Structure and Evolution of Stars

Lecture 12: Energy Transport

- Equations of stellar structure for case when radiative diffusion dominates energy transport
- Eddington Limit
  - Consequences of the Eddington Limit, “Eddington Luminosity”
- Energy transport mechanisms
- Condition for the onset of convection
- Equations of stellar structure for case where convection dominates energy transport

Equations of Stellar Structure

$$\frac{dP}{dr} = -\rho \frac{Gm}{r^2}$$

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

$$\frac{dT}{dr} = -\frac{3}{4ac} \frac{\kappa\rho}{T^3} \frac{L}{4\pi r^2}$$

$$\frac{dL}{dr} = 4\pi r^2 \rho\varepsilon$$

The Eddington Limit

• Have derived 4 equations of stellar structure that involve 7 unknowns – P, M, T, L, ρ, ε and κ

• Have investigated the behaviour of P, ε and κ:

$$P = P(\rho, T, composition)$$

$$\varepsilon = \varepsilon(\rho, T, composition)$$

$$\kappa = \kappa(\rho, T, composition)$$

• Therefore have 3 of the variables in terms of ρ and T, with tractable power-law dependencies, and, for fixed composition, have 4 equations and 4 unknowns - M, T, L, ρ

• The stellar structure equation derived from consideration of radiative transfer is predicated on energy transport via radiative diffusion

• When is this assumption valid?

The Eddington Limit

Take the expression for radiation pressure and substitute for *dT* into the stellar structure equation derived from consideration of radiative diffusion (in Lecture 9)

$$P_{rad} = \frac{1}{3} a T^4 \Rightarrow dT = \frac{3 dP_{rad}}{4 a T^3}$$

$$\frac{dT}{dr} = -\frac{3}{4ac} \frac{\kappa \rho}{T^3} \frac{L}{4\pi r^2}$$

to give

$$\Rightarrow \frac{dP_{rad}}{dr} = -\frac{\kappa \rho}{c} \frac{L}{4\pi r^2}$$

Take equation for hydrostatic equil.:

$$\frac{dP}{dr} = -\rho \frac{Gm}{r^2}$$

divide to give relationship between the relative change in *P* in terms of the luminosity, mass and opacity

$$\Rightarrow \frac{dP_{rad}}{dP} = \frac{\kappa L}{4\pi c G m}$$

The Eddington Limit

Total pressure is sum of radiation and gas contributions and thus lhs of the equation must be less than unity

$$\frac{dP_{rad}}{dP} = \frac{\kappa L}{4\pi c G m}$$

$$P = P_{rad} + P_{gas}$$

$$\Rightarrow \frac{dP_{rad}}{dP} < 1$$

Immediately sets a condition for the product of the opacity and luminosity in terms of the mass

$$\Rightarrow \kappa L < 4\pi c G m$$

Stars that obey this condition are said to be in *radiative equilibrium*. If condition not satisfied then cannot have equilibrium with energy transport via radiative diffusion

The Eddington Limit

At centre of a star take equation of thermal equilibrium (Lecture 9)

$$\frac{dL}{dr} = 4\pi r^2 \rho \varepsilon \Rightarrow \frac{dL}{dm} = \varepsilon$$

combined with the boundary condition

$$L(0) = 0$$

$$\Rightarrow L/m \to \varepsilon_c \text{ as } m \to 0$$

where $\varepsilon_c = \varepsilon(m = 0)$

Providing a condition on the maximum energy generation rate in the centre of a star that can be transported by radiative diffusion

$$\varepsilon_c < \frac{4\pi c G}{\kappa}$$

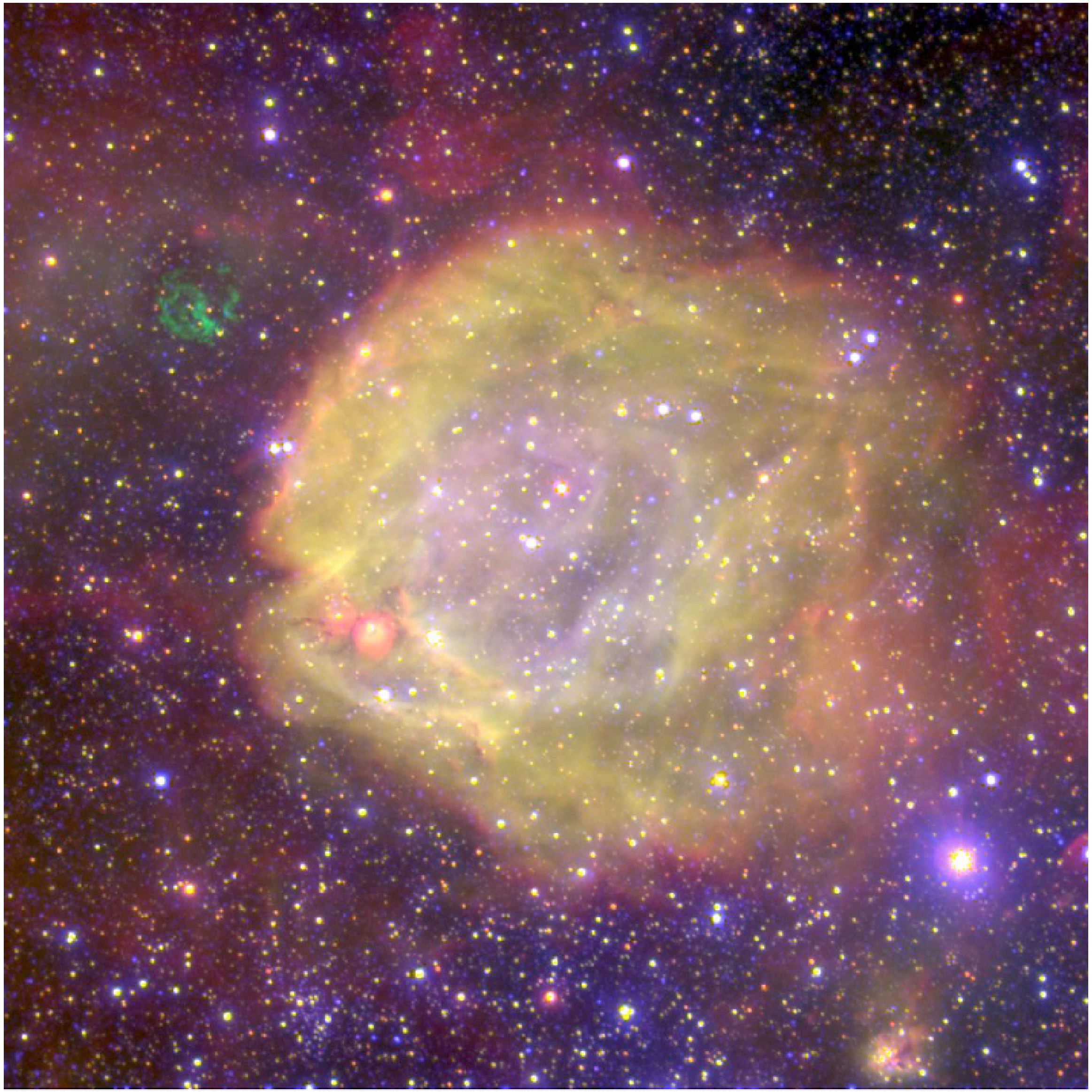
The Eddington Limit

At the surface of a star both the luminosity and mass applicable are those for the star as a whole and thus:

$$L < \frac{4\pi cGM}{\kappa}$$

Eddington pointed out that the rhs gives a critical luminosity (the Eddington Luminosity) above which material will be removed from the star via radiation pressure. Material is normally ionised and the Thomson cross-section of electrons dominates the opacity – nuclei are dragged along with the electrons as momentum transfer from the photons pushes material outwards

$$L_{Edd} = \frac{4\pi cGM}{\kappa} = 3.2\times 10^4 \left(\frac{M}{M_{Sun}}\right)\left(\frac{\kappa_{es}}{\kappa}\right) L_{Sun}$$



Nebula Around the Hot Binary Star AB7 in the SMC
(VLT MELIPAL + FORS 1)

ESO PR Photo 09a/03 (8 April 2003)

Energy Transport Mechanisms

• Three important mechanisms for transporting energy from the inner regions of stars outward towards the surface

- **Radiative diffusion** of photons (already met in Lecture 8)

- Particle diffusion, or **conduction**, where electrons are most important (treatment as for radiation)

- Bulk motion of material, or **convection**

• In the case of radiative diffusion and conduction the critical parameter for the efficiency of the process is the mean free path (mfp) of the carrier. In normal stars, the mfp of photons is larger than for particles and the mfp of electrons is greater than for nuclei. Thus, radiative diffusion is more important than conduction, although electron conduction is dominant in white dwarfs (later lecture)

Energy Transport Mechanisms

• In Lecture 8 we introduced the opacity, κ, of material to radiation such that mfp=1/($\kappa\rho$), where high opacity (small mfp) indicates substantial resistance to the passage of radiation, deducing that

• Treatment for energy transport via conduction (particles rather than photons) is identical in form and in principle the opacity, κ, can be written:

$$\frac{1}{\kappa} = \frac{1}{\kappa_{rad}} + \frac{1}{\kappa_{cond}}$$

although conduction is not important in most stars

• Having considered the case where radiative diffusion dominates the energy transport, still need to determine when convection is dominant

Convection: Qualitative Outline

• mass element Δm, density ρ_1 and pressure P_1

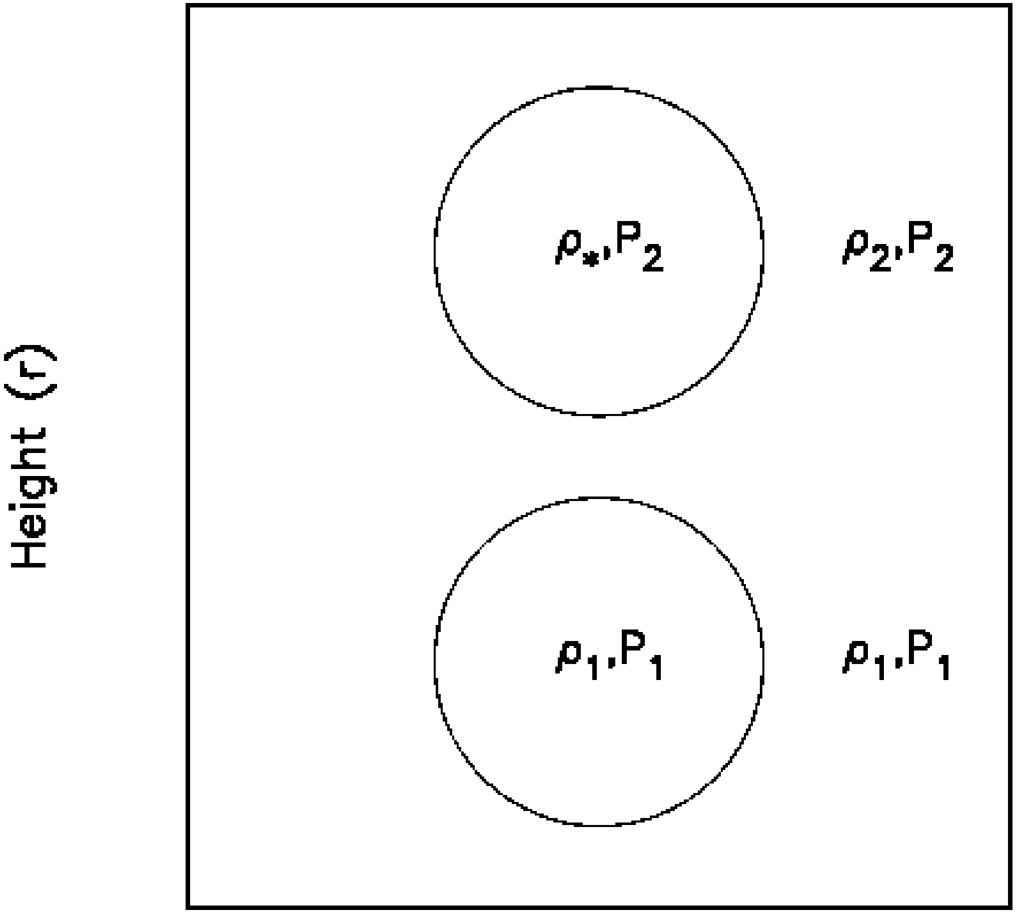
• element moves small distance upward into region with density ρ_2 and pressure P_2, where $P_2 < P_1$ and surrounding pressure now lower than within the mass element

• Dynamical timescale < thermal timescale – can assume no heat exchange between element and surroundings (adiabatic)

• mass element expands until achieves density ρ_* where in general $\rho_* \neq \rho_2$, and pressures are equal

• If $\rho_* > \rho_2$ mass element descends and motions are damped – stable

• If $\rho_* < \rho_2$ mass element ascends (Archimedes), motions are amplified and region is unstable to convection


Height (r)
ρ*,P2
ρ2,P2
ρ1,P1
ρ1,P1

Condition for Convective Instability

Pressure and density for adiabatic process, where γ is the ratio of specific heats

$$P = const \times \rho^{\gamma} \Rightarrow P\big/\rho^{\gamma} = const$$

Relation between pressure and density in the starting and finishing locations

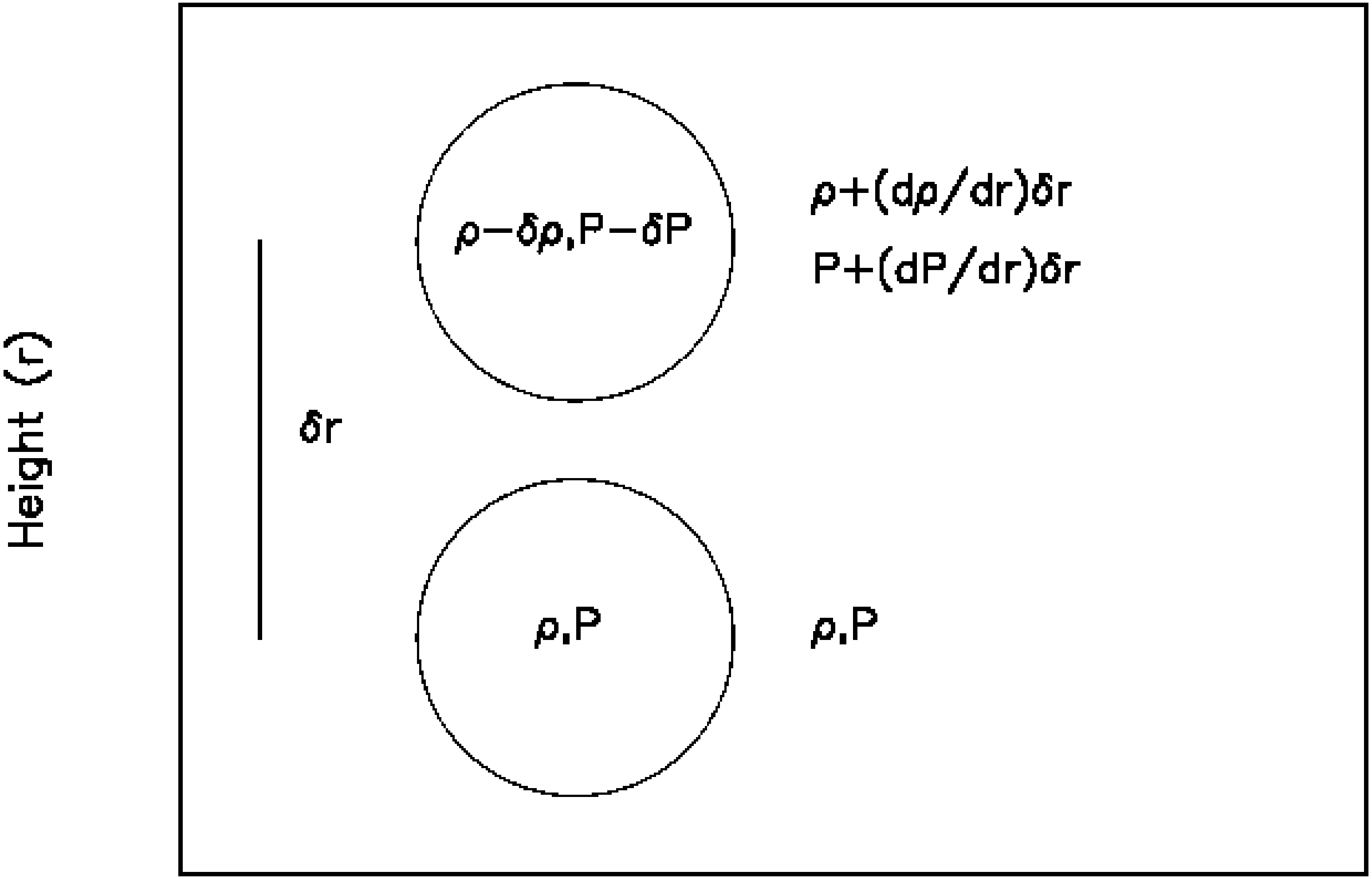
$$\frac{P - \delta P}{(\rho - \delta\rho)^{\gamma}} = \frac{P}{\rho^{\gamma}}$$

If $\delta\rho << \rho$ then make expansion:

$$(\rho - \delta\rho)^{\gamma} \approx \rho^{\gamma} - \gamma\rho^{\gamma-1}\delta\rho$$

giving:

$$\Rightarrow \frac{P - \delta P}{\rho^{\gamma} - \gamma\rho^{\gamma-1}\delta\rho} = \frac{P}{\rho^{\gamma}}$$


Height (r)
δr
ρ−δρ,P−δP
ρ+(dρ/dr)δr
P+(dP/dr)δr
ρ,P
ρ,P

Condition for Convective Instability

rearrange: $$\rho^{\gamma}(P - \delta P) = P(\rho^{\gamma} - \gamma\rho^{\gamma-1}\delta\rho)$$

$$\Rightarrow \delta P = \frac{\gamma P\,\delta\rho}{\rho}$$

equating pressure in the mass element with the surrounding pressure

$$\delta P = \left(-\frac{dP}{dr}\right)\delta r$$

eliminate δP from the 2 expressions:

$$\Rightarrow \frac{\gamma P\,\delta\rho}{\rho} = -\frac{dP}{dr}\delta r$$

giving the change in density for the mass element

$$\Rightarrow \delta\rho = \left(\frac{\rho}{\gamma P}\right)\left(-\frac{dP}{dr}\right)\delta r$$

Condition for Convective Instability

mass element will continue to rise if less dense than new surroundings and convection will occur

$$\rho - \delta\rho < \rho + \left(\frac{d\rho}{dr}\right)\delta r$$

substitute in expression for $\delta\rho$ from last slide

$$\rho - \left(\frac{\rho}{\gamma P}\right)\left(-\frac{dP}{dr}\right)\delta r < \rho + \left(\frac{d\rho}{dr}\right)\delta r$$

$$\Rightarrow \left(\frac{\rho}{\gamma P}\right)\left(\frac{dP}{dr}\right) < \frac{d\rho}{dr}$$

divide both sides by dP/dr to give condition for convection to take place

$$\frac{\rho}{\gamma P} < \frac{d\rho}{dP} \quad \text{or} \quad \frac{1}{\gamma} < \frac{P}{\rho}\frac{d\rho}{dP}$$

Condition for Convective Instability

Examine form of the condition for convective instability for a perfect gas

$$P = nkT = \frac{\rho}{\mu m_H} kT$$

$$\Rightarrow \log P = \log \rho + \log T + const$$

differentiate to give

$$\Rightarrow \frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

substitute in relation for convective instability on last slide

$$\frac{1}{\gamma} < \frac{P}{dP}\left(\frac{dP}{P} - \frac{dT}{T}\right)$$

rearrange

$$\Rightarrow \frac{1}{\gamma} < 1 - \frac{P}{T}\frac{dT}{dP}$$

Condition for Convective Instability

One final rearrangement gives:

$$\frac{\gamma - 1}{\gamma} > \frac{P}{T}\frac{dT}{dP}$$

which may be re-written in terms of the temperature and pressure gradients in the star

$$\left|\frac{dT}{dr}\right| < \frac{\gamma - 1}{\gamma}\frac{T}{P}\left|\frac{dP}{dr}\right|$$

For a monatomic gas, $\gamma = 5/3$

We have derived an expression for the condition necessary for energy transport via convection to be important

Stellar Structure Equations with Convection

When energy transport via convection must replace equation for the temperature gradient using new relation for convective instability

$$\left|\frac{dT}{dr}\right| < \frac{\gamma - 1}{\gamma}\frac{T}{P}\left|\frac{dP}{dr}\right|$$

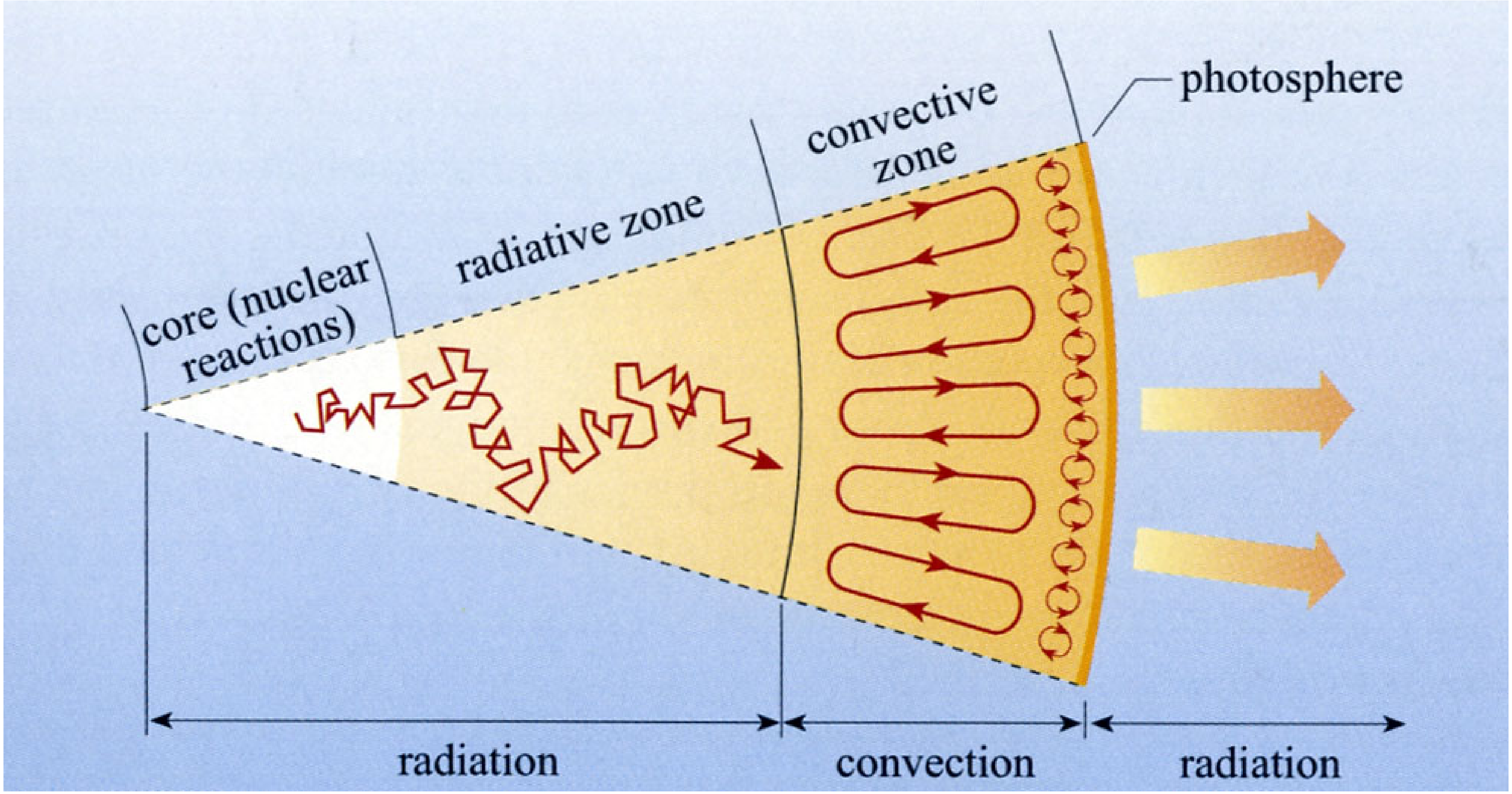
Convection is very efficient and setting the two sides equal provides a good approximation. Substitute for *P* assuming a perfect gas

$$\frac{dT}{dr} = \frac{\gamma - 1}{\gamma}\frac{\mu m_H}{k\rho}\frac{dP}{dr}$$

Then use hydrostatic equilibrium equation to substitute for *dP/dr*

$$\frac{dT}{dr} = -\frac{\gamma - 1}{\gamma}\frac{\mu m_H}{k}\frac{Gm}{r^2}$$

Energy Transport in the Sun



Lecture 12: Summary

- Eddington Limit provides a constraint on the stability of star with energy transport via radiative diffusion
- Application of Eddington Limit at surface of a star produces important concept of Eddington Luminosity, setting a limit to the luminosity of an object as a function of mass
- Three processes can enable transport of energy – radiative diffusion, conduction and convection
- Can derive condition on the size of the temperature gradient necessary for convective instability with the transport of energy dominated by convection
- Replace equation of stellar structure derived assuming radiative diffusion with expression for the temperature gradient derived for case where convection dominates
- Examples of where convection will be important include the cores of massive stars and in the outer regions of stars like the Sun

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