51] around a dS spacetime. The particle analysis can also be applied to other modified gravity theories to ensure that no new degrees of freedom are introduced when transitioning from a Minkowski to a dS background, a phenomenon called strong-coupling [79]. Scalar-tensor theories can also be studied to ensure that spontaneous scalarisation does not occur when moving to a dS background, i.e. that the theories do not produce tachyonic instabilities because of this background [80]. This will allow for the identification of theories which have a healthy particle spectrum in dS so that their potential at producing a theory of quantum gravity, or solving some cosmological tensions can be further investigated.

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A Deriving the a matrices

In order to solve for the a coefficient matrices which then define the operator $\hat{\mathcal{O}}$ via Eq. (3.22) we used Eq. (3.23). However, in their paper Lin, Hobson, and Lasenby [39] suppress the explicit Local Lorentz indices which are crucial for understanding which indices are contracted. For this reason, Eq. (3.23) was derived again from first principles starting from Eq. (3.22) and using the orthonormality properties of the SPOs. The property which is of particular relevance for this derivation is the orthogonality one;

$$P_{ik}(J^P)_{\acute{\alpha}}{}^{\acute{\mu}} P_{lj}(J'^{P'})_{\acute{\mu}\acute{\beta}} = \delta_{JJ'}\delta_{PP'}\delta_{kl} P_{ij}(J^P)_{\acute{\alpha}\acute{\beta}}. \tag{A.1}$$

Explicitly including Lorentz indices in Eq. (3.22) gives

$$\hat{\mathcal{O}}^{\acute{\alpha}\acute{\beta}} = \sum_{i,j,J,P} a_{ij} \hat{P}_{ij}(J^P)^{\acute{\alpha}\acute{\beta}}. \tag{A.2}$$

From here onward we are dropping the $\hat{\mathcal{O}}$ notation and noting that the $\mathbf{e}_a$ vectors as seen in Eq. (3.17) are responsible for picking out the right part of the $\hat{\mathcal{O}}$ operator as highlighted in Eqs. (3.25a) - (3.25d).

Pre- and post- multiplying Eq. (A.2) by diagonal SPOs

$$P_{ii}(J^P)_{\acute{\mu}\acute{\nu}} \mathcal{O}^{\acute{\alpha}\acute{\beta}} P_{jj}(J^P)_{\acute{\gamma}\acute{\rho}} = \sum_{k,l,J',P'} a_{kl}(J'^{P'}) P_{ii}(J^P)_{\acute{\mu}\acute{\nu}} P_{kl}(J'^{P'})^{\acute{\alpha}\acute{\beta}} P_{jj}(J^P)_{\acute{\gamma}\acute{\rho}}. \tag{A.3}$$

Because of $\delta_{JJ'}\delta_{PP'}$ in Eq.(A.1) the only terms that remain in the sum over J' and P' in Eq. (A.3) are those with $J^P = J'^{P'}$ so we can simplify this to

$$P_{ii}(J^P)_{\acute{\mu}\acute{\nu}} \mathcal{O}^{\acute{\alpha}\acute{\beta}} P_{jj}(J^P)_{\acute{\gamma}\acute{\rho}} = \sum_{k,l} a_{kl}(J^P) P_{ii}(J^P)_{\acute{\mu}\acute{\nu}} P_{kl}(J^P)^{\acute{\alpha}\acute{\beta}} P_{jj}(J^P)_{\acute{\gamma}\acute{\rho}}. \tag{A.4}$$

Next we focus on $P_{kl}(J^P)^{\acute{\alpha}\acute{\beta}}P_{jj}(J^P)_{\acute{\gamma}\acute{\rho}}$. In order to apply Eq. (A.1), the inner Lorentz index must match and so we set $\acute{\gamma} = \acute{\beta}$ to obtain

$$P_{kl}(J^P)^{\acute{\alpha}\acute{\beta}}P_{jj}(J^P)_{\acute{\beta}\acute{\rho}} = \delta_{lj}P_{kj}(J^P)^{\acute{\alpha}}{}_{\acute{\rho}}. \tag{A.5}$$

Because of δ_{lj}, the summation over l in Eq. (A.4) sets $l = j$.

Next we focus on $P_{ii}(J^P)_{\acute{\mu}\acute{\nu}}P_{kj}(J^P)^{\acute{\alpha}}{}_{\acute{\rho}}$. Using a similar argument we require the inner indices to match and so $\acute{\nu} = \acute{\alpha}$ and

$$P_{ii}(J^P)_{\acute{\mu}\acute{\alpha}}P_{kj}(J^P)^{\acute{\alpha}}{}_{\acute{\rho}} = \delta_{ik}P_{ij}(J^P)_{\acute{\mu}\acute{\rho}}. \tag{A.6}$$

The summation over k then sets $k = i$ to give the final expression

$$P_{ii}(J^P)_{\acute{\mu}\acute{\alpha}}\mathcal{O}^{\acute{\alpha}\acute{\beta}}P_{jj}(J^P)_{\acute{\beta}\acute{\rho}} = a_{ij}P_{ij}(J^P)_{\acute{\mu}\acute{\rho}}\,. \tag{A.7}$$

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