Structure and Evolution of Stars
Lecture 10: Thermonuclear Reactions

- Thermonuclear fusion and liberation of energy
- Classical and Quantum Mechanical estimates of the temperature to initiate nuclear fusion
- Reaction rate for fusion reactions
  - Cross-section for interactions
  - Gamow Peak
- Star of the Week #2: Identification of the Progenitor Star of a Type II core-collapse supernova

Thermonuclear Reactions

• Binding energy per nucleon – increases to peak at iron

• Fusing together light nuclei to form nucleus closer to the iron-peak results in the release of energy

• Energy released given by Einstein's formula $E=mc^2$, where "m"$=\Delta m$ the difference in mass between the constituent light nuclei and the nucleus of the new element

• Nuclei are positively charged and in order to fuse, necessary to overcome the **coulomb barrier** in order for protons and neutrons to be close enough that the attractive **strong nuclear force** dominates

• In classical physics, coulomb barrier is formidable

Binding Energy per Nucleon

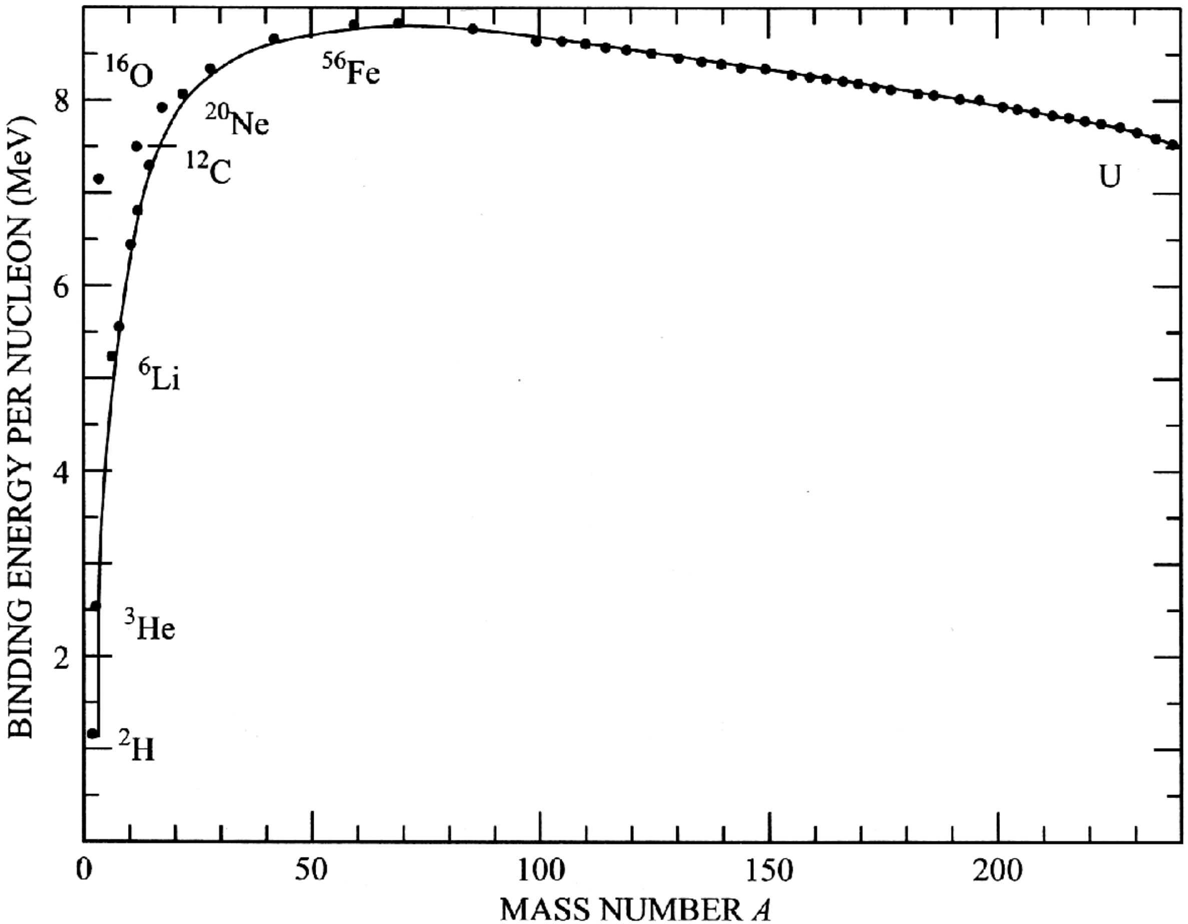


Fig. 1.3 Binding energy per nucleon for atomic nuclei. There is a broad maximum at mass number 56 which implies that energy is normally released when two light nuclei fuse to form a heavier nucleus provided the nucleus formed has a mass number less than 56

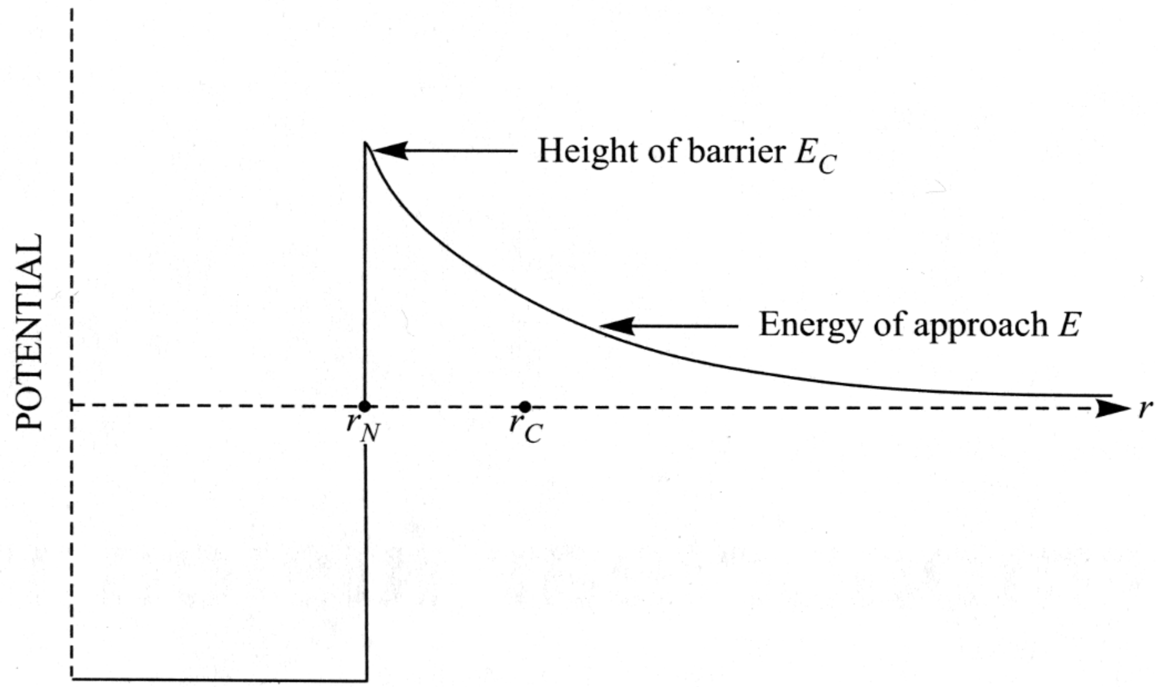


Fig. 4.1 A representation of the Coulomb and nuclear potentials between two nuclei of charge Z_A and Z_B. The distance r_C is the classical distance of closest approach for nuclei with an energy of approach equal to E. The distance r_N represents the range of short-range nuclear forces. E_C is the height of the Coulomb barrier keeping the nuclei apart

Temperature Required to Overcome Coulomb Barrier

Estimate T required to initiate fusion by equating KE of nuclei with the Coulomb barrier energy. Using the reduced mass, μ, of the particles

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Where Z is the number of protons, e the charge of the electron and r the separation

$$\frac{1}{2}\mu v^2 = \frac{3}{2}kT = \frac{Z_1 Z_2 e^2}{r}$$

Leads to estimate of T some 3-orders of magnitude larger than our estimate in Lecture 6!

$$\Rightarrow T = \frac{2}{3}\frac{Z_1 Z_2 e^2}{kr} \approx 10^{10}\,\mathrm{K}$$

Quantum Tunneling

• In QM, once a particle comes within a radius of another that is small enough for the uncertainty principle to be important then there is a small, but finite, probability the Coulomb barrier will be penetrated

• The scale, *r*, is set by the de Broglie wavelength, λ, of the particle $\lambda = h/p$ where *h* is Planck's constant and *p* is the particle momentum

• Happens when:

$$T = \frac{2}{3}\frac{Z_1 Z_2 e^2}{k\lambda}$$

• Equate KE of particles with height of the Coulomb barrier to produce KE=0, i.e. particles can (classically) get no closer

• KE of well-separated particles:

$$\frac{1}{2}\mu v^2 = \frac{p^2}{2\mu}$$

Quantum Tunneling

Particle KE equal to Coulomb barrier energy at separation λ:

$$kT = \frac{Z_1 Z_2 e^2}{\lambda} = \frac{p^2}{2\mu} = \frac{(h/\lambda)^2}{2\mu}$$

$$\Rightarrow Z_1 Z_2 e^2 = \frac{h^2}{2\mu\lambda}$$

$$\Rightarrow \lambda = \frac{h^2}{2\mu Z_1 Z_2 e^2}$$

Substitute expression for λ in expression for *T* on last slide gives:

$$T = \frac{2}{3} \frac{Z_1 Z_2 e^2}{k} \frac{2\mu Z_1 Z_2 e^2}{h^2} = \frac{4}{3} \frac{\mu Z_1{}^2 Z_2{}^2 e^4}{kh^2} \approx 10^7\,\mathrm{K}$$

with numerical value for protons, i.e. Hydrogen

Reaction Rate for Fusion

Number density of nuclei n_i and n_j with energy of encounters $=\mu v(E)^2/2$

Cross-section for encounters is $\sigma(E)$

Nucleus sweeps out an interaction volume:

$$\sigma(E)v(E)dt$$

where $v(E)=(2E/\mu)^{1/2}$ is the velocity.

Number of encounters:

$$n_j\sigma(E)v(E)dt$$

If fractional number of nuclei with energies between E and $E+dE$ is $f_E dE$, rate of encounters as a function of E:

$$\frac{dN_E}{dt} = n_j\sigma(E)v(E)\,f_E dE$$

Include all nuclei, n_i, and integrate over all energies to calculate the reaction rate per unit volume:

$$r_{ij} = \int_0^\infty n_i n_j \sigma(E)v(E)\,f_E dE$$

Reaction Rate for Fusion

We have the velocity in terms of the particle energies already and the distribution of particle energies is given by the Maxwell-Boltzmann distribution:

$$f_E dE = \frac{2}{\pi^{1/2}} \frac{1}{(kT)^{3/2}} E^{1/2} e^{-E/kT} dE$$

Remaining requirement to understand behaviour of the reaction rate is to calculate the energy-dependence of the cross-section *σ(E)*

Three factors are important: the de Broglie wavelength of the nuclei; the probability for quantum tunneling and a compendium of additional physical processes

Reaction Rate for Fusion

Dependence of the cross-section on the de Broglie wavelength λ

$$\sigma(E) \propto \pi\lambda^2 \propto \left(\frac{h}{p}\right)^2$$

$$\mathrm{KE} = \frac{1}{2}\mu v^2 = \left(\frac{p^2}{2\mu}\right)$$

Simple inverse dependence on the energy of the particles

$$\Rightarrow \sigma(E) \propto \frac{1}{E}$$

Tunneling Probability

Can be shown that:

$$\sigma(E) \propto \exp(-2\pi^2 u_c / E)$$

where u_c is the energy of the Coulomb barrier and for $r=\lambda=h/p$ can write:

$$\frac{u_c}{E} = \frac{Z_1 Z_2 e^2 / r}{\mu v^2 / 2} = \frac{2 Z_1 Z_2 e^2}{h v}$$

Thus

$$\frac{2\pi^2 u_c}{E} = \frac{2\pi^2}{E^{1/2}} E^{1/2} \frac{u_c}{E}$$

$$= \frac{2\pi^2}{E^{1/2}} \frac{\mu^{1/2} v}{2^{1/2}} \frac{2 Z_1 Z_2 e^2}{h v}$$

leading to a simple dependence on energy alone plus product of many constants

$$= \left[\frac{2^{3/2} \pi^2 \mu^{1/2} Z_1 Z_2 e^2}{h} \right] \frac{1}{E^{1/2}}$$

Tunneling Probability

Define:
to include all the constants

$$b = \frac{2^{3/2}\pi^2\mu^{1/2}Z_1Z_2e^2}{h}$$

Can then write:

$$\sigma(E) \propto \exp(-bE^{-1/2})$$

Producing a particularly simple dependence of the cross-section on the energy of the interacting nuclei to describe the importance of quantum tunneling.

The "constant" *b*, subsumes all the dependence on the composition of the material

Additional Factors

• Additional physical factors affect the cross-section and hence the reaction rate:

Nuclear resonance: when energy of colliding particles is close to the energy ΔE between two internal nuclear energy levels the cross-section is increased substantially

Electron screening: the many free electrons combine to reduce the effective positive charge of the nuclei, allowing the nuclei to get slightly closer, enhancing the reaction rate

Angular momenta: of the nuclei also has a modest effect on the reaction rate

Reaction Rate for Fusion

Putting the dependences of the three contributions to the cross-section – de Broglie wavelength, quantum tunneling, miscellaneous effects gives:

$$\sigma(E) = S(E)\frac{1}{E}\exp\left[-bE^{-1/2}\right]$$

Where have combined constants and miscellaneous effects into the first term. Substitute expression for cross-section into equation for the reaction-rate (bottom of Slide 8) with velocity and Maxwell-Boltzmann distribution (Slide 9) gives:

$$r_{ij} = \left(\frac{2}{kT}\right)^{3/2}\frac{n_i n_j}{(\mu\pi)^{1/2}}\int_0^\infty S(E)\exp[-bE^{-1/2}]\exp[-E/kT]dE$$

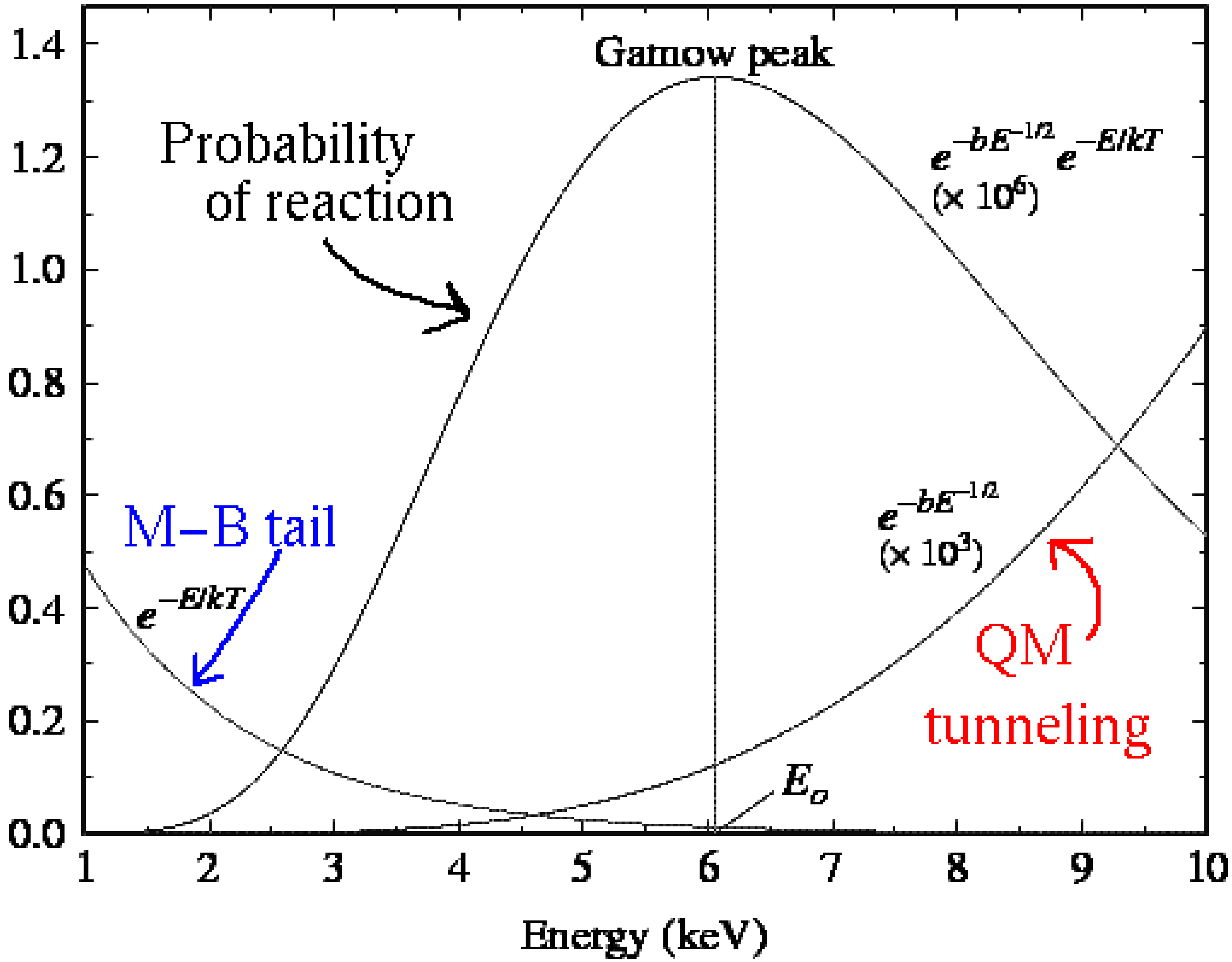
Reaction Rate for Fusion

Tunneling probability term: $$\exp[-bE^{-1/2}]$$

Maxwell-Boltzmann term: $$\exp[-E/kT]$$

Combine to give strongly peaked distribution with peak at: $$E_0 = (bkT/2)^{2/3}$$

Most reactions take place at the **Gamow peak**



Gamow peak
Probability of reaction
$e^{-bE^{-1/2}} e^{-E/kT}$ ($\times 10^6$)
M−B tail
$e^{-E/kT}$
$e^{-bE^{-1/2}}$ ($\times 10^3$)
QM tunneling
E_o
Energy (keV)
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
1
2
3
4
5
6
7
8
9
10

(Ex)-Star of the Week: #2

- Identifying the progenitor star of a supernova

- Smartt, Maund, Hendry, Tout, Gilmore…

  2004, *Science*, 303, 499; Maund et al. 2005, MNRAS, 364, L33 and now many more searches, e.g. Cau et al. 2013, ApJ, 775, L7, summary Smartt 2015, PASA, 32, 16

- Key test of stellar evolution models for short-lived high-mass stars would be to identify exact nature of star when supernova occurs

- Supernovae in Galaxy – how often?

(Ex)-Star of the Week: #2

- SN2003gd in M74 – a type II, core-collapse supernova (SN), expected to result from a ~10 Solar mass progenitor in the red-supergiant phase of evolution
- Want L, T and mass to compare to stellar models
- Utilise deep pre-explosion images of region where SN occurs
- Hubble Space Telescope images key as angular resolution (~0,05”) still far superior to typical ground-based images and SN likely to occur in crowded regions
- Probability of having observed star-forming regions in nearby galaxies where supernovae expected to occur?

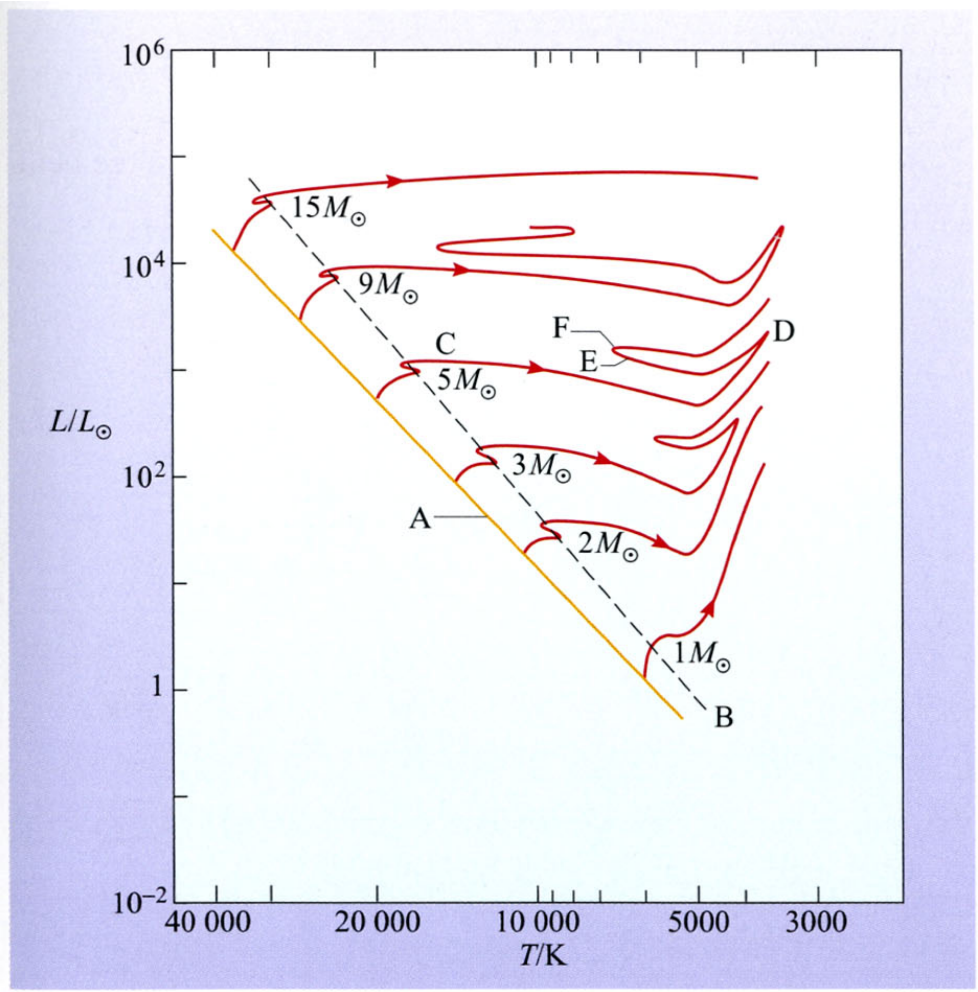


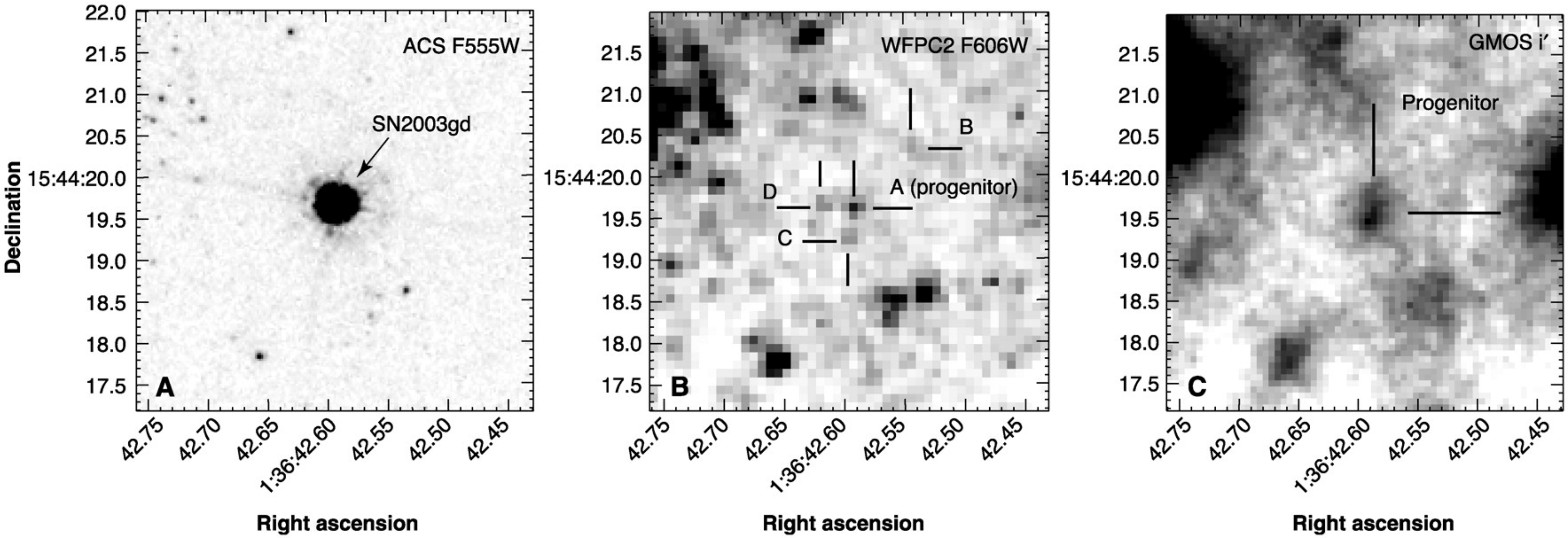
Figure 7.2 The predicted paths of stars on the H–R diagram as they evolve off the main sequence to the red giant (or supergiant) phase. The letters on the $5M_\odot$ track refer to different stages of nuclear reactions in the star. The line marked A denotes the onset of hydrogen core fusion – the start of main sequence life. The dashed line B denotes the cessation of hydrogen core fusion – the end of main sequence life, and the onset of hydrogen shell fusion. Subsequent stages are labelled on the $5M_\odot$ track only: (C) hydrogen shell fusion continues; (D) helium core fusion starts; (E) helium core fusion continues; (F) helium shell fusion starts. The small loops to the left for the $1M_\odot$ and $2M_\odot$ stars have been omitted for clarity.

Spiral Galaxy NGC 7424
(VLT MELIPAL + VIMOS)

ESO PR Photo 33c/04 (1December 2004)

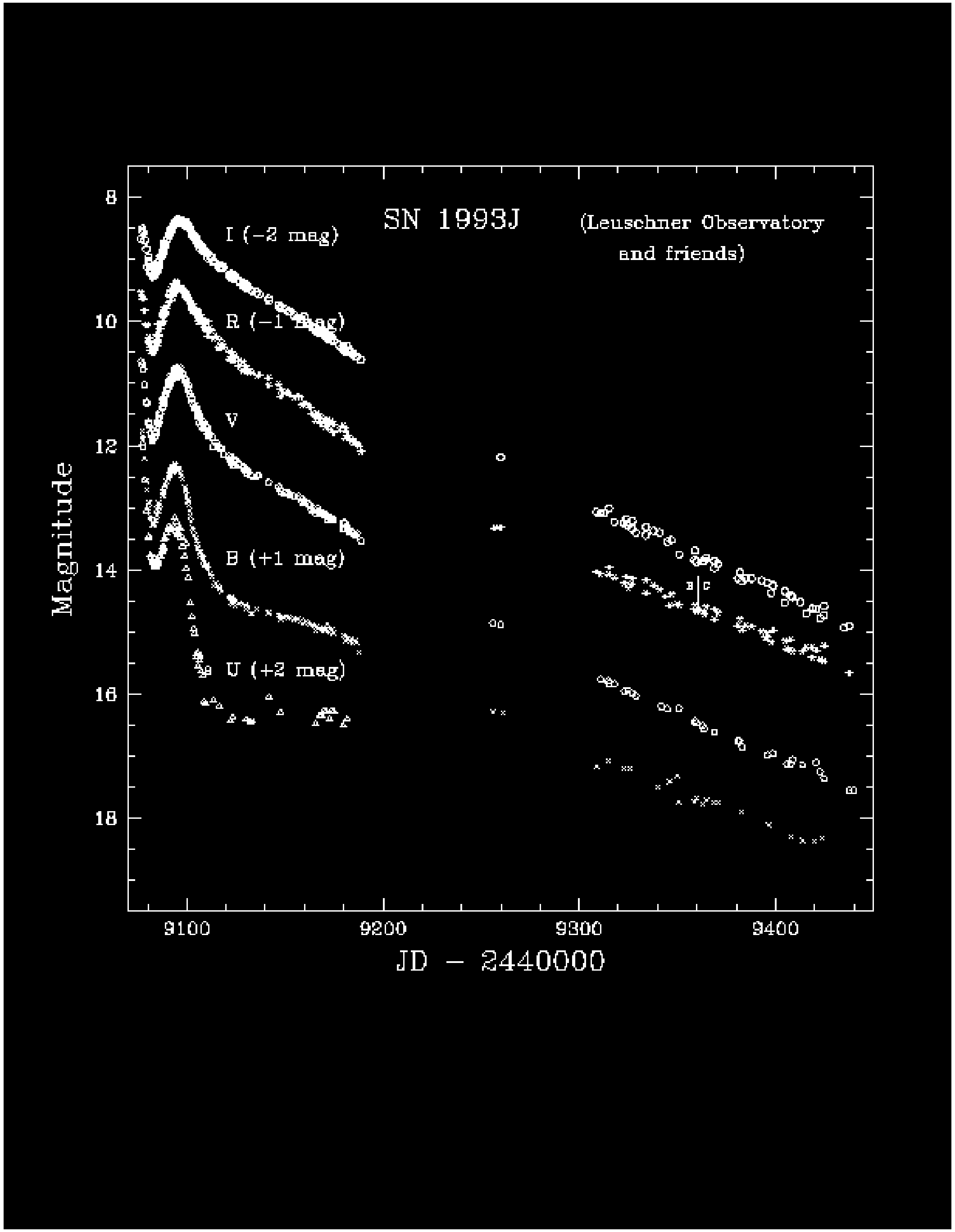
Spiral Galaxy NGC 6118 and SN 2004dk
(VLT MELIPAL + VIMOS)

ESO PR Photo 33b/04 (1 December 2004)



Images, 5 arcseconds on a side, of the supernovae (HST) in the V-band at left (V=17.6 for SN). Pre-SN HST image at centre, ground-based pre-SN image

Note “composite”image of A+C+D in ground-based image


SN 1993J
(Leuschner Observatory
and friends)
I (−2 mag)
R (−1 mag)
V
B (+1 mag)
U (+2 mag)
Magnitude
JD − 2440000
8
10
12
14
16
18
9100
9200
9300
9400

(Ex)-Star of the Week: #2

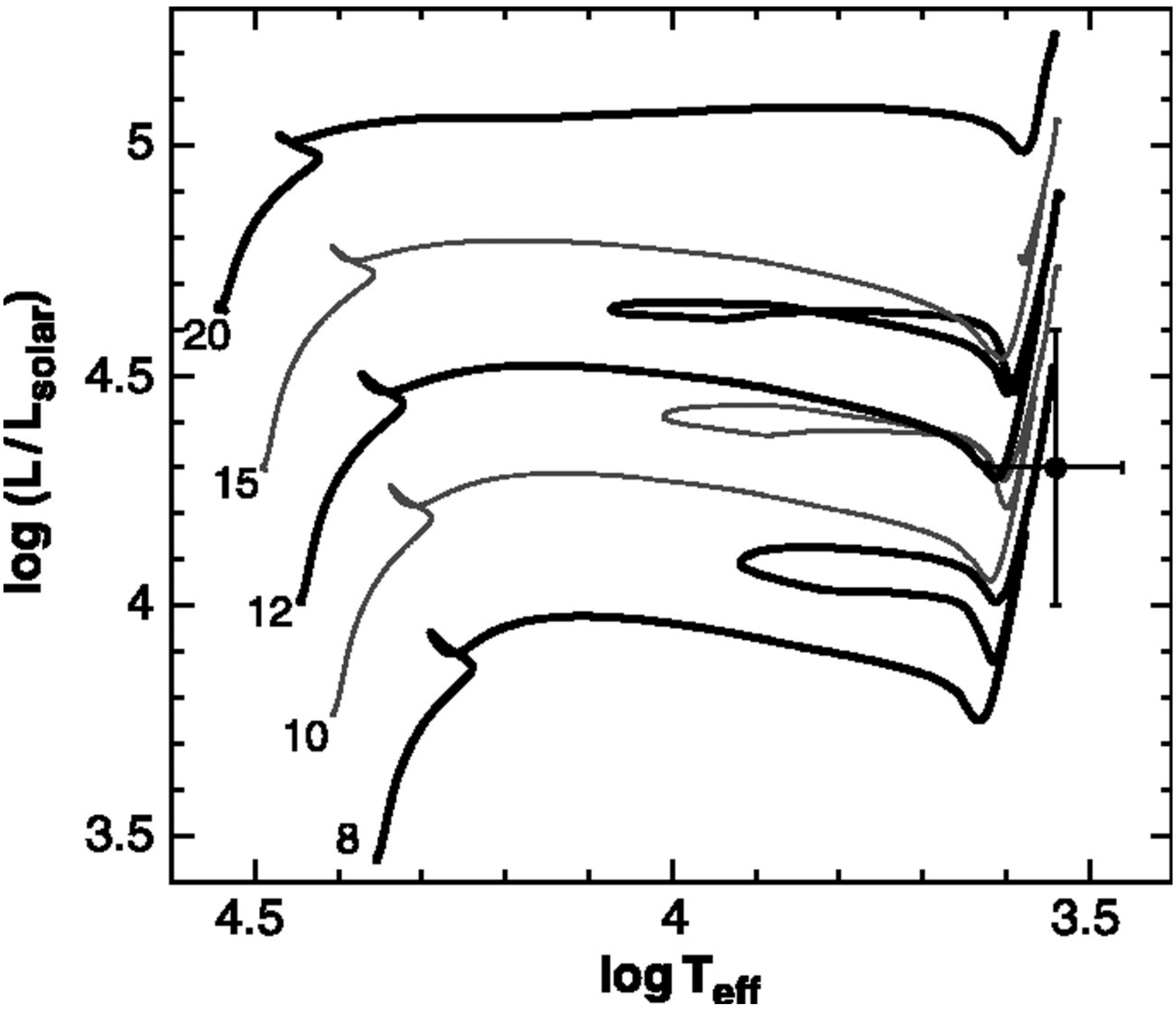
Distance to M74 has to be estimated via standard candles

Brightest supergiant stars	7.5±2.8 Mpc
HII regions	10.2±3.4 Mpc
SN itself	9.7±1.9 Mpc

Results in mean distance of 9.1±1.9 Mpc – provides estimate of absolute magnitude (=Luminosity) in HR-diagram

Measure colour (B-V) of progenitor and hence deduce spectral-type or T_{eff} of the progenitor star

Principal uncertainty is the amount of dust and hence change in colour of the progenitor star

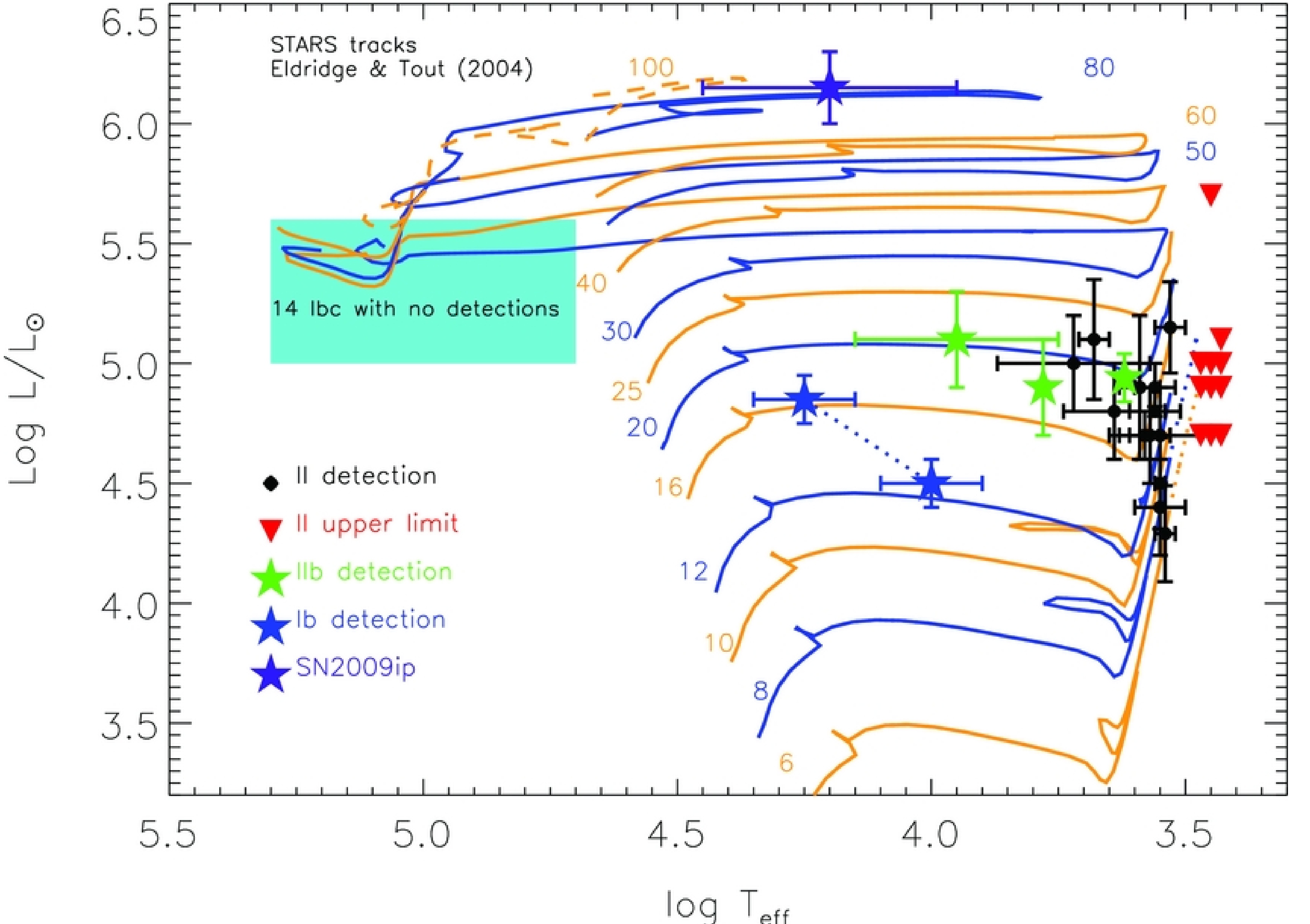

log (L / L_{solar})
5
4.5
4
3.5
20
15
12
10
8
4.5
4
3.5
log T_{eff}

(Ex)-Star of the Week: #2

• First detection of a Type II Supernova progenitor star, the most common supernova in the Universe

• A red supergiant with mass 8(+4/-2) Solar masses near the lower limit to the theoretical predictions for the progenitors of such objects

• Result is consistent with similar analyses of 2 other nearby Type II SN that produced limits of <12-15 Solar masses



S Smartt (2015, PASA, 32, 16 >30 progenitors)

Lecture 10: Summary

- Quantum mechanical considerations lead to the possibility of nuclear fusion at temperatures typical of the cores of stars
- The reaction rate depends on a combination of the energy distribution of the nuclei (Maxwell-Boltzmann distribution) and the interaction cross-section
- Interaction cross-section depends on the product of the dependence of the de Broglie wavelength and the tunneling probability as a function of energy
- Implication for the reaction rate is to produce a strongly peaked distribution as a function of particle energy – **Gamow Peak**
- Star of the Week – direct test of the evolutionary models for high mass stars

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