Structure and Evolution of Stars

Lecture 8: The Equation of State (cont.) and the Opacity of material

- Degeneracy Pressure
  - Quantum mechanical origin
  - Importance of electrons
  - Non-relativistic case
  - Relativistic case
- The log ρ – log T plane
  - Regimes where gas, radiation and degeneracy pressure dominate
- Opacity of material and mean free path
  - Definitions

Degeneracy Pressure

Heisenberg's Uncertainty Principle constrains the location of a particle within volume and momentum phase space (6 dimensions; 3 spatial and 3 momentum):

$$\Delta V \Delta p \geq h^3$$

Pauli's Exclusion Principle states that no 2 electrons can occupy the same quantum state. Electron has only 2 spin states – *up* and *down* – and therefore, only 2 electrons can occupy each element of phase space as set by Heisenberg's Uncertainty Principle, i.e. single electron cannot occupy a volume smaller than:

$$h^3 / 2$$

If the density becomes high enough, ΔV small, the resulting **degeneracy pressure,** as electrons are forced into higher momentum states, can be important, dominating the classical gas pressure

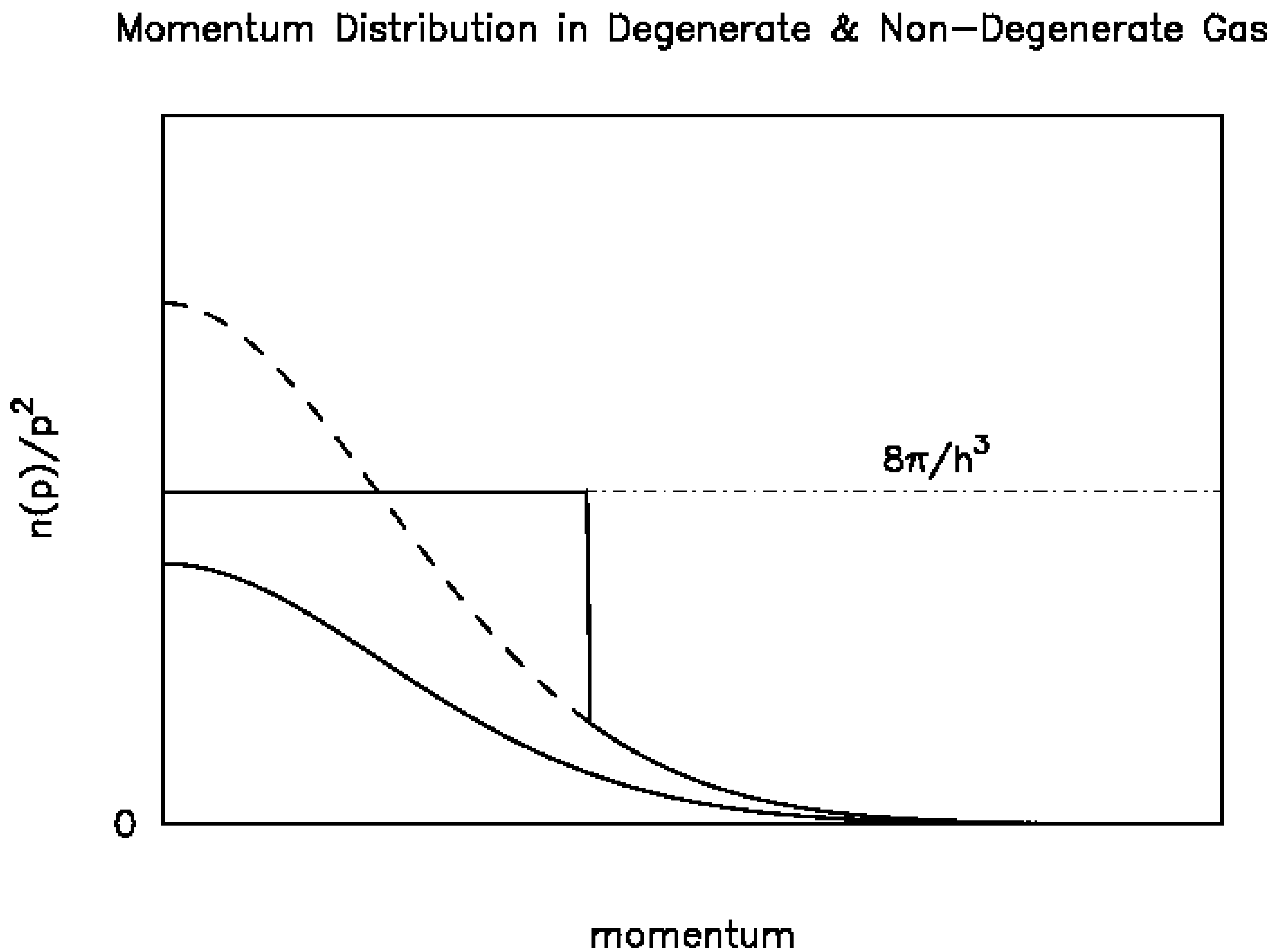
Degeneracy Pressure

• Perfect gas, with $P=nkT$ and particles have Maxwellian velocity distribution, which determines the distribution of particle momenta, dependent on T only

• Now increase density, shrinking volume available per particle, $\Delta V \alpha \rho^{-1}$, and, for electrons, Heisenberg's Uncertainty Principle leads to an increase in particle momenta that exceeds that due to gas temperature alone

• Pressure is higher than that inferred from temperature – and **degeneracy pressure** is important

• *Complete degeneracy* occurs when all available momentum states up to some maximum momentum are occupied and the HUP gives an equality $\Delta V \Delta p = h^3$, minimising $\Delta V \Delta p$. Complete degeneracy only occurs at zero temperature but in practice assuming complete degeneracy provides a very good approximation for calculation of degeneracy pressure


Momentum Distribution in Degenerate & Non-Degenerate Gas
n(p)/p^2
8π/h^3
0
momentum

Degeneracy Pressure

Volume of shell in phase-space is: $4\pi V p^2 dp$

#particles with momentum $p \rightarrow p+dp$ per unit volume:

$$n_e(p)dp = \frac{2}{\Delta V} = \frac{2}{h^3} 4\pi p^2 dp \quad p \le p_0$$

Find maximum momentum p_0 by integrating:

$$n_e = \int_0^{p_0} n_e(p)dp = \frac{8\pi}{3h^3} p_0{}^3$$

and rearranging

$$\Rightarrow p_0 = \left(\frac{3h^3 n_e}{8\pi} \right)^{1/3}$$

Degeneracy Pressure

Calculate pressure using the pressure integral with expression for *n(p)dp* from previous slide

$$P = \frac{1}{3}\int_0^\infty vpn(p)dp; \quad v = \frac{p}{m_e}$$

$$P_{e,\deg} = \frac{8\pi}{3m_e h^3}\int_0^{p_0} p^4 dp; \quad \mu_e = \frac{\rho}{n_e m_H}$$

Integrate and then substitute in expression for p_0 from previous slide

$$P_{e,\deg} = \frac{8\pi}{15 m_e h^3} p_0^{\;5} = \frac{h^2}{20 m_e}\left(\frac{3}{\pi}\right)^{2/3} \frac{1}{m_H^{5/3}}\left(\frac{\rho}{\mu_e}\right)^{5/3}$$

[You should work out the value of the constant]

$$P_{e,\deg} = K_1^{'}\left(\frac{\rho}{\mu_e}\right)^{5/3}; \text{ for fixed } \mu_e \;\; P_{e,\deg} = K_1 \rho^{5/3}$$

Why are electrons so much more important than nuclei?

Degeneracy Pressure

As density increases, p_0 increases until $p_0/m_e \approx c$ and relativistic effects become important

For limiting case, where $v=c$, replace v by c in the pressure integral and repeat analysis just as for the non relativistic case

$$P_{e,\deg} = \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3} \frac{1}{m_H^{4/3}}\left(\frac{\rho}{\mu_e}\right)^{4/3} \quad ; \quad P_{e,\deg} = K_2'\left(\frac{\rho}{\mu_e}\right)^{4/3}$$

Key change in the relativistic case is that the dependence of the pressure on the density has decreased from 5/3 to 4/3

The log ρ – log T Plane

For a given combination of density and Temperature, consider which form of the equation of state dominates – gas, radiation or degeneracy pressure

$$P_{gas} = \frac{\rho}{\mu m_H} kT \quad \text{(gas)}$$

$$P_{rad} = \frac{1}{3} a T^4 \quad \text{(radiation)}$$

Determine boundaries of regions in the log ρ – log T plane where particular equation of state appropriate by demanding equality (i.e. equal contributions to the pressure) between the relations

$$P_{e,\deg} = K_1 \rho^{5/3} \text{ (degeneracy)}$$

$$P_{e,\deg} = K_2 \rho^{4/3} \quad \text{(degeneracy} - v \approx c)$$

The log ρ – log T Plane

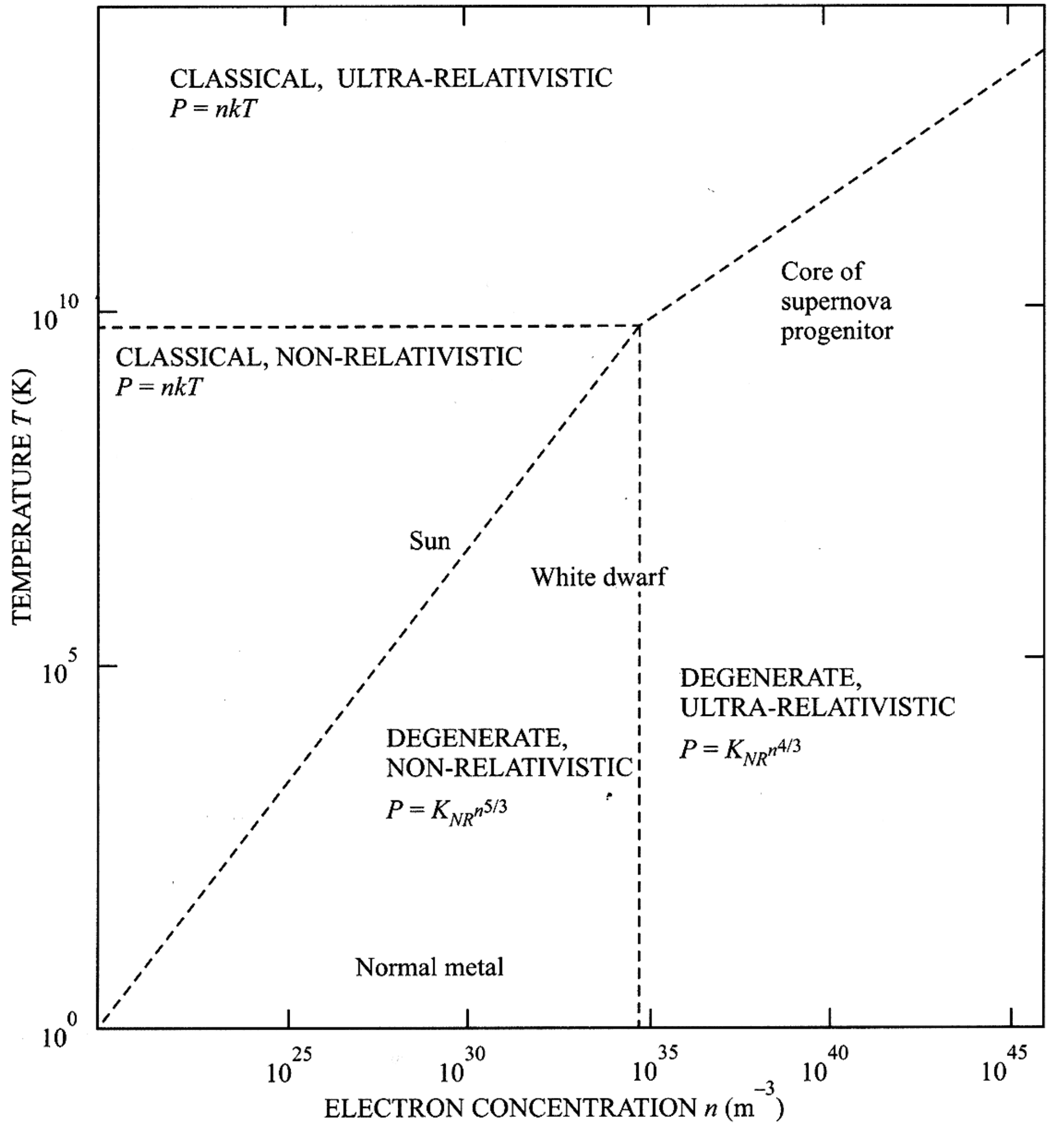


Fig. 2.2 Equation of state regimes for an ideal electron gas at a temperature T and at a density of n electrons per cubic metre. Typical values are shown for the temperature and density for electrons in a normal metal, in the sun, in a white dwarf and in the iron core of an evolved star just prior to a supernova

The log ρ – log T Plane

$$P_{rad} = P_{gas} \quad \Rightarrow \frac{1}{3} a T^4 = \frac{\rho}{\mu m_H} kT \quad \Rightarrow \mathrm{T} \propto \rho^{1/3}$$

above this line radiation pressure dominates

$$P_{e,\deg} = P_{gas} \quad \Rightarrow K_1 \rho^{5/3} = \frac{\rho}{\mu m_H} kT \quad \Rightarrow T \propto \rho^{2/3}$$

above this line gas pressure dominates

$$P_{e,\deg} = P_{e,\deg,rel} \quad \Rightarrow K_1 \rho^{5/3} = K_2 \rho^{4/3} \quad \Rightarrow \rho = \mathrm{const}$$

vertical line at critical density, non-relativistic regime to the left and relativistic regime to the right

The log ρ – log T Plane

Finally:

$$P_{e,\deg,rel} = P_{gas} \quad \Rightarrow K_2 \rho^{4/3} = \frac{\rho}{\mu m_H} kT \quad \Rightarrow T \propto \rho^{1/3}$$

relevant at very high densities and temperatures. The cores of Type II Supernovae have an equation of state that lies close to this line in the log ρ – log T plane

Opacity

• Energy generated within the star via nuclear reactions – important to understand the microscopic processes that determine the interaction of radiation with matter

• **Opacity** – describes the resistance of material to the passage of radiation

• Consider a plane parallel slab of material, density ρ, thickness dr, with a radiation flux I (energy per unit area per unit time) incident on one face

• How much radiation reaches the other side of the slab? or, alternatively, how much of the radiation flux is absorbed by the slab?

Opacity

• Amount absorbed is directly proportional to the density of material and also directly proportional to the slab thickness

• Amount absorbed clearly depends on the incident flux and can write:

$$dI = -\kappa I \rho dr$$

where κ is the **opacity coefficient** with dimensions of area per unit mass ($m^2 kg^{-1}$)
κ depends on the properties of the material such as the composition and temperature. Integrate to give:

$$I = I_0 e^{-\kappa \rho r}$$

the radiation flux at distance r from a source producing I_0
The characteristic absorption length=$1/\kappa\rho$ is the **mean free path** of a photon

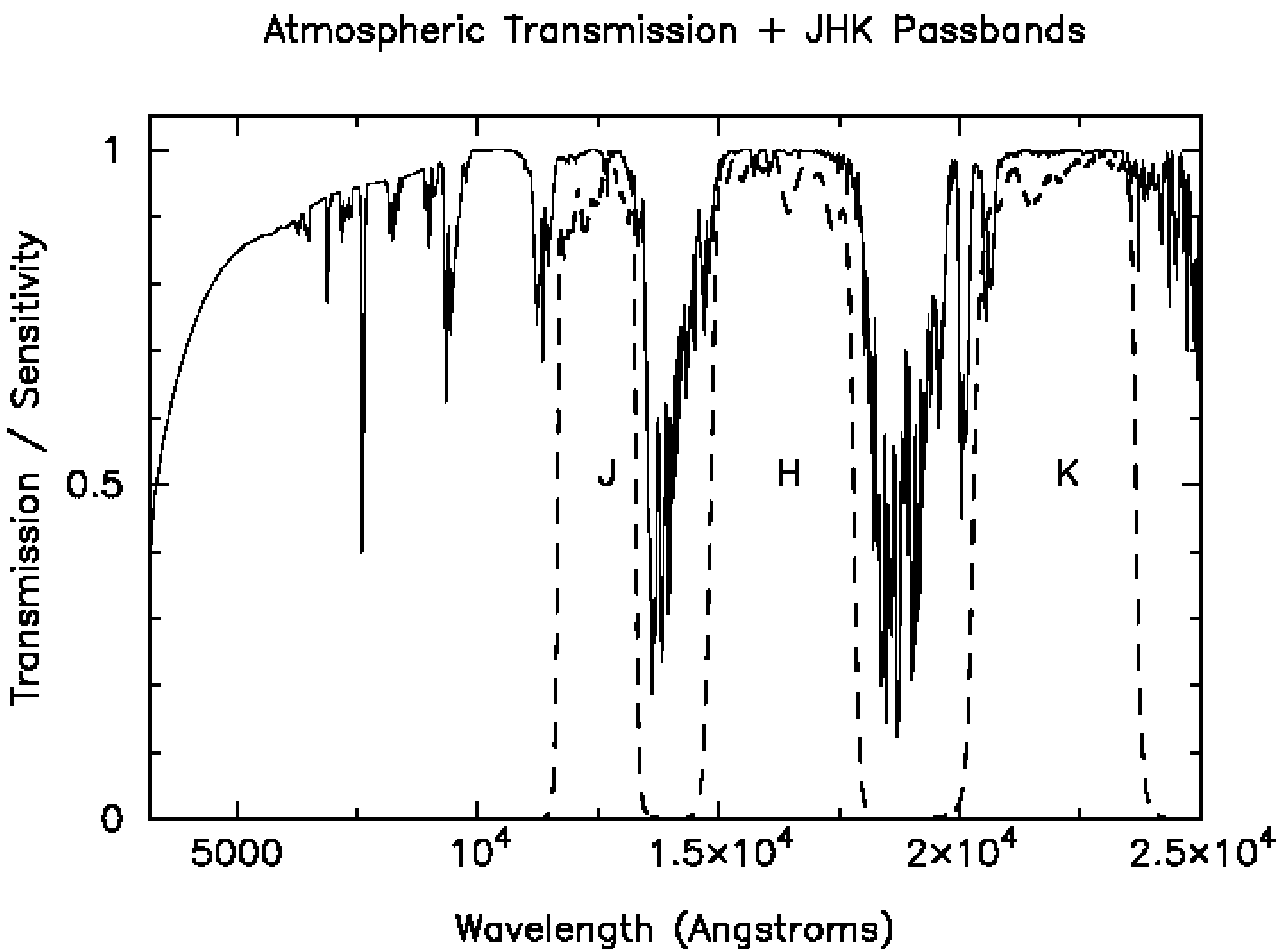
Opacity

• The dimensionless quantity τ, defined by $d\tau = -\kappa\rho dr$ is called the **optical depth**

• Optical depth is a measure of transparency of a medium to radiation and both opacity and optical depth are frequency (or equivalently wavelength) dependent

• Opaque material has high optical depth, $\tau >> 1$, due to combination of density, thickness and opacity.

•What about the atmosphere at optical, near-infrared and X-ray wavelengths?
•What defines the photosphere of the Sun?
•How does the radius of the Sun appear to change if we observe at 500nm, 656nm (Hα line), and with a neutrino telescope?


Atmospheric Transmission + JHK Passbands
Transmission / Sensitivity
1
0.5
0
J
H
K
5000
10^4
1.5×10^4
2×10^4
2.5×10^4
Wavelength (Angstroms)

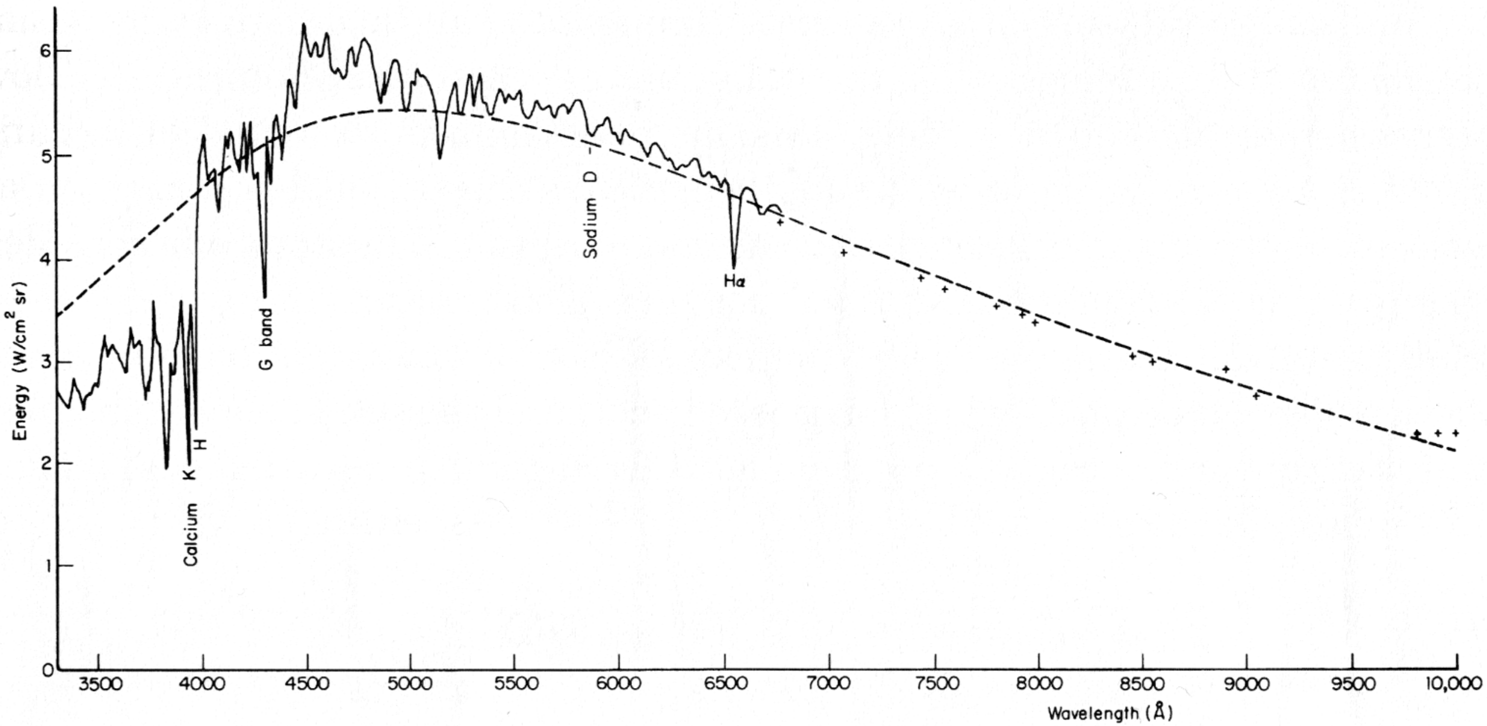


Figure 9.5 The spectrum of the Sun. The dashed line is the curve of an ideal blackbody having the Sun's effective temperature. (Figure from Aller, *Atoms, Stars, and Nebulae*, Revised Edition, Harvard University Press, Cambridge, MA, 1971.)

Limb Darkening

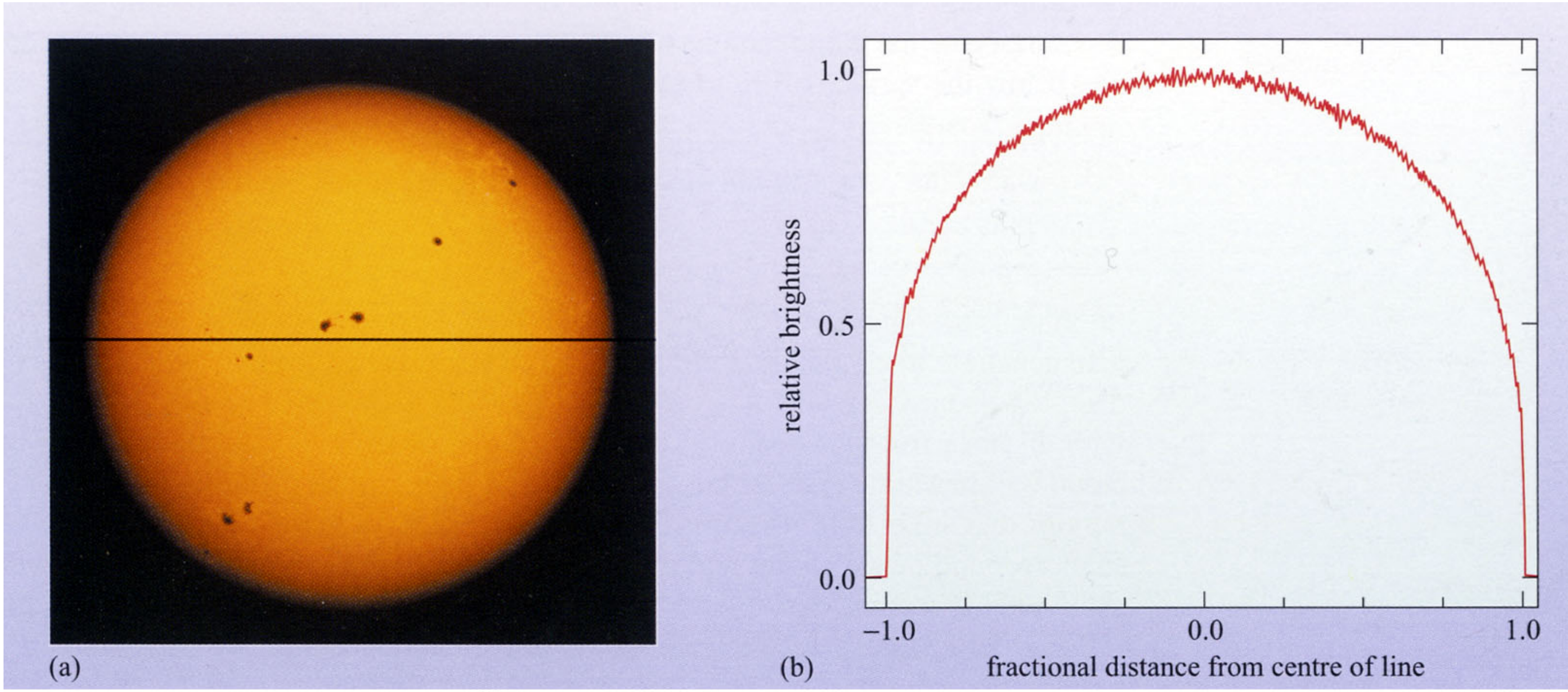


Figure 1.3 (a) The visible solar disc, crossed by a straight line. (b) The relative brightness of the photosphere at various points along the straight line shown in (a). ((a) NOAO; (b) Foukal, 1990)

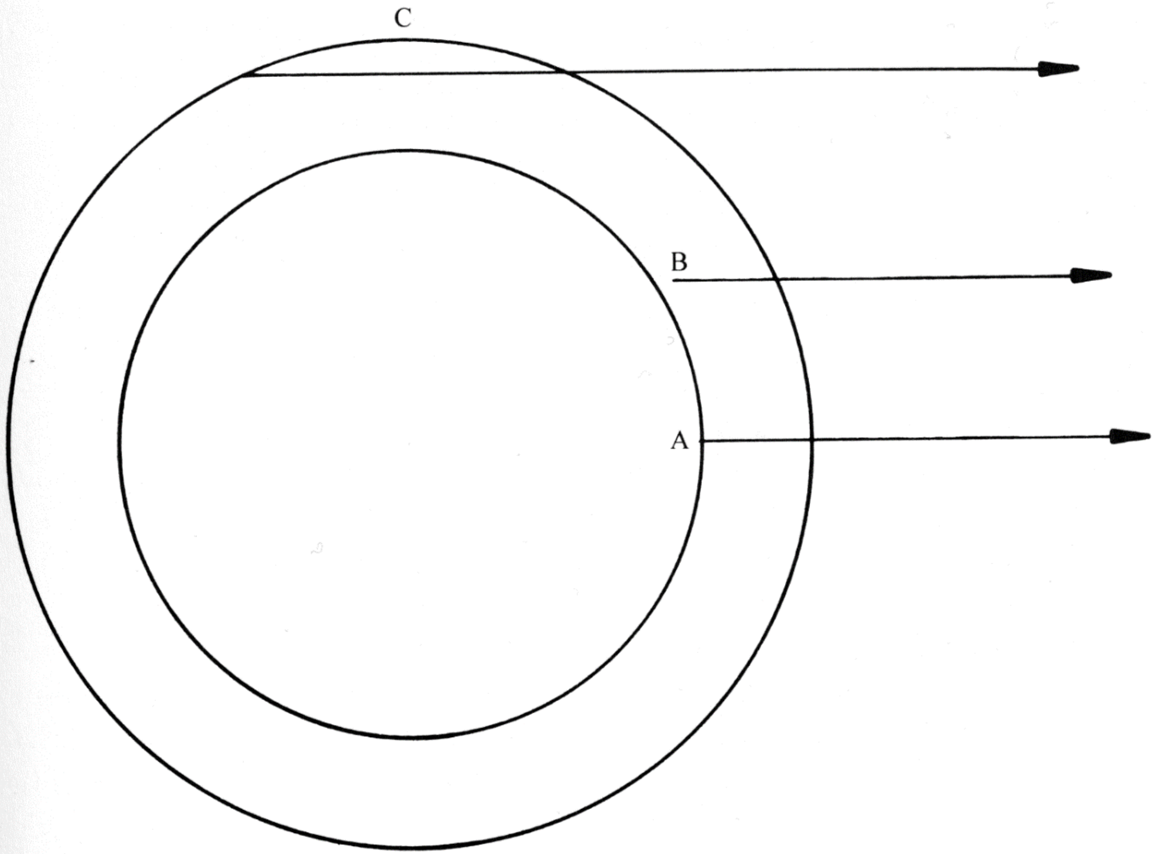
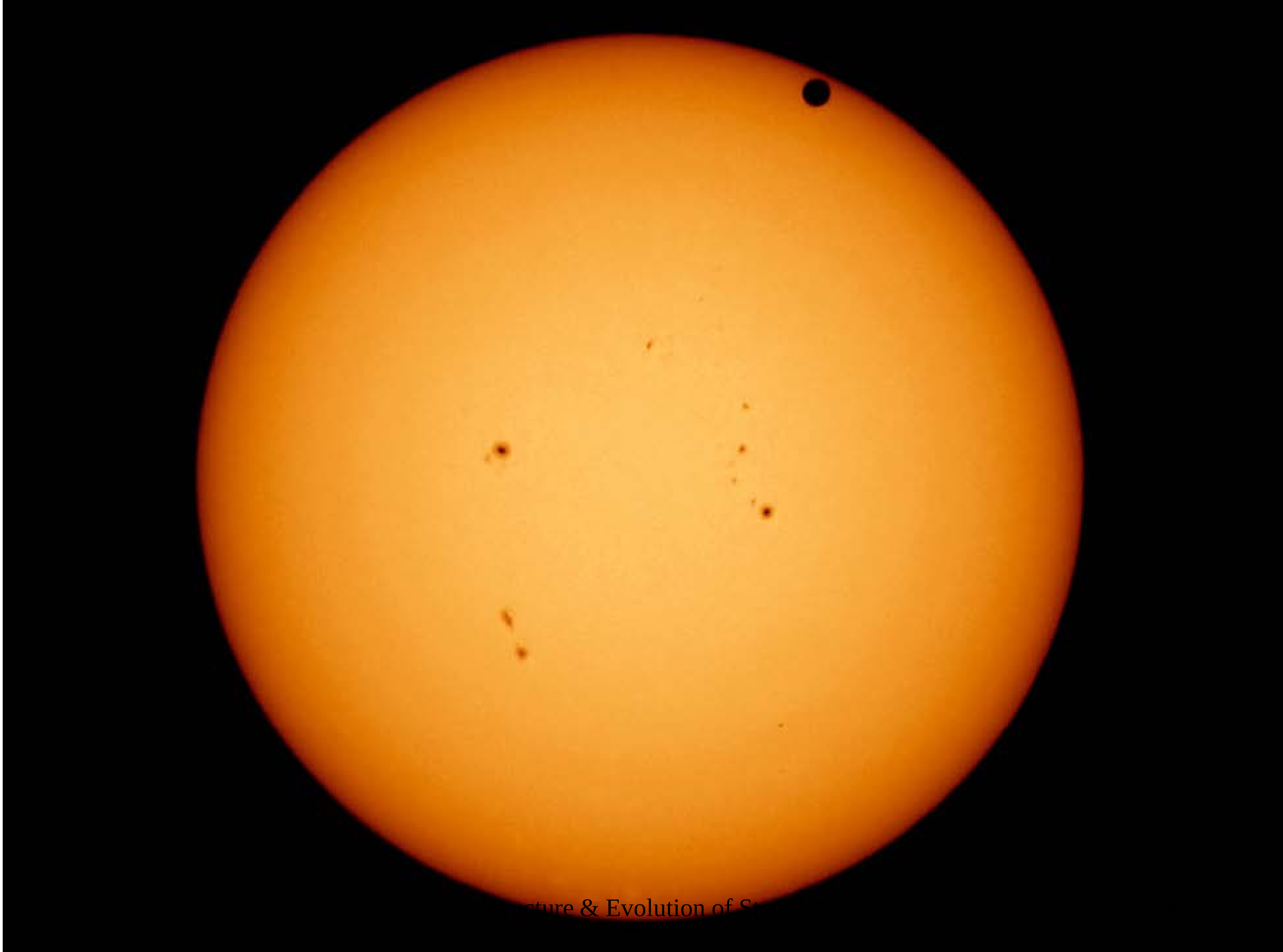
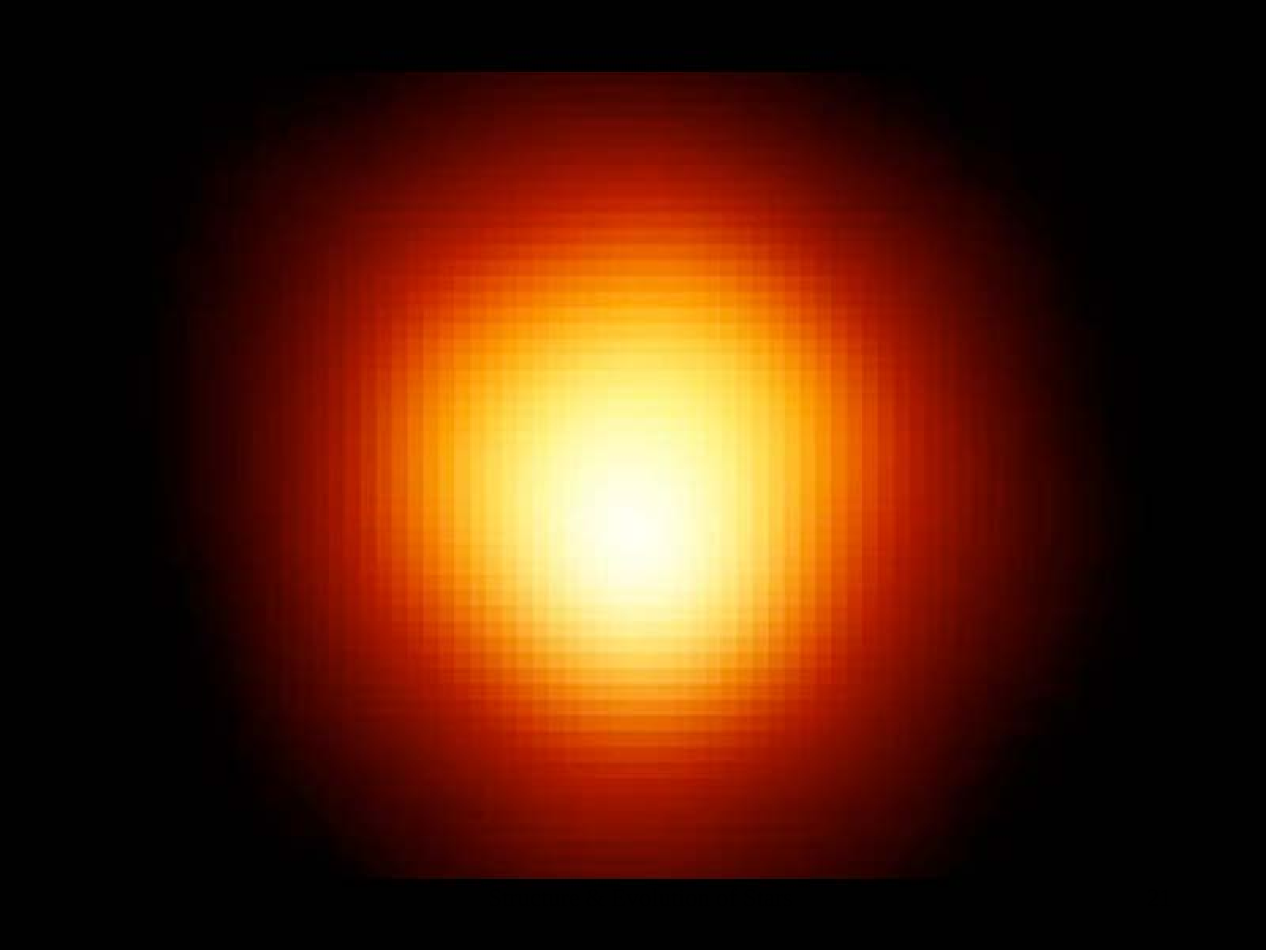


Fig. 39. Limb darkening. Light travelling to an observer from the centre of a stellar disk comes on average from the spherical layer passing through A. Light from B, which is at a lower temperature than A, can also just reach the observer; it travels through more absorbing material than if it were at the same level in the disk centre. At C radiation can pass right through the limb, but it all comes from matter at a lower temperature than at either A or B. As a consequence the stellar disk looks brighter at the centre than at the limb.





Opacity

$\int_R^\infty \kappa \rho dr \cong 1$ defines the "surface" of an object, which is a function of wavelength

- Given radiation of different frequencies interacts differently with material need always to consider opacity as a function of frequency or wavelength: $\kappa = \kappa_\nu$

- In practice, calculation of opacity therefore involves an integration over a range of frequencies

- Material has a cross-section, σ, that quantifies the effective size the material presents for interaction with radiation – again, σ is a function of frequency (or wavelength), i.e. $\sigma = \sigma_\nu$. Cross-section self-evidently has dimensions of area (m^2)

Lecture 8: Summary

- **Degeneracy pressure** depends on density and is *independent* of T

- log ρ – log T plane important for determining whether gas pressure, radiation pressure or degeneracy pressure dominates the equation of state

- **Opacity** is a measure of the resistance of material to the passage of radiation. Important concept of **mean free path** and the dimensionless quantity **optical depth**. "surface" of an object defined by optical depth equal to unity

- Very different physical interactions between matter and radiation contribute to the opacity. In general, opacity exhibits very strong dependence on frequency (or wavelength) of radiation – more in Lecture 9

Picture Credits

- Slide 17 © from Green and Jones, CUP
- Slides 16 © Ostlie and Carroll
- Slide 9 © Phillips, Wiley
- Slide 18 © Tayler “Sun as a Star”, CUP