Structure and Evolution of Stars

Lecture 15: Protostars and Reaching the ZAMS

- Pre-Main Sequence Evolution
  - Jeans’ Mass for collapse
  - Estimate radius and temperature for fully ionised protostar
  - Calculate timescale for collapse
  - Importance of convection

- Polytropic solution for convective star

- Hayashi Track of a protostar in the HR-diagram

Pre-Main Sequence Evolution

- Gravitational PE and thermal KE for gas cloud:

$$E_{grav} \approx \frac{GM^2}{R}; \quad E_{KE} = \frac{3}{2} NkT$$

- Gravitational PE must exceed pressure due to thermal motion:

$$\left| E_{grav} \right| > E_{KE}$$

- Gives condition for mass of cloud of radius *R* that can collapse – Jeans Mass:

$$\Rightarrow M_J = \frac{3kT}{2G\mu m_H} R$$

- Also provides value for critical density for cloud of mass *M* for collapse to take place:

$$\Rightarrow \rho_J = \frac{3}{4\pi M^2} \left[\frac{3kT}{2G\mu m_H} \right]^3$$

Pre-Main Sequence Evolution

- Interstellar medium includes many diffuse clouds consisting predominantly of Hydrogen and Helium, T~30K, n~5×10^{8}m^{-3}, with masses up to M~100M_{sun}

- For any reasonable composition of H and He the Jeans Mass is an order of magnitude larger than the maximum cloud mass, so no collapse

- Giant molecular clouds, T~30K, n~10^{14}m^{-3}, with masses extending over range M≈10-3000M_{sun} also exist and the Jeans Mass is ~5M_{sun}

- Giant molecular clouds are unstable to gravitational collapse and constitute the principal regions where star formation can occur

Star-Birth Clouds · M16 HST · WFPC2

PRC95-44b · ST ScI OPO · November 2, 1995

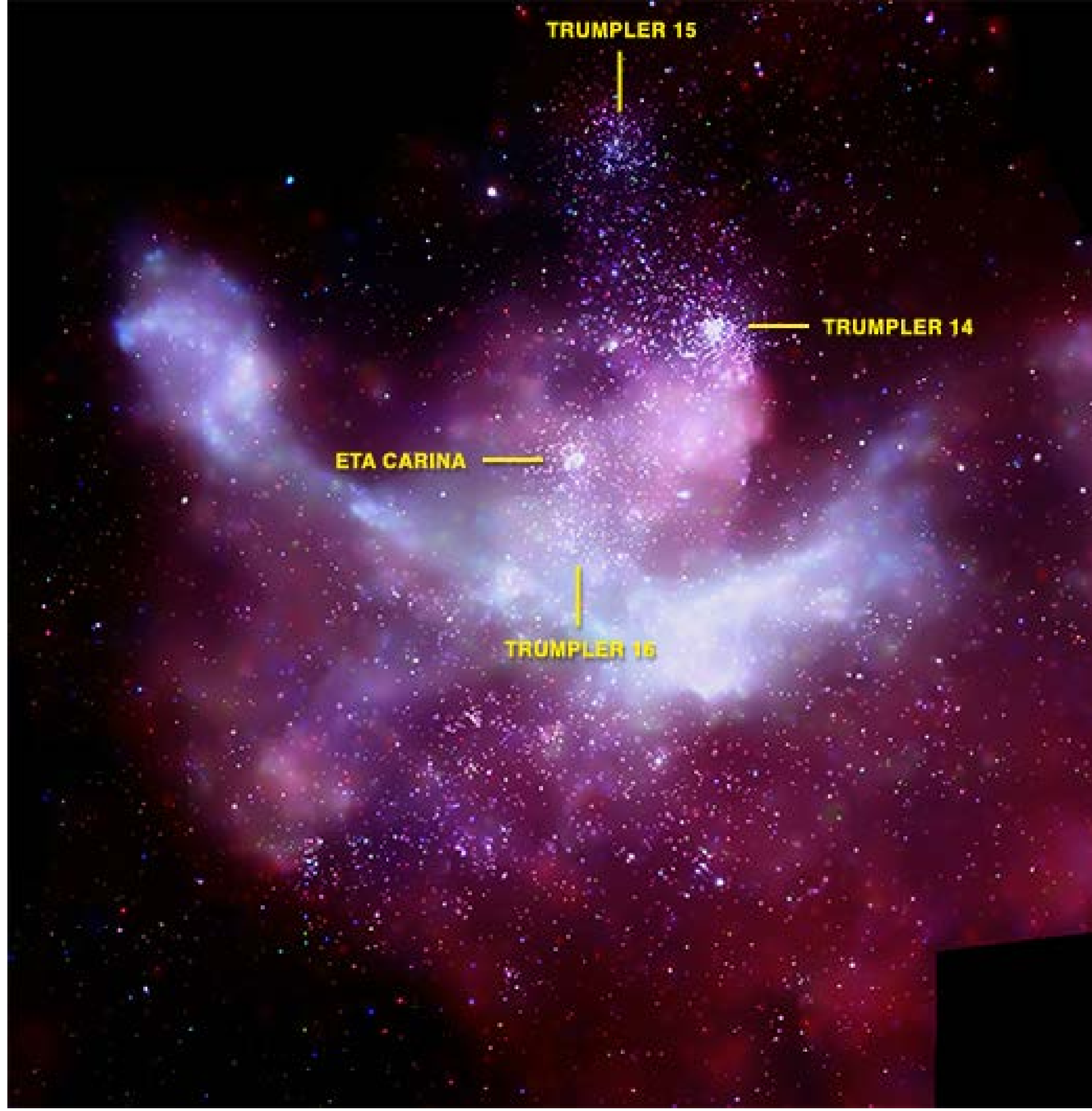
J. Hester and P. Scowen (AZ State Univ.), NASA

Carina nebula: 2.5kpc distant, 100pc across. Sequence of SF with time

Chandra X-ray colour composite RGB (from 0.6/0.8/0.9keV X-rays) Image 1.1 deg on a side

Star-formation propagating from North-west (top right – slide 6; top left here) towards the south-east at ~30kms^{-1}

X-rays from very hottest young stars and SN remnants



Pre-Main Sequence Evolution

- Initially, radius of cloud $R >> R_{ps}$, and, given low T, kinetic energy is not significant. Thus, can consider cloud contracting from infinity with gravitational PE so liberated available to generate luminosity (energy radiated away) or ionise the material in the cloud
- Initially, collapse proceeds on a dynamical timescale, $\tau_{dyn} \sim 1/\sqrt{G\rho}$ as density of material is low and no significant pressure to resist collapse and timescales are much longer than τ_{dyn} for stars, where density is many orders of magnitude higher
- The fraction of the gravitational PE liberated as radiation is small and majority of the energy is available to, first, break the H_2 molecules into H atoms, and then ionise the H and He atoms
- With these well-founded assumptions, can estimate key physical properties of collapsing cloud producing a proto-star

Pre-Main Sequence Evolution

• Obtain upper limit to radius of protostar by assuming collapse from infinity with energy liberated all into ionising material (i.e. assume no energy radiated away – pretty good approximation)

$$E_{grav} = \alpha \frac{GM^2}{R} \approx \frac{M}{m_H}\left(\frac{X}{2}\chi_{H_2} + X\chi_H + \frac{Y}{4}\chi_{He}\right)$$

$$Y \approx 1 - X;\ \alpha \approx 0.5;\quad \chi = \text{disruption energy}$$

$$(\chi_{H_2} = 4.5\text{eV};\ \chi_H = 13.6\text{eV};\ \chi_{He} = 79\text{eV})$$

• Radius is large ($\approx 60 R_{\text{sun}}$ for Solar mass object):

$$\frac{R_{ps}}{R_{sun}} \approx \frac{50}{1 - 0.2X}\frac{M}{M_{sun}};\qquad X = 0.7$$

• Estimate average T using Virial Theorem just as in Lecture 6

$$\bar{T} = \frac{\alpha}{3}\frac{\mu}{k}\frac{GMm_H}{R_{ps}} \approx 60000\text{K}$$

T is independent of object mass

Pre-Main Sequence Evolution

- Once cloud collapsing in hydrostatic equilibrium, half of energy is radiated away while half heats up constituent material

$$L = 4\pi R_{ps}^2 \sigma T_{eff}^4 ; \quad L = -\frac{1}{2}\frac{dE_{grav}}{dt}$$

Equate surface luminosity with gravitational energy liberated during collapse

$$\Rightarrow L = 4\pi R_{ps}^2 \sigma T_{eff}^4 = -\frac{GM^2}{2R_{ps}^2}\frac{dR_{ps}}{dt}$$

Estimate timescale from initial radius divided by rate of collapse and obtain familiar thermal (Kelvin-Helmholtz timescale)

$$\tau_{pms} \approx \frac{R_{ps}}{dR_{ps}/dt} = \frac{GM_*^2}{2R_* L_*} \approx \tau_{thermal}$$

Pre-Main Sequence Evolution

- Radius of protostellar object is large $R_{ps} \approx 60 R_{sun} \quad \text{for } 1 M_{sun}$
- Low T means opacity is very high with contribution from H^- ions important $\bar{\kappa} \approx \kappa_0 \rho T^4$
- Luminosity is high because of large surface area $L_{ps} = 4\pi R_{ps}^2 \sigma T_{eff}^4$
- Radiative diffusion incapable of transporting energy liberated to surface
- Protostar is essentially fully convective with just a thin outer radiative layer $\frac{P}{T}\left(\frac{dT}{dP}\right) > \frac{\gamma - 1}{\gamma}$

Hayashi Tracks

• Model protostar as fully convective (r=0→R) with thin radiative envelope. At r=R Hydrostatic Equilibrium implies:

$$\frac{dP}{dr} \approx -\rho \frac{GM}{R^2}$$

• Integrate to find pressure at R:

$$P_R = \frac{GM}{R^2} \int_R^\infty \rho dr$$

• Photons escape from where optical depth is approximately unity (sets thickness of radiative envelope):

$$\int_R^\infty \kappa \rho dr = \overline{\kappa} \int_R^\infty \rho dr = 1$$

• Approximate mean (ie Rosseland Mean type) opacity as power-laws of density and T:

$$\overline{\kappa} = \kappa_0 \rho^a T_{eff}^b$$

$$\Rightarrow \kappa_0 \rho^a T_{eff}^b \int_R^\infty \rho dr = 1$$

Hayashi Tracks

Combine equations to give: $P_R = \frac{GM}{R^2\kappa_0}\rho_R^{-a}T_{eff}^{-b}$

write logarithmically:

$$\Rightarrow \log P_R = \log M - 2\log R - a\log\rho_R - b\log T_{eff} + C$$

Also know: $L = 4\pi R^2\sigma T_{eff}^4$

logarithmically: $\Rightarrow \log L = 2\log R + 4\log T_{eff} + C$

and $P_R = \frac{\rho_R}{\mu m_H}kT_{eff}$

logarithmically: $\Rightarrow \log P_R = \log\rho_R + \log T_{eff} + C$

- Have 3 logarithmic equations relating P,M,R,ρ and T – need a 4th

Polytropes

• It is possible to take the first two stellar structure equations – hydrostatic equilibrium and mass conservation – and, if the pressure is independent of temperature, derive analytic solutions

• Equations of state of the form: $$P = K\rho^{\gamma} = K\rho^{(1+1/n)}$$

are termed **polytropic equations of state**, with K, n (the **polytropic index**) and γ all constants

• Can obtain polytropic solutions because either, pressure is independent of T (e.g. degenerate matter) and K is not model dependent, or, if pressure does have a T dependence but a second relation between density and T means that in combination an effective polytropic equation state exists but K is model dependent

Polytropes

• Important example of 2nd case is a perfect gas:

$$P = \frac{\rho}{\mu m_H} kT$$

where energy transport is via convection so that the condition on the temperature gradient is: (Lecture 12)

$$\frac{\gamma - 1}{\gamma} = \frac{P}{T}\frac{dT}{dP}$$

Gives relation between *P* and *T*

$$\Rightarrow P \propto T^{5/2} \quad \text{for} \quad \gamma = 5/3$$

Equate 2 expressions for *P* and write *T* in terms of the density:

$$\Rightarrow \rho T \propto T^{5/2} \Rightarrow T \propto \rho^{2/3}$$

Substitute for *T* in 1st equation:

$$\Rightarrow P \propto \rho^{5/3} \quad \text{polytrope with } n = 1.5$$

Polytropic Solutions

- Equation of hydrostatic equilibrium:

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}; \quad \times \frac{r^2}{\rho} \quad \text{and differentiate}$$

$$\Rightarrow \frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dm}{dr}$$

second stellar structure equation:

substitute for $\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}$ on rhs

(Poisson's equation)

$$\Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

take equation of state of form:
where K, n, and hence γ, constants

$$P = K\rho^\gamma \quad \text{with} \quad \gamma = 1 + 1/n$$

Polytropes

• You will be expected to be able to manipulate the equations of stellar structure for cases where pressure is a simple power-law of density, i.e. satisfy a polytropic equation of state

• Analytic solutions [of the Lane-Emden equation] exist only for n=0, 1, 5 with the last corresponding to a star of essentially infinite size. As n becomes smaller the mass distribution is more centrally complicated

• A model with n=3 does have some connection to the actual physical structure of certain stars

• Derivation and use of the Lane-Emden equation is not examinable – next four slides provided for completeness [mathematicians may like!]

Polytropic Solutions

- Substituting the polytropic equation of state gives a 2nd order differential equation

$$\frac{(n+1)K}{4\pi G n}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho^{\frac{n-1}{n}}}\frac{d\rho}{dr}\right) = -\rho$$

- Define dimensionless variable θ:

$$\rho = \rho_c \theta^n \quad \text{with} \quad 0 \le \theta \le 1$$

- Allows equation to be expressed in a simpler form:

$$\left[\frac{(n+1)K}{4\pi G \rho_c^{\frac{n-1}{n}}}\right]\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n$$

- Where coefficient in square brackets is a constant with dimensions length squared:

$$\frac{(n+1)K}{4\pi G \rho_c^{\frac{n-1}{n}}} = \alpha^2$$

Polytropic Solutions

Can replace r by a dimensionless variable ξ

$$r = \alpha\xi$$

Allowing simplification of the penultimate equation on previous slide to give the **Lane-Emden** equation of index n:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

with boundary conditions:

$$\theta = 1; \quad d\theta/d\xi = 0 \ \text{ at } \xi = 0$$

Solutions $\theta(\xi)$ decrease monotonically, reaching zero at $\xi=\xi_1$, corresponding to the stellar radius $R=\alpha\xi_1$

Polytropic Solutions

Total mass of polytropic star:

$$M = \int_0^R 4\pi r^2 \rho dr = 4\pi\alpha^3 \rho_c \int_0^{\xi_1} \xi^2 \theta^n d\xi$$

Using Lane-Emden equation:

$$= 4\pi\alpha^3 \rho_c \int_0^{\xi_1} \frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) d\xi = -4\pi\alpha^3 \rho_c \xi_1^2 \left(\frac{d\theta}{d\xi}\right)_{\xi_1}$$

$$\text{eliminate } \rho_c \text{ using} \left[\frac{(n+1)K}{4\pi G \rho_c^{\frac{n-1}{n}}}\right] = \alpha^2; \quad \alpha = R/\xi_1$$

To obtain a relation between *M*, *R* and *K* for index *n*

$$\Rightarrow \left(\frac{GM}{M_\mathrm{n}}\right)^{\mathrm{n-1}} \left(\frac{R}{R_n}\right)^{\mathrm{3-n}} = \frac{[(n+1)K]^n}{4\pi G}$$

Polytropic Solutions

$$\left(\frac{GM}{M_{\mathrm{n}}}\right)^{\mathrm{n-1}}\left(\frac{R}{R_n}\right)^{\mathrm{3-n}} = \frac{[(n+1)K]^n}{4\pi G}$$

where constants $M_n = -\xi^2 (d\theta / d\xi)_{\xi_1}$

and $R_n = \xi_1$ vary with n

$$P = K\rho^{(1+1/n)}$$

$$K^n \propto M^{n-1} R^{3-n}$$

- Appears singularly un-illuminating at first sight but the key point is that for a system that can be modelled as a polytrope we now have a relationship between *P*, *M*, *R* and ρ. Thus, for our fully convective protostar we can derive a 4th logarithmic relation between variables

$$\Rightarrow n \log P_R = (n-1)\log M + (3-n)\log R + (n+1)\log \rho_R + C$$

Hayashi Tracks

• Can now take the 3 logarithmic equations from Slide 10 that were derived using the properties of the thin radiative protostar atmosphere

• Combine with the 4th logarithmic equation derived from the polytropic model of the convective protostar and eliminate log R, log ρ_R and log P_R to obtain a relation between L, T_{eff} and M:

$$\log L = A \log T_{eff} + B \log M + C$$

$$A = \frac{(7-n)(a+1) - 4 - a + b}{0.5(3-n)(a+1) - 1}; \quad B = \frac{(n-1)(a+1) + 1}{0.5(3-n)(a+1) - 1}$$

• Can now place track of protostar, mass M, in the HR-diagram.

Hayashi Tracks

• Taking a=1 – a good approximation – which simplifies the expressions for A and B

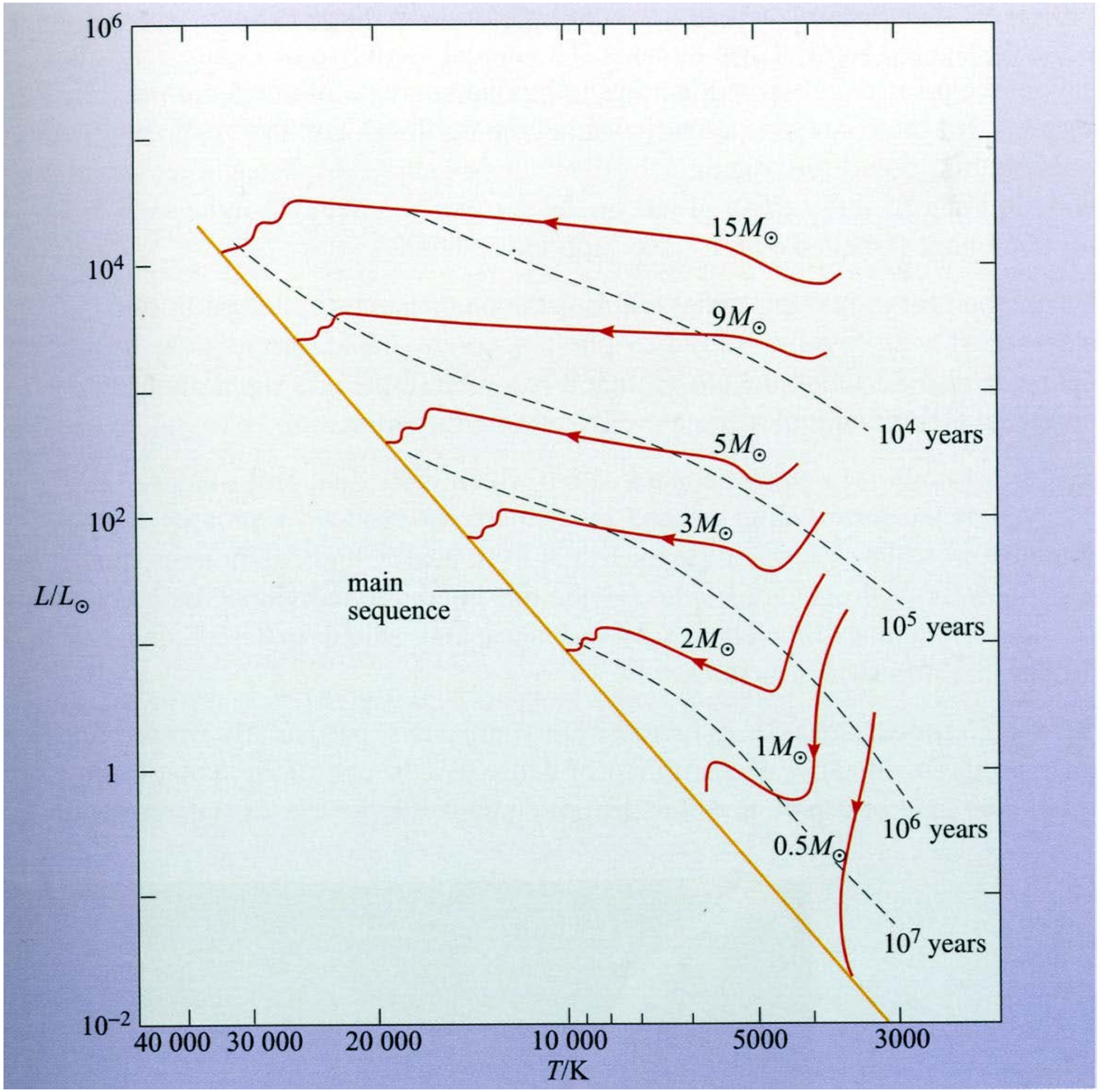
• The T-dependence of the opacity has values of b≈4 at low T and polytropic index for fully convective star is n=1.5 and A=20 – an almost vertical track in the HR-diagram

• As the mass of the protostar increases the tracks lie further to the left on the HR-diagram

$$A = \frac{9 - 2n + b}{2 - n}$$

$$B = \frac{2n - 1}{2 - n}$$

Hayashi tracks on the HR-diagram for stars of different mass. Note near vertical form of tracks when protostars in convective stage. As radiative diffusion becomes important the tracks move to the right onto the ZAMS



Hayashi Tracks

• The derivation of the behaviour in the *L*-*T* plane has made the most extreme assumptions and the Hayashi Tracks for a given *M* and represent the limiting case where the protostar is radiating the gravitational energy released by contraction as efficiently as possible (via convection)

• Regions to the right of the tracks represent objects that are unstable, no longer in hydrostatic equilibrium, and this portion of the HR-diagram is thus a forbidden zone. Energy transport would be taking place with a superadiabatic temperature gradient

• Explains why one cannot have stars with extremely low surface temperatures

• At high masses, stars increasingly radiative as they contract and motion in HR-diagram is almost horizontal – Henyey Track

Lecture 15: Summary

- Have examined basic physical properties of collapsing gas cloud that will form a star on the zero-age-main sequence
- Large radii, high luminosity and high opacity at low T results in convective energy transport in order for luminosity to reach stellar surface

- Can use polytropic model for fully convective star to provide relationship between P, M, R and ρ that enables behaviour of contracting protostar in the HR-diagram to be defined.

- Hayashi Track of a protostar in the HR-diagram is an almost vertical track that defines the limiting locus for an object in hydrostatic equilibrium. Regions to the right of the Hayashi Track for a given mass are forbidden

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