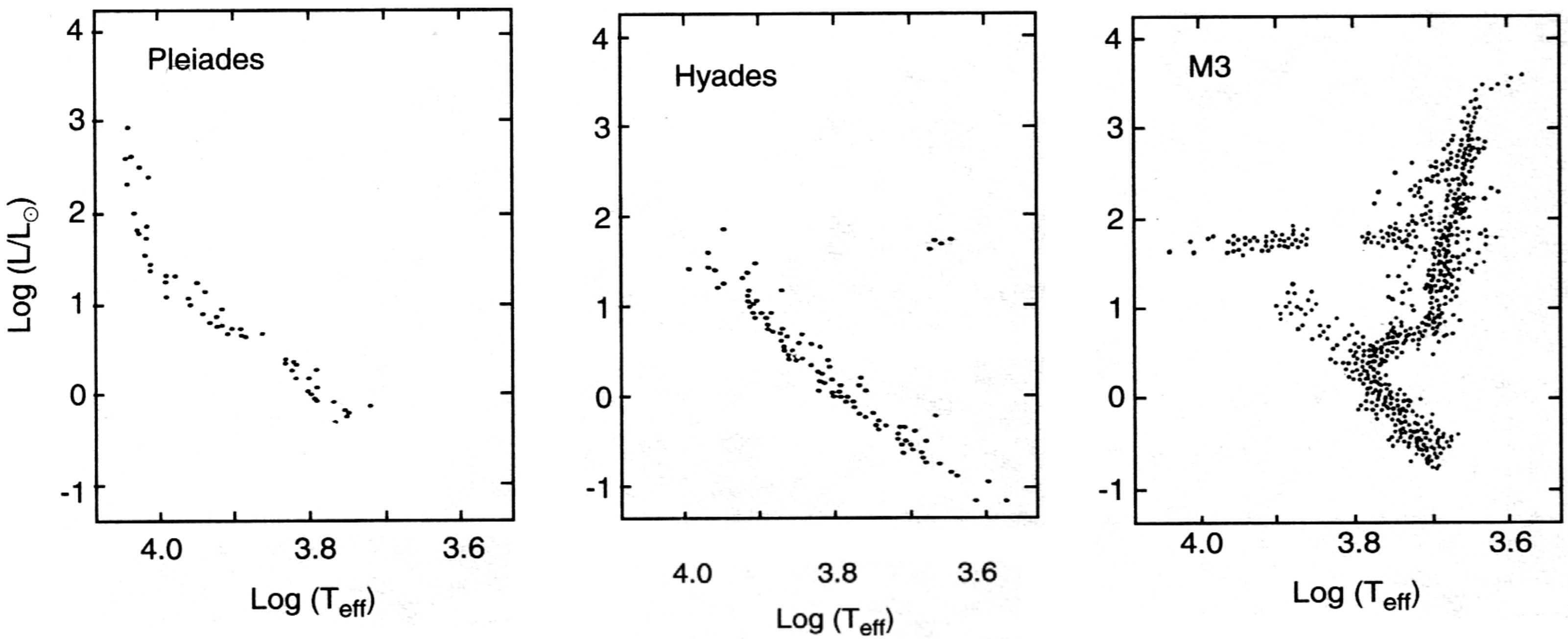
Structure and Evolution of Stars

Lecture 6: The Equations of Stellar Structure

- Mass Conservation
- Equation of Hydrostatic Equilibrium
- Check on validity of assumptions
- Boundary Conditions
- Virial Theorem
- First estimates of pressure and temperature in stellar interiors

Lecture 4: Sparsely Populated Regions of the HR-diagram

HR-diagrams for 3 stellar clusters of very different ages



Radically different morphology of HR-diagrams as a function of age. An "isochrone" describes location of stars of fixed age in HR-diagram. (M3 ≈ 11 × 10^9yr; Hyades ≈ 6 × 10^8yr)

Physical Properties to be Determined

- Will assume that star is close to spherical so that behaviour as a function of radial distance, r, is sufficient to specify physical properties. Rotation and magnetic fields not significant
- Wish to determine:
  - Mass $M=M(r)$
  - Density $\rho=\rho(r)$
  - Pressure $P=P(r)$
  - Temperature $T=T(r)$
  - Luminosity $L=L(r)$
  - Energy $\varepsilon=\varepsilon(r)$ — energy production rate per unit volume
  - Composition $X_H=X_H(r)$ — Hydrogen, Helium,…fraction

Can Effects of Rotation be Ignored?

- Consider small mass element at radius R on the equator of star rotating with angular velocity, ω. Gravitational and centripetal forces act on the mass element. Require gravitational force to dominate:

$$F_{Grav} = -\frac{GMm}{R^2}; \quad F_{Rot} = m\omega^2 R$$

$$F_{Rot} << F_{Grav}$$

In terms of the angular velocity:

$$\Rightarrow \omega^2 << \frac{GM}{R^3}$$

In terms of the period of rotation, $P=2\pi/\omega$, and the dynamical timescale (Lecture 5) have condition:

$$\Rightarrow P^2 >> 8\pi^2 \frac{R^3}{2GM} = 8\pi^2 \tau_{Dyn}^2$$

$$\Rightarrow P >> 9\tau_{Dyn}$$

For the Sun, $P \approx 1$ month $\approx 2.6\times10^6$ s, and condition is safely satisfied

Mass Conservation

- Shell of material at radius *r* and thickness *dr*.

  Mass in the shell is *dm* = volume × density

$$= 4\pi r^2 dr \times \rho$$

$$\Rightarrow \frac{dm}{dr} = 4\pi r^2 \rho$$

Hydrostatic Equilibrium

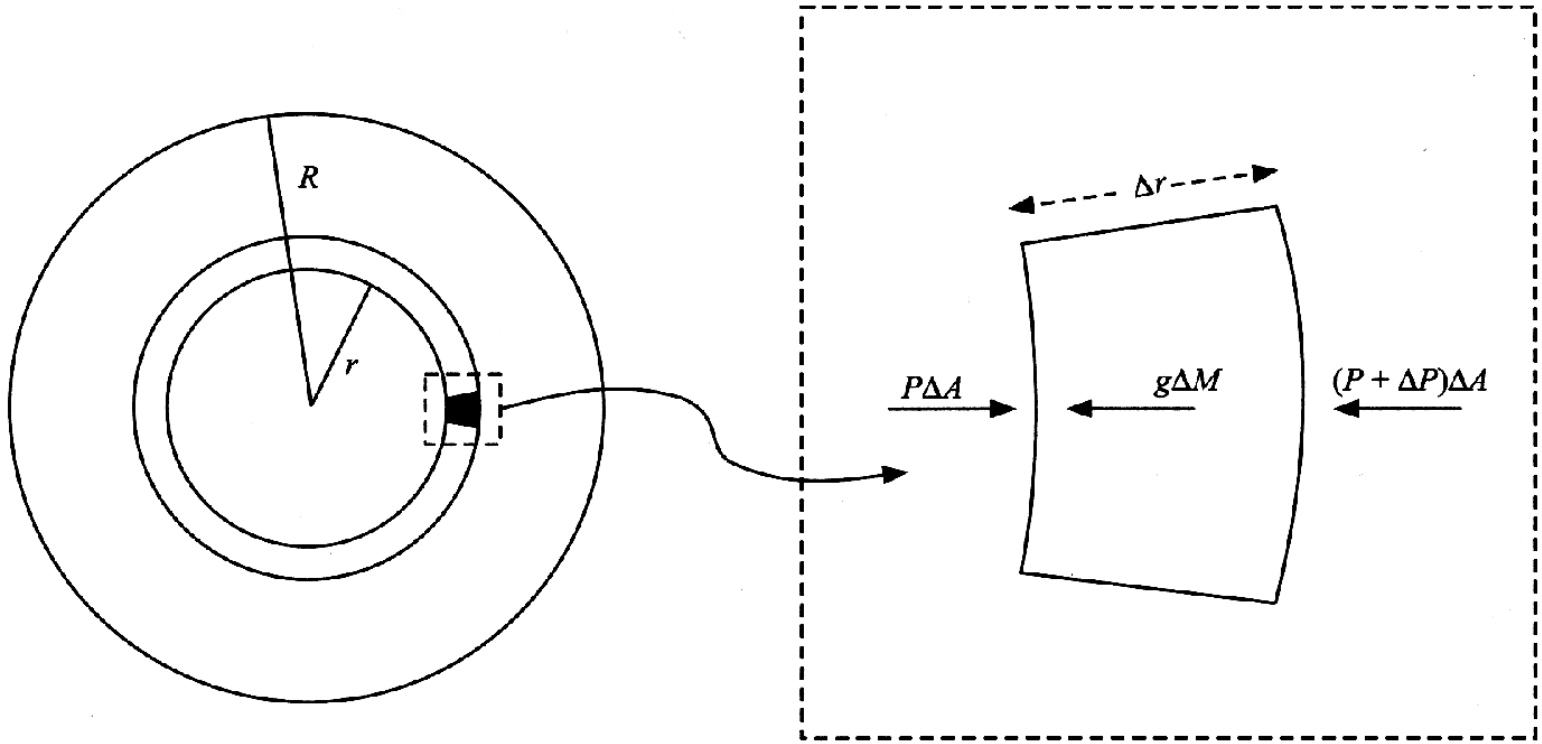


Fig. 1.1 A spherical system of mass M and radius R. The forces acting on a small element with volume $\Delta r\ \Delta A$ at distance r from the centre due to gravity and pressure are indicated. The gravitational attraction of the mass $m(r)$ within r produces an inward force which is equal to $g(r)\,\rho(r)\,\Delta r\,\Delta A = g(r)\,\Delta M$. If there is a non-zero pressure gradient at r, the difference in pressure on the inner and outer surfaces leads to an additional force which can oppose gravity

Hydrostatic Equilibrium

Consider the forces acting on a small cylindrical volume element, cross-sectional area, dS, height, dr, at radius r in a star. Height dr is small so density, ρ, can be taken to be constant

There is a gravitational force acting inward due to the mass of the star interior to the radius r

There is a force resulting from the pressure, P, of the gas surrounding the element. Spherical symmetry of the system means that the forces acting on the sides of the element (perpendicular to r) cancel

Need to consider only the gravitational force inward and the pressure on the near and far faces of the cylinder

Hydrostatic Equilibrium

Mass of cylindrical element: $\Delta m = \rho dr dS$

Net radial force due to gravity and pressure difference:

$$\ddot{r}\Delta m = \frac{Gm\Delta m}{r^2} + P(r)dS - P(r+dr)dS$$

Rewrite pressure on outer face:

$$P(r+dr) = P(r) + (\partial P / \partial r)dr$$

Substitute for mass from above:

$$\ddot{r}\Delta m = \frac{Gm\Delta m}{r^2} - \frac{\partial P}{\partial r}\frac{\Delta m}{\rho}$$

Simplify by dividing by the mass of the element:

$$\ddot{r} = -\frac{Gm}{r^2} - \frac{1}{\rho}\frac{\partial P}{\partial r}$$

Hydrostatic Equilibrium

When the acceleration vanishes we have balance between gravity and pressure and system is said to be in "hydrostatic equilibrium". rhs is always negative so pressure increases inwards

$$\frac{dP}{dr} = -\rho \frac{Gm}{r^2}$$

Can also express as a function of mass rather than radius, noting:

$$dr = dm/(4\pi r^2 \rho)$$

to give:

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$$

If hydrostatic equilibrium applies at all *r*, then star is in hydrostatic equilibrium

Hydrostatic Equilibrium Valid for the Sun?

Suppose there is a net force equal to a fraction, λ, of the gravitational acceleration at *R*:

$$\ddot{r} = -\frac{Gm}{r^2} - \frac{1}{\rho}\frac{\partial P}{\partial r} = -\lambda \frac{GM}{R^2}$$

After time, *t*, material has moved distance Δ*r*:

$$\Delta r = \frac{1}{2}\lambda \frac{GM}{R^2} t^2$$

Rearranging to give time taken to move distance Δ*r:*

$$t = \sqrt{\frac{2R^2 \Delta r}{\lambda GM}}$$

Say $\Delta r \approx 0.1R$, then for Solar mass and radius we have:

$$t \approx \frac{10^3}{\sqrt{\lambda}}\,\mathrm{s}$$

Know age is $\geq 10^9$yr and thus $\lambda \leq 10^{-27}$, and assumption must be very good!

Boundary Conditions

- Mass conservation and the condition for hydrostatic equilibrium produce first two equations of stellar structure. We will end up with a set of differential equations and in order to make effective use of the equations need some **boundary conditions**
- Unsurprisingly, the most obvious places to consider the boundary conditions are at the centre of a star (r=0) and the surface of a star ($r=R$)
- At the surface of the star; $r=R$, $m=M$, ρ=0, $P=0$
- At the centre of the star; $r=0$, $m=0$, $dP/dr=0$
- Can immediately start making progress using only the condition of hydrostatic equilibrium to constrain the central pressure in a star

Constrain Central Pressure

Starting from the condition for hydrostatic equilibrium:

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$$

Integrate from centre to surface of the star:

$$P(M) - P(0) = -\int_0^M \frac{Gmdm}{4\pi r^4}$$

Note that $P(M)\approx 0$ and $P(0)=P_{cen}$ and that $r \leq R$:

$$P_{cen} = \int_0^M \frac{Gmdm}{4\pi r^4} > \int_0^M \frac{Gmdm}{4\pi R^4}$$

Integrate rhs and evaluate for Solar mass and radius:

$$P_{cen} > \frac{GM^2}{8\pi R^4} = 4.4 \times 10^{13}\,\mathrm{Nm}^{-2}$$

The Virial Theorem

Take equation for hydrostatic equilibrium and expression for Volume:

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}; \quad V = \frac{4}{3}\pi r^3$$

Multiply by volume:

$$\int_0^{P(R)} V dP = -\frac{1}{3}\int_0^M \frac{Gmdm}{r}$$

Note that integral on rhs is the gravitational potential energy:

$$E_{Grav} = -\int_0^M \frac{Gmdm}{r}$$

Integrate lhs by parts, finding first term vanishes (boundary conditions), and thus we obtain the Virial Theorem:

$$\int_0^{P(R)} V dP = \left[PV\right]_0^R - \int_0^{V(R)} P dV$$

$$\Rightarrow -3\int_0^{V(R)} P dV = E_{Grav}$$

The Virial Theorem

As $dV=dm/\rho$, relation can also be written:

$$-3\int_0^M \frac{P}{\rho}dm = E_{Grav}$$

Relation derived for the entire star but have similar relation for any radius $R_s<R$:

$$P_S V_S - \int_0^{M_s} \frac{P}{\rho}dm = \frac{1}{3}E_{Grav}(r \leq s)$$

Have not discussed nature of material making up star but suppose consists of an ideal gas:

$$P = nkT = \frac{\rho}{m_p}kT$$

Kinetic energy per unit mass is then:

$$KE = u = \frac{3}{2}\frac{kT}{m_p} = \frac{3}{2}\frac{P}{\rho}$$

The Virial Theorem

The integral over mass of the internal energy per particle is simply the total internal energy, *U*, for the system and have the Virial Theorem in familiar form

$$2\int_0^M u\,dm = 2U = -E_{Grav}$$

The exact value of the gravitational energy of the star depends on the density profile, $\rho(r)$, and hence the mean particle separation. Can express the energy in terms of the mass and radius of the star scaled by a constant, α, of order unity

$$E_{Grav} = -\alpha \frac{GM^2}{R}$$

The Virial Theorem

Can now calculate the average temperature within stars:

From Virial Theorem: $$U = \int_0^M \frac{3}{2}\frac{kT}{m_p}dm = \frac{3}{2}\frac{k}{m_p}\overline{T}M = \frac{1}{2}\alpha\frac{GM^2}{R}$$

Taking α=0.5, assume composition is atomic hydrogen and use values of Solar mass and radius gives:

$$\overline{T} = \frac{\alpha}{3}\frac{m_p}{k}\frac{GM}{R} \approx 4\times10^6\,\mathrm{K}$$

Substituting for the average density gives scaling with mass and density:
Temperature increases with mass and density

$$\overline{\rho} = 3M / 4\pi R^3$$

$$\Rightarrow \overline{T} \propto M^{2/3}\overline{\rho}^{1/3}$$

Lecture 6: Summary

- Begun to assemble set of differential equations that describe structure of a spherical, stable star
- Mass conservation
- Hydrostatic Equilibrium from balance between gravity and pressure
- Virial Theorem: $U = -1/2E_{Grav}$
- Performed some simple checks on assumptions
- HE and VT with appropriate boundary conditions at center and surface of star enable quantitative estimates of physical quantities
- Central pressure and average temperature for the Sun both impressively large

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