EXERCISES: Set 3 of 4

Q1:

(i) Show that the equation of hydrostatic equilibrium may be written as:

$$\frac{\mathrm{d}\ln P}{\mathrm{d}\ln \tau} = \frac{g\tau}{\kappa P} = \frac{\mathrm{d}\log P}{\mathrm{d}\log \tau}$$

where τ is the optical depth, κ is the opacity and g is the gravitational acceleration.

(ii) We know from geological and fossil records that the Sun's luminosity has remained constant over the last billion years. From this statement, deduce the accuracy of the approximation of hydrostatic equilibrium as applied to the Sun. (At the surface of the Sun, the gravitational acceleration is $g = 2.5 \times 10^2$ m s^{-2}).

Q2: A star is composed of H (mass fraction $X = 0.7$), He (mass fraction $Y = 0.3$) and negligible amounts of heavier elements.

(i) Calculate the mean molecular weight immediately above and below the radius in the star where hydrogen becomes ionized. Assuming the transition between ionized and neutral hydrogen takes place over a very small radial distance, such that the pressure and temperature can be considered constant across the zone, what would this imply about the dynamical stability of the zone?

(ii) Assuming that the pressure P has contributions βP from gas pressure and $(1-\beta)P$ from radiation pressure, where $0 \leq \beta \leq 1$, show that:

$$\beta^4 \left(\frac{P^3}{\rho^4}\right) = \left(\frac{\mathcal{R}}{\mu}\right)^4 \frac{3}{a}(1-\beta)$$

(iii) A polytrope of index n has central pressure $P_c = W_n G M^2/R^4$ and central density $\rho_c = X_n M/(\frac{4}{3}\pi R^3)$, where W_n and X_n are dimensionless constants that depend only on n. Write down the equation for β_c, the value of β at the centre of the polytrope of index n, and show that β_c depends only on M and n.

Q3: The gravitational binding energy of a star of mass M and radius R is given by:

$$E = -\frac{\alpha G M^2}{R}$$

where α is a constant. Such a star contracts homologously at constant effective temperature, radiating at a rate $L(t)$.

(i) Derive expressions for $L(t)$ and $R(t)$ for a star of fixed mass which at time $t = 0$ had $L = L_0$ and $R = R_0$.

(ii) Show that at late times L and R display power law dependences on time.

(iii) Where would such a star be found in the Hertzsprung-Russell diagram?

Q4: Consider a convective star.

(i) Give an approximate derivation for the boundary condition at the photosphere in the form:

$$\kappa P \sim \frac{GM}{R^2}$$

In fully convective low-mass main-sequence stars, the equation of state is that of an ideal gas and the opacity is of the form $\kappa = \kappa_0 \rho T^8$. Show that the adiabatic constant $K \equiv P T^{-5/2}$ is such that $K \propto M^{1/2} R^{-1} T_{\rm eff}^{-6}$. The energy generation rate is of the form $\mathcal{E} = \mathcal{E}_0 \rho T^6$.

(ii) Show that for a group of such stars of the same chemical composition the members satisfy the following relations:

$$R \propto M^{11/17}; \qquad L \propto M^{37/17}$$

(iii) Sketch the locus of such stars on the H-R diagram, marking the locus of the main sequence and the position of the Sun.

Q5: The H II region around an O -star has radius $R = 5\,\mathrm{pc}$, temperature $T = 10\,000$ K, and density $n = 100\ \mathrm{cm}^{-3}$.

(i) Estimate whether such an interstellar cloud is stable against gravitational collapse.

(ii) If such an H II region were visible at redshift $z = 3$ and you recorded its spectrum with a ground-based telescope, which spectral feature would you expect to be strongest? Give reasons for your answer.

Q6: An astronomer claims that in a distant galaxy the stellar Initial Mass Function is 'top-heavy'.

(i) Explain what is meant by such a statement.

(ii) Put forward some observational tests that may verify the validity of this claim.

Q7: Estimate the time the Sun will spend on the horizontal branch, if 15% of its mass is converted from ^{4}He to ^{12}C via the triple-alpha reaction.