Structure and Evolution of Stars

Lecture 9: Opacity Sources and additional equations of stellar structure

- Opacity
  - Dependence on density, Temperature and composition
  - Rosseland mean opacity
  - Kramers opacity
- Thermal equilibrium
  - 3rd differential equation of stellar structure
- Simple radiative transfer
  - 4th differential equation of stellar structure
- Synopsis of where we are to date

Sources of Opacity

• In stars, the interaction between radiation and electrons are the most important contributors to the opacity

• *Electron scattering* – scattering of photons by free electrons without change of photon energy (Thomson scattering). At very high velocities, electron velocity close to *c*, Compton scattering (photon energy does change) also occurs but not usually important

$$\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 = 6.7\times 10^{-25}\,\mathrm{cm}^2$$

Thomson cross-section is in fact independent of photon frequency

Why is scattering from ions not important?

Sources of Opacity

• *Free-free absorption* – absorption of a photon by a free electron, mediated by short interaction with a nucleus or ion, increases the energy of an electron

• *Bound-free absorption* (photoionisation) – absorption of a photon by a bound electron removes the electron from the ion (or atom)

• *Bound-Bound absorption* – absorption of a photon by a bound electron results in transition of the electron into a higher energy state

Of the four main sources of opacity, which do you think will dominate in the cores of stars? and which are likely to be important in the outer atmospheres of stars?

Specifying Opacity

• To calculate opacity, need to allow for contribution from each radiation-matter process:

$$\kappa_\nu = \kappa_{\nu,es} + \kappa_{\nu,ff} + \kappa_{\nu,bf} + \kappa_{\nu,bb}$$

and often the value depends strongly on frequency (or wavelength). Need to allow for all possible interactions between photons and all elements in all states of ionisation. A major undertaking

• A degree of simplification possible for our purposes. In stellar interiors, material is close to fully ionised and $\kappa_{\nu,bf} \approx 0$ and $\kappa_{\nu,bb} \approx 0$

• Conventional to adopt power-law approximations for the opacity of the form:

$$\kappa = \kappa_0 \rho^a T^b$$

Electron Scattering Opacity

• Electron scattering depends on the Thomson cross-section for the electron, which is independent of the photon wavelength and also has no temperature or density dependence. Thus in the power-law approximation for opacity *a*=*b*=0 and the opacity depends only on the number of electrons per nucleon, $1/\mu_e$, in the material

$$\kappa_{es} = \frac{\kappa_{es,0}}{\mu_e} \approx \frac{1}{2}\kappa_{es,0}(1+X)$$

In terms of mass fraction of hydrogen, *X*

A is nuclear mass and *Z* is nuclear charge

$$\frac{1}{\mu_e} = \sum_j F_j \frac{Z_j}{A_j}$$

Weighted average of *Z*/*A* is ≈ 0.5

$$= X + \frac{1}{2}Y + (1 - X - Y)\left\langle \frac{Z}{A} \right\rangle \approx \frac{1}{2}(1+X)$$

Kramers Opacity

• Electron scattering provides a background opacity that is normally not a major contributor to the total opacity because the Thomson cross-section is small. However, at very high temperature electron scattering can become important

• Free-free opacity is dominant in many situations within the interiors of stars and Hendrik Kramers first showed that opacity due to free-free absorption can be well approximated by a power-law dependence on density and temperature

• Specifically, Kramers showed that a=1 and b=-7/2 (or -3.5) and

$$\kappa \propto \rho T^{-3.5}$$ is known as **Kramers opacity** law

• In fact the relationship also works well for lower T where bound-free opacity is important and Kramers opacity has wide applicability

Detailed calculations are required to determine behaviour of opacity as a function of frequency but it is usual to employ an average over frequency and such averages are termed **Rosseland mean** opacities

Then need to determine dependence on density, temperature and composition, e.g. Kramers opacity

Note power-law like behaviour of opacity as a function of *T* at *T*>10000K

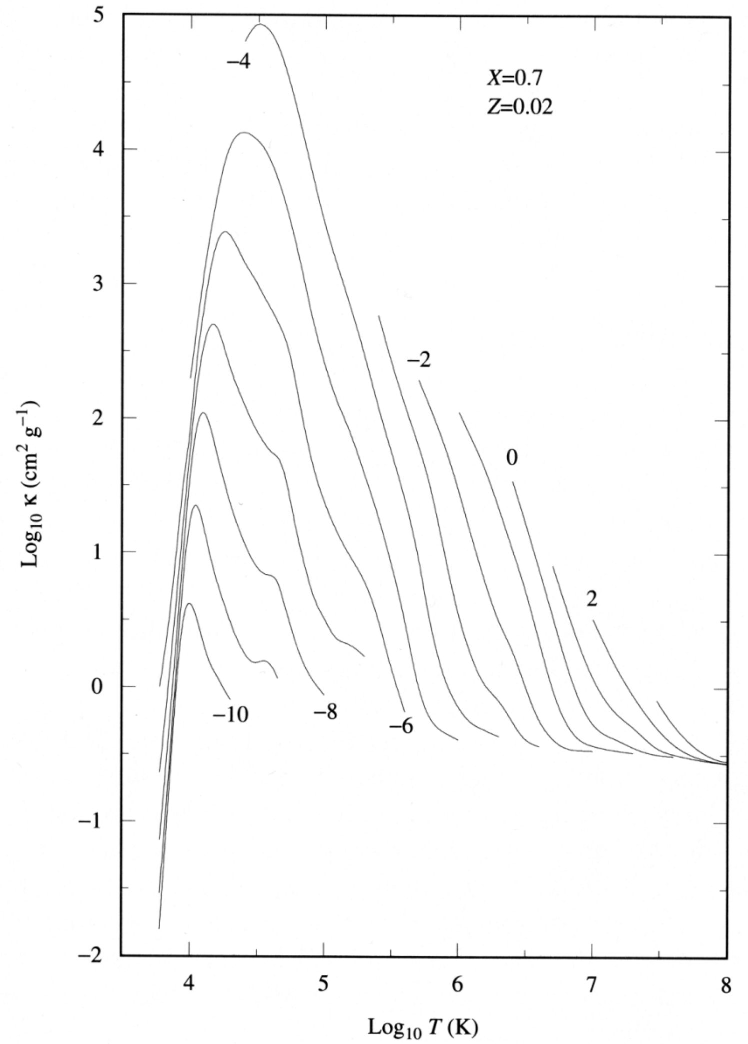
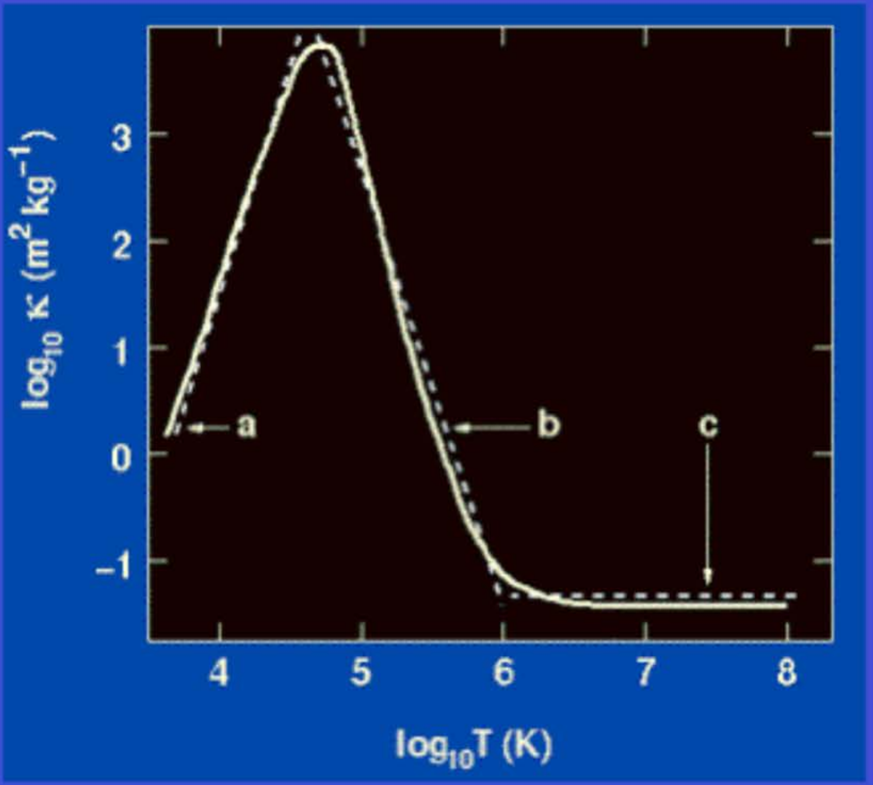


Figure 9.10 Rosseland mean opacity. The curves are labeled by the value of the density ($\log_{10}\rho$ in g cm^{-3}). (Data from Rogers and Iglesias, *Ap. J. Suppl.*, *79*, 507, 1992.)

Figure shows opacity as a function of temperature for a star of given ρ (10^{-1} kgm^{-3}). Solid curve is from detailed opacity calculations. Dotted lines are approximate power-law forms.



At high *T*: κ is low and remains constant. Most atoms fully ionised, high photon energy, hence free-free absorption unlikely, Dominant mechanism is electron scattering, independent of *T*, α=β=0

$$\kappa = \kappa_0 \text{ (curve c)}$$

Opacity is low a low T, and decreases with T. Most atoms not ionised, few electrons available to scatter photons or for free-free absorption. Approx analytical form is α=1/2 , β=4

$$\kappa = \kappa_0 \rho^{1/2} T^4 \quad \text{(curve a)}$$

At intermediate T, κ peaks, when bound-free and free-free absorption are very important, then decreases with T (Kramers opacity law)

$$\kappa = \kappa_0 \rho T^{-3.5} \quad \text{(curve b)}$$

Opacity: Summary

• In general the opacity has significant dependence on the frequency of the radiation. In practice, and always for this course, employ averages over frequency to give **Rosseland mean** opacities

• Parameterisation of the opacity in terms of powers of density and temperature provides a good approximation to the Rosseland mean opacities

$$\kappa = \kappa_0 \rho^a T^b$$

• At very high *T*, electron scattering dominates, with $a=b=0$

• **Kramers opacity**, with $a=1$ and $b=-3.5$, is applicable for a wide range of *T* where free-free and bound-free contributions dominate the total opacity

• At low temperatures, applicable in outer atmospheres, $a=0.5$ and $b=+4$ (note sign for *b*) is applicable

Examples of opacity and mean-free-path inside the Sun

Radius (Solar Units)	Opacity ($m^2\ kg^{-1}$)	Density ($kg\ m^{-2}$)	Mean-free-path (mm)
0.0	0.1	1.5×10^5	0.07
0.6	1.0	3.5×10^2	3.0
0.9	10.0	1.2×10^1	8.0

Values from Phillips

Check on Basic Assumptions

From our own calculations, given only the average opacity of material in the Sun, we can perform check on validity of the thermodynamic equilibrium assumption that we have made

Average opacity in Sun ≈0.1 $m^2 kg^{-1}$. Know average density is 1400 kg m^{-3} and thus the mean free path (mfp) is ≈0.07 m

Average drop in T over 1-mfp ≈ $T_{average}/R$ ≈ 0.001K (with $T_{average}$ from Virial Theorem in Lecture 6)

Demonstrates that thermodynamic equilibrium and blackbody form for radiation are both very good assumptions

Thermal Equilibrium

• Consider star in equilibrium with an internal energy generation mechanism, producing $\varepsilon(r)$ Joules per unit mass per unit time

• Equilibrium implies no change with time and straightforward to derive a continuity relation for the change in luminosity, L, as a function of radius without considering how energy is transported

Volume of shell at radius r is $4\pi r^2 dr$, with density ρ, the total energy produced within the shell is $4\pi r^2 dr\, \rho\, \varepsilon$ and the rate of change of luminosity with radius is:

$$\frac{dL}{dr} = 4\pi r^2 \rho \varepsilon$$

providing 3rd equation of stellar structure (already have Hydrostatic Equilibrium, and mass conservation from Lecture 6)

Radiative Diffusion

• If we integrate the equation to obtain the total luminosity as a function of radius, expect L to grow rapidly from r=0 and then to reach a maxium (equal to L_{star}) at radius where energy generation ceases
• Have argued that material and radiation within the interior of stars is in thermodynamic equilibrium and that the radiation has a blackbody spectrum
• Blackbody radiation is isotropic and the mean free path for radiation is small, so how does the radiation get from the interior of the star to the surface?
• For the case in which energy transport takes place via the diffusion of radiation can derive a relation between the luminosity, temperature and radius

Radiative Diffusion

• Consider the same slab of material we used to develop the ideas about opacity and the mean free path.

• The absorption of radiation by material in the slab means that momentum is transferred to the material

Momentum absorbed per unit time: $\frac{|dI|}{c}$

Momentum transfer is balanced by net force exerted by the radiation field on the inner and outer faces of slab:

$$P_{rad}(r) - P_{rad}(r+dr) = -\left(\frac{dP_{rad}}{dr}\right)dr$$

Therefore we have:

$$\frac{I\kappa\rho}{c} = -\frac{dP_{rad}}{dr}$$

Radiative Diffusion

Blackbody nature of radiation means radiation pressure is:

$$P_{rad} = \frac{1}{3} a T^4$$

Substitute in last equation on previous slide to give relationship between the radiative flux and *T*

$$\Rightarrow I = -\frac{4acT^3}{3\kappa\rho} \frac{dT}{dr}$$

Multiply by surface area of sphere at radius *r* to give total luminosity, *L*, at radius *r*:

Provides 4th equation of stellar structure relating *L*, *T* and the temperature gradient

$$\Rightarrow L = -4\pi r^2 \frac{4acT^3}{3\kappa\rho} \frac{dT}{dr}$$

Radiative Diffusion

Can invert the equation to specify the temperature gradient in terms of *L* and *T*

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{L}{4\pi r^2}$$

Can also use mass, *m*, as the independent space variable instead of the radius *r*

$$\frac{dT}{dm} = -\frac{3}{4ac}\frac{\kappa}{T^3}\frac{L}{(4\pi r^2)^2}$$

Stellar Structure: Where we are so far?

$$\frac{dP}{dr} = -\rho \frac{Gm}{r^2}$$ Hydrostatic Equilibrium

$$\frac{dm}{dr} = 4\pi r^2 \rho$$ Mass continuity

Have four differential equations

$$\frac{dT}{dr} = -\frac{3}{4ac} \frac{\kappa\rho}{T^3} \frac{L}{4\pi r^2}$$ Radiative transfer

$$\frac{dL}{dr} = 4\pi r^2 \rho \varepsilon$$ Thermal equilibrium

Stellar Structure: Where we are so far?

Also now know that *P* has 3 very different contributions

$$P = nkT + P_{e,\deg} + \frac{1}{3}aT^4$$

Rosseland mean opacity given by power-law relations of density and temperature

$$\kappa = \kappa_0 \rho^a T^b$$

Kramers opacity applicable over wide range of density and temperature

$$\kappa = \kappa_0 \rho T^{-3.5}$$

Stellar Structure: Where we are so far?

- However, other than appealing to the existence of an internal energy generation source, which, via our consideration of the characteristic timescales, has to be nuclear in origin, have said nothing about the nature of the energy generation rate per unit mass, $\varepsilon(r)$

- Also, we have considered only the case where the mechanism for transport of energy from the interior to the surface of a star is mediated by diffusion of photons. What other mechanisms can result in the transfer of energy from the interior to the surface of a star?

Lecture 9: Summary

- Opacity – have looked at the dependence on density, temperature and composition. Although opacity is a function of frequency, will use Rosseland mean opacity and find that a particular form, Kramers opacity, is applicable of wide range of ρ and T
- Condition for overall thermal equilibrium provides the 3rd differential equation of stellar structure
- Consideration of simple radiative transfer provides 4th differential equation of stellar structure
- To date we have assembled four differential equations that a model of a star must satisfy. Also have outlined importance of perfect gas, radiation and degeneracy to the total pressure
- Next, must consider origin of energy generation within stars

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