Structure and Evolution of Stars

Lecture 14: Homology Solutions (#2)

- Main-sequence properties from homology
  - luminosity-temperature relation
  - minimum main-sequence mass, luminosity and temperature (scaling from the Sun)
  - compare to minimum mass derived by considering collapsing gas cloud in hydrostatic equilibrium
  - maximum main-sequence mass, luminosity and temperature (scaling from the Sun)
- Summary of homology results
- Comments on solution of equations of stellar structure

Explaining the Main Sequence

- Observationally we have seen that there is a well-defined relation between luminosity and effective temperature for stars in the HR-diagram

$$\log L = \alpha \log T_{eff} + \text{constant}$$

where the slope, α, becomes steeper at higher L

- From measurement of stellar masses in binary systems have also established the existence of a mass-luminosity relation that is well approximated by

$$L \propto M^{\beta}$$

- Hypothesis is that the main-sequence is the locus of stars of different masses burning hydrogen in their cores. Can we now use the stellar structure equations to reproduce the observational data?

Homology

• Repeat the exact same procedure for all 4 stellar structure equations along with $P=nkT$ for a perfect gas to give a set of 5-pairs of equations ($P=nkT$ gives the 3rd equation as listed here)

• lh set consists of nonlinear differential equations independent of M for the functions $f_1, f_5 \ldots$

• rh set consists of *algebraic* equations that relate the dimensional coefficients $P_*, R_* \ldots$

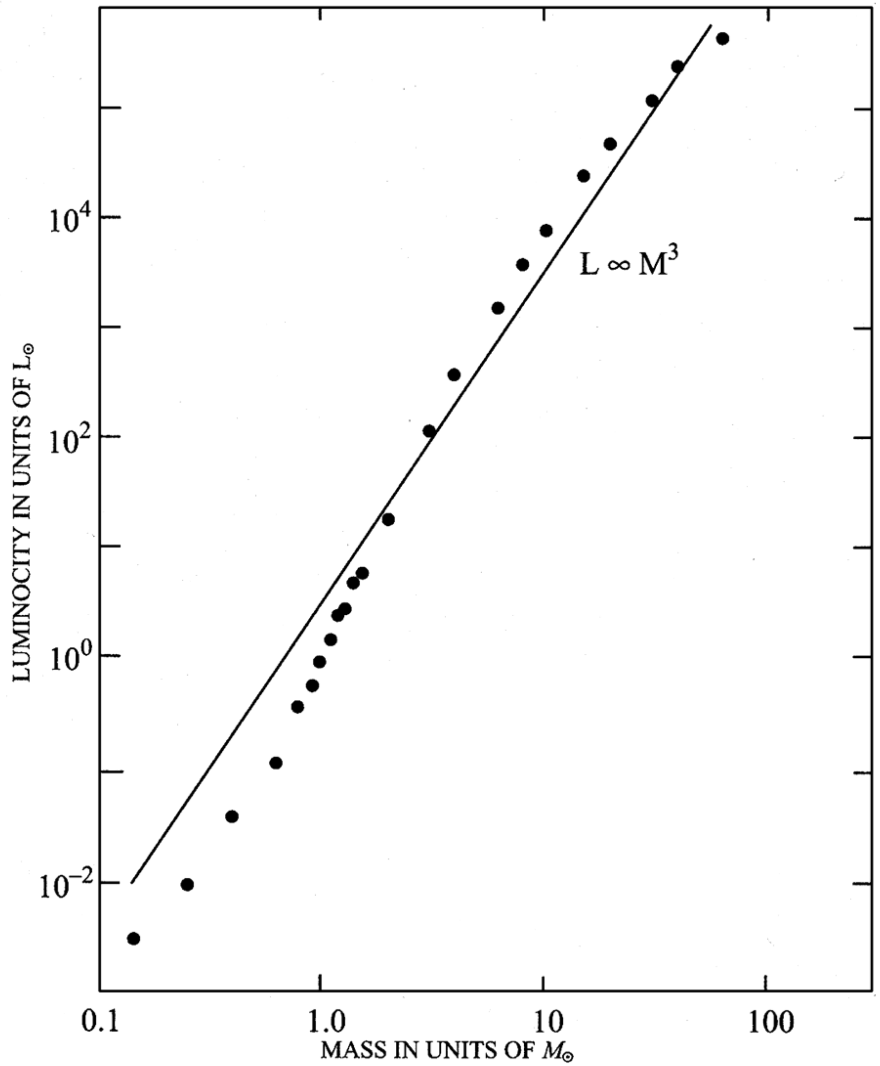
$$\frac{df_2}{dx} = -\frac{x}{4\pi f_1^{\,4}}; \quad P_* = \frac{GM^2}{R_*^{\,4}}$$

$$\frac{df_1}{dx} = \frac{1}{4\pi f_1^{\,2} f_3}; \quad \rho_* = \frac{M}{R_*^{\,3}}$$

$$f_2 = f_3 f_4; \qquad T_* = \frac{\mu m_H}{k}\frac{P_*}{\rho_*}$$

$$\frac{df_4}{dx} = -\frac{3 f_5}{4 f_4^{\,3}(4\pi f_1^{\,2})^2}; \quad L_* = \frac{ac}{\kappa}\frac{T_*^{\,4} R_*^{\,4}}{M}$$

$$\frac{df_5}{dx} = f_3 f_4^{\,n}; \qquad L_* = \varepsilon_0 \rho_* T_*^{\,n} M$$



The mass–luminosity relation for hydrogen burning stars with a chemical composition similar to the sun. The data on representative main sequence stars is taken from Table 3.13 in the Astronomy and Astrophysics section of the *Physics Vade Mecum* compiled by Fredrick (1989)

Homology: Main Sequence Properties

Take the two equations for L_* from Lecture 13 (slide #3 here):

$$L_* = \frac{ac}{\kappa}\left(\frac{\mu m_H}{k}\right)^4 M^3; \quad L_* = \varepsilon_0 \rho_* T_*^n M$$

equate and then substitute for pressure and density in the homology relation derived using $P=nkT$:

$$P_* = \frac{GM^2}{R_*^4}; \ \rho_* = \frac{M}{R_*^3}; \ T_* = \frac{\mu m_H}{k}\frac{P_*}{\rho_*}$$

eliminate T from the equations and rearrange to find the radius in terms of mass

$$R_* \propto M^{(n-1/n+3)}$$

True for any value of the fractional mass, x, including x=1, so have radius-mass relation on main sequence

Homology: Main Sequence Properties

• Form of the relation depends on the value of *n* and hence the energy-generation process. For low mass stars where the p-p chain dominates ($n \approx 4$):

$$R_* \propto M^{(n-1/n+3)}$$

$$\Rightarrow R_* \propto M^{3/7}$$

at high masses where the CNO cycle dominates ($n \approx 18$):

$$\Rightarrow R_* \propto M^{17/21}$$

• Easy to obtain density dependence on mass by substituting for R_* in equation:

$$\rho_* = \frac{M}{R_*^{\,3}}$$

to give density as a function of mass:

$$\rho_* \propto M^{(6-2n/n+3)}$$

Homology: Main Sequence Properties

• As n>3 for even the p-p chain, the density of stars decreases as the mass increases

$$\rho_* \propto M^{(6-2n/n+3)}$$

• Now determine dependence of L on $T_{\rm eff}$ by taking the two earlier equations:

$$L \propto M^3; \quad R_* \propto M^{(n-1/n+3)}$$

substitute into the familiar relation:

$$L = 4\pi R^2 \sigma T_{eff}^4$$

to give dependence of L on T as a function of n:

$$L^{1-\frac{2(n-1)}{3(n+3)}} \propto T_{eff}^4$$

Homology: Main Sequence Properties

$$L^{1-\frac{2(n-1)}{3(n+3)}} \propto T_{eff}^4$$

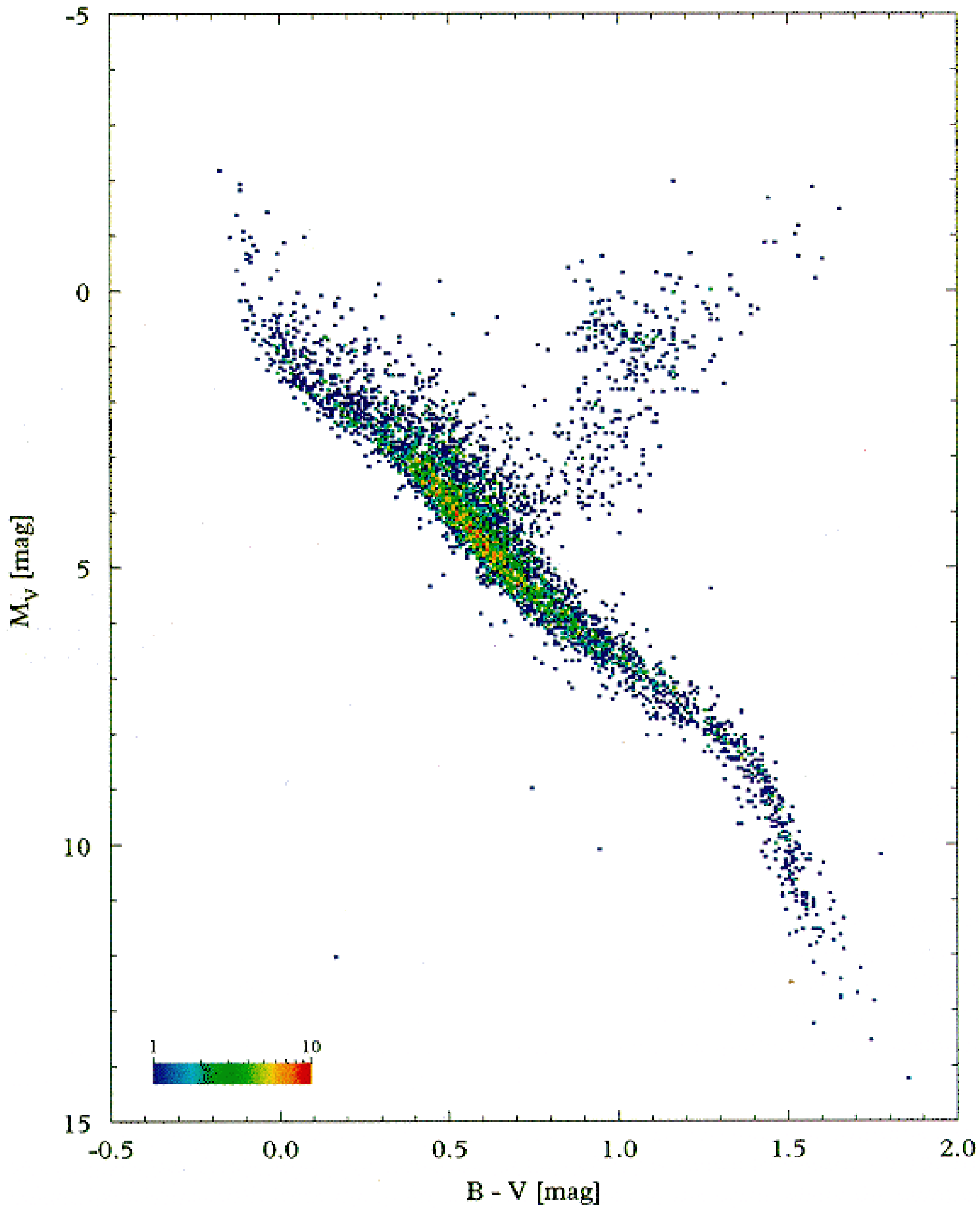
• For the p-p chain, $n \approx 4$, at low mass (=Luminosity) have:

$$\log L = 5.6 \log T_{eff} + \text{const}$$

• At higher masses, where $n \approx 18$, have:

$$\log L = 8.7 \log T_{eff} + \text{const}$$

in very good agreement with the observed form of the main sequence in the HR-diagram


-5
0
5
10
15
M_V [mag]
-0.5
0.0
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1.0
1.5
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10

Main Sequence Properties

• For stars with the same composition, the nuclear fuel available, i.e. Hydrogen, is directly proportional to mass of the star

• From homology solution have shown that: $L \propto M^3$

thus, lifetime of star on main sequence: $\tau_{ms} \propto \frac{M}{L} \propto M^{-2}$

• What are the mass limits that define the minimum and maximum masses of stars on the main sequence?

• For the minimum mass we can use two different schemes to estimate the mass from what we have learned

Minimum MS-Mass: from Sun + Homology

- Minimum central temperature to initiate p-p chain $T_c \approx 4\times10^6$K

- Detailed models of the Sun give a value for the central temperature of 15 $\times10^6$K

Homology relations from earlier:

$$T_* = \frac{\mu m_H G}{k}\frac{M}{R_*}; \quad R_* \propto M^{n-1/n+3}$$

p-p chain appropriate:

$$\Rightarrow T_c \propto M^{4/7} \quad \text{for } n = 4$$

Scale relative to M and T for the Sun:

$$\Rightarrow \frac{T_c}{T_{c,sun}} = \left(\frac{M}{M_{sun}}\right)^{4/7} \Rightarrow M_{\min} \approx 0.1 M_{sun}$$

Minimum MS-Mass L & T: from Sun + Homology

- Have corresponding L from mass-luminosity relation

$$L_* \propto M^3 \text{ that } L_{\min} \approx 0.001 L_{sun}$$

- Take homology scaling of mass-radius relation for n=4 (p-p chain) and familiar L-R-T_{eff} relation:

$$\frac{R}{R_{sun}} = \left(\frac{M_{\min}}{M_{sun}} \right)^{3/7} \text{ and } L = 4\pi R^2 \sigma T_{eff}^4$$

substitute mass-radius scaling into L-R-T_{eff} relation

$$\Rightarrow L_{\min} = 4\pi R_{sun}^2 \left(\frac{M_{\min}}{M_{sun}} \right)^{6/7} \sigma T_{eff,\min}^4$$

evaluate for known Solar quantities (T_{eff}=5800K)

$$\Rightarrow T_{eff,\min} \approx 1700\text{K}$$

Minimum MS-Mass: from Collapse of Cloud in Hydrostatic Equilibrium

Collapsing cloud, classical gas of *N* particles, mean molecular weight, μ, with gravitational and kinetic energy:

$$E_{Grav} \approx \frac{GM^2}{R}; \quad E_{KE} = \frac{3}{2}NkT$$

Condition for hydrostatic equilibrium:

$$\Rightarrow 2E_{KE} + E_{Grav} = 0$$

Shows that the gas temperature increases with both the cloud mass and the density, which increases as cloud collapses

$$\Rightarrow kT \approx \frac{\mu m_H GM}{3R} = \mu m_H GM^{2/3} \rho^{1/3}$$

Minimum MS-Mass: from Collapse of Cloud in Hydrostatic Equilibrium

de Broglie wavelength for the electrons

$$\lambda = \frac{h}{p}; \quad KE \approx kT \Rightarrow p \approx (m_e kT)^{1/2}$$

Pressure due to classical gas providing electrons separated with distance >>λ, i.e. density:

$$\rho << \frac{\mu m_H}{\lambda^3} \approx \mu m_H \frac{(m_e kT)^{3/2}}{h^3}$$

Degeneracy pressure becomes dominant when electrons are separated by distance $\approx\lambda$, giving a critical density, ρ_{degen} above which the temperature of the gas does not increase

$$\rho_{degen} = \mu m_H \frac{(m_e kT)^{3/2}}{h^3}$$

Minimum MS-Mass: from Collapse of Cloud in Hydrostatic Equilibrium

Take expression for critical density and substitute into expression for the temperature derived from hydrostatic equilibrium condition

$$\rho_{degen} = \mu m_H \frac{(m_e kT)^{3/2}}{h^3}$$

$$kT \approx \mu m_H G M^{2/3} \rho^{1/3}$$

to give

$$kT \approx \mu m_H G M^{2/3} (\mu m_H)^{1/3} \frac{(m_e kT)^{1/2}}{h}$$

Rearrange to give dependence on mass and then evaluate for minimum *T* for p-p chain:

$$kT \approx \left[\frac{(\mu m_H)^{8/3} m_e G^2}{h^2} \right] M^{4/3}$$

$$\Rightarrow M_{\min} \approx 0.1 M_{sun} \quad \text{for} \quad T = 4 \times 10^6 \,\text{K}$$

Maximum MS-Mass *L* & *T*: from Sun + Homology

- Derived constraint on stellar luminosity (Eddington Limit) in Lecture 12, which can be re-written:

$$L < \frac{4\pi cGM}{\kappa}$$

$$\frac{L}{L_{sun}} < \frac{4\pi cGM_{sun}}{\kappa L_{sun}} \frac{M}{M_{sun}}$$

- Incorporate the mass-luminosity relation from our homology solution:

$$\frac{L}{L_{sun}} = \left(\frac{M}{M_{sun}}\right)^3$$

to give estimate of the maximum main-sequence mass, using opacity due to electron scattering (appropriate for radiation pressure dominant)

$$\frac{M}{M_{sun}} < \left(\frac{4\pi cGM_{sun}}{\kappa L_{sun}}\right)^{1/2} \approx 180$$

Maximum MS-Mass L & T: from Sun + Homology

• Combine maximum mass with the homology mass-luminosity relation to give the luminosity for the most massive stars $L<5.8\times10^6 L_{sun}$

• Use the homology mass-radius relation with n=18 (appropriate for energy generation via the CNO cycle in high mass stars) and the L-R-T_{eff} relation, exactly as used to calculate T_{eff} for the low-mass end of the main sequence and obtain $T_{eff}\approx 40000$K

• Have now obtained the luminosity-mass and luminosity-temperature relation for the main sequence. Have mass limits at lower and upper ends with estimates of associated temperatures and luminosities in good agreement with observations

• Can refine predictions using more realistic opacity relation, e.g. Kramers Opacity relation $\kappa=\kappa_0\rho T^{-3.5}$, instead of κ=constant

Solving the Equations of Stellar Structure

- Homology has allowed the derivation of the key relationships between physical quantities of interest

- Strictly, analysis applies only to stars of homogeneous composition as they lie on the **zero age main sequence** (ZAMS)

- Variation in composition results in small spread in the width of the ZAMS as the opacity depends on the exact composition

- Once nuclear burning takes place the composition changes and it is necessary to incorporate the time-dependent change in composition to predict the exact properties of the stars

- The full set of stellar structure equations can not be solved analytically, except in a few special cases

Homology (Reminder)

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Solving the Equations of Stellar Structure

- One class of special cases arises when the pressure depends only on the density so that there is no *T*-dependence and the equation of hydrostatic equilibrium and mass conservation may be solved analytically – termed **polytropic equations of state**

$$P = K\rho^{\gamma} \quad \text{with} \quad \gamma = 1 + \frac{1}{n}$$

where n is the polytropic index: n=1.5 corresponds to γ=5/3 and n=3 to γ=4/3, representing the equation of state for a non-relativistic and relativistic degenerate gas respectively.

- Even with such an equation of state there are analytic solutions only for n=0, 1 or 5
- Polytropes provide useful insight for aspects of pre-main sequence evolution (later Lecture)

Solving the Equations of Stellar Structure

• Real stellar models are produced by solving the stellar structure equations numerically on a computer:

- for a given composition the equations are solved, with detailed consideration of the evaluation of the opacity, at time $t=t_0$, producing a value for the energy liberated by fusion
- composition is adjusted, to take account of the fusion processes, to give composition at $t=t_0+\delta t$
- new composition used to solve the stellar structure equations again
- sequence repeats through lifetime of the star

• Even this procedure is not straightforward, with sophisticated numerical algorithms and small time-steps employed. Active field of research in order to model behaviour of stars at various stages of evolution

Lecture 14: Summary

- Self-similar, homology, solutions allow the key dependences between parameters for stars on the main-sequence to be determined, including:
  - luminosity-temperature relation
  - minimum and maximum main-sequence mass, luminosity and temperature (scaling from the Sun)
- Good agreement with observational data and relations can be refined using more sophisticated treatment for the opacity
- Solutions to the equations of stellar structure not possible analytically and detailed computer models required

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