EXERCISES: Set 2 of 4

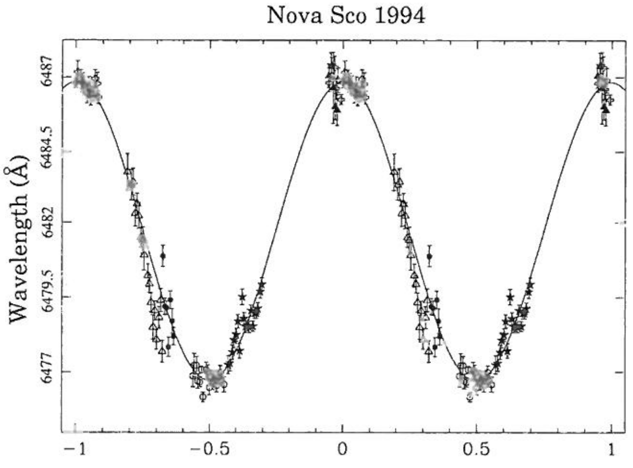
Q1: In 1994 a new bright star was found, Nova Sco 1994. It turned out that it was a sudden brightening of an X-ray binary in which an F star is orbiting a compact, unseen, object. The mass function of the unseen companion is defined as

$$f(M_1, M2) = \frac{(M_1 \sin i)^3}{(M_1 + M_2)^2},$$

and can be estimated from the radial velocity amplitude of the F star v_2, the binary period P, and the inclination of the plane of the orbit to the line of sight, $\sin i$. The mass function has the units of mass, and is the minimum mass of the companion should the star for which we have orbital information be a test particle (or effectively massless). When additional information is available about the mass of the star with the orbital information, more accurate estimates of the companion mass can be obtained.

(i) Derive the formula giving the mass function of the unseen companion in terms of v_2, P and $\sin i$, assuming that the orbit is circular.

(ii) The Figure below shows the observed wavelength of the Fe II λ6485.10 line as function of the orbital period.



Justify the assumption that the orbit is circular using the data shown in the Figure, and deduce the value of v_2. Why is the mean wavelength of the Fe line not equal to its laboratory wavelength?

(iii) When the system returns to quiescence, the light of the binary system is dominated by the F-type star. In such close binaries, the stars become deformed by the gravitational pull of the companion. This leads to their projected surface on the sky—and therefore the total brightness—varying with the orbital period. From such variations, the orbital inclination can be deduced. In the case shown here, $i \simeq 70°$ and $P = 2.62\,\mathrm{days}$. From these parameters, determine the value of the mass function.

(iv) Knowing that the mass of an F-type star is $M_2 \simeq 2.4 M_\odot$, deduce M_1. It is generally thought that in X-ray binaries where the mass of the unseen companion is $M_1 \gtrsim 3 M_\odot$, the compact object is likely to be a black hole. Is the unseen companion in Nova Sco 1994 a black hole?

Q2: The gas density within a star decreases from the centre to the surface as a function of radial distance r according to

$$\rho(r) = \rho_c \left[1 - \left(\frac{r}{R} \right)^2 \right]$$

where ρ_c is the core density and R is the star's radius.

(i) Find $m(r)$.

(ii) Derive the relation between the total mass of the star M and R.

(iii) What is the average density of the star in units of the core density ρ_c?

(iv) The gravitational potential energy of a star of mass M and radius R is given by:

$$U_g = -\alpha \frac{GM^2}{R}$$

where α is a constant of order unity determined by the distribution of matter within the star.

Find the value of α for the density profile given above.

Q3: The Lorentz profile:

$$\phi_\nu = \frac{\frac{\Gamma}{4\pi^2}}{(\nu - \nu_0)^2 + \left(\frac{\Gamma}{4\pi}\right)^2}$$

describes the '*natural*' broadening of an absorption line due to the inherent widths of the energy levels between which the atomic transition takes place following the absorption of a photon of rest frequency ν_0. In the above expression $\Gamma = 1/\Delta t$ is the radiative damping constant, inversely proportional to the mean lifetime Δt of the excited energy level to which absorption takes place.

(i) Show the the full width at half maximum of a Lorentzian line profile is $\Delta\nu_{FWHM} = \Gamma/2\pi$.

(ii) At what interval in units of $\Gamma/4\pi$ (in other words, in units of the half-width at half maximum) from the rest frequency does the Lorentz profile have a value of 1% of its central intensity?

(iii) Thermal broadening is described by a Gaussian distribution. For a Gaussian distribution with the same FWHM as the Lorentz profile, what is the probability of absorption at the same $\Delta\nu$ from the line centre as in part (ii)?

Q4: The temperature dependence of energy generation by the triple-alpha process is:

$$\mathcal{E}_{3\alpha} = k_0 T_8^{-3} e^{-(44/T_8)}$$

where T_8 is the temperature in units of 10^8 K and k_0 is a constant.

(i) By considering the energy generation near $T_8 = 1$ to scale as $\mathcal{E}(T) = k_1 T_8^\gamma$, show that: $\mathcal{E}_{3\alpha} \approx k_2 T_8^{41}$, where k_1 and k_2 are constants.

(ii) Calculate the change in energy output resulting from a 10% change in temperature.

Q5: An eclipsing-binary system has a parallax of 0.1 arcsec (with negligible error) and consists of two solar-type stars with a semi-major axis of $500 R_\odot$. The period is known very accurately.

(i) What is the angular size of each of the stars and of the semi-major axis? If you can measure angles on the sky with a 1σ rms accuracy of 0.01 arcsec, what is the percentage accuracy of the measurement of the semi-major axis and of the radius of each star ?

(ii) Assume that the stars emit as blackbodies:

$$F_\nu(T) = \frac{2h\nu^3}{c^2} \cdot \frac{1}{e^{h\nu/kT} - 1},$$

where ν is the frequency is Hz, with an effective temperature $T_{\text{eff}} \simeq 5800\,\text{K}$. If you measure the flux ratio between $\log \nu = 14.0$ and 15.0 with an accuracy of 10%, with what precision can you determine the value of T_{eff}?

(iii) If we now include an error in the measurement of the parallax of $\sigma_\Pi = 0.01$ arcsec, what is the percentage accuracy in the mass of the system?

Q6: In a certain temperature regime, the p-p chain can be approximated by the following four reactions:

$$\begin{aligned}
{}^1\text{H} + {}^1\text{H} &\rightarrow {}^2\text{H} + \beta^+ + \nu, \\
{}^1\text{H} + {}^2\text{H} &\rightarrow {}^3\text{He} + \gamma, \\
{}^3\text{He} + {}^4\text{He} &\rightarrow {}^7\text{Li}, \\
{}^7\text{Li} + {}^1\text{H} &\rightarrow {}^4\text{He} + {}^4\text{He} \ .
\end{aligned}$$

(i) Write down the rate equations for the abundances N_1, N_2, N_3, N_4, and N_7 of the five nuclear species involved in terms of the rates r_{11}, r_{12}, r_{34}, and r_{17} of the four reactions.

(ii) If ^{2}H, ^{3}He and ^{7}Li are all in transient equilibrium, show that the equations can be simplified to

$$\dot{N}_1 = -4 r_{11} N_1^2\,, \qquad \dot{N}_4 = r_{11} N_1^2\,,$$

and find equations for the equilibrium abundances of the other species.

(iii) At lower temperatures, the last two reactions in the chain are replaced by the single reaction:

$$^3\mathrm{He} + {}^3\mathrm{He} \rightarrow {}^4\mathrm{He} + 2\ {}^1\mathrm{H} \quad .$$

Making equivalent assumptions about transient equilibrium, show that the corresponding simplified rate equations are:

$$\dot{N}_1 = -2r_{11}N_1^2\,, \qquad \dot{N}_4 = \frac{1}{2}r_{11}N_1^2\,.$$

(iv) What is the underlying reason why these differ from the previous equations by a factor of 2?

Q7(i): Two early-type stars in the same cluster start their lives on the H-burning main sequence with the same mass: $M_0(\mathrm{A}) = M_0(\mathrm{B}) = 5\,M_\odot$. Star A is single. Star B is a member of a binary system and throughout its life on the main sequence loses mass to a compact companion at an average rate $\dot{M} = 1 \times 10^{-8} M_\odot\ \mathrm{yr}^{-1}$. Which of the two stars do you think will leave the main sequence first and why?

Q7(ii): An astronomer armed with a photometer and two broad-band filters, V and B, measures the following magnitudes for two stars in the constellation of Pegasus: $m_V(\alpha\,\mathrm{Peg}) = 2.45$, $m_B(\alpha\,\mathrm{Peg}) = 2.45$; and $m_V(\beta\,\mathrm{Peg}) = 2.40$, $m_B(\beta\,\mathrm{Peg}) = 4.04$. On the basis of this information alone, which of the two stars would you consider more likely to be the closer one to the Sun? What other information would you require to definitely establish which is closer?

Q8: Using the tabulation of solar photospheric abundances by Asplund et al. 2009 (ARAA, 47, 481) on the next page calculate the mass fractions of H, He, C, N, O, and Ne in the Sun.

Table 1: Element abundances in the present-day solar photosphere. Also given are the corresponding values for CI carbonaceous chondrites (Lodders, Palme & Gail 2009). Indirect photospheric estimates have been used for the noble gases (Sect. 3.9).

	Elem.	Photosphere	Meteorites		Elem.	Photosphere	Meteorites
1	H	12.00	8.22 ± 0.04	44	Ru	1.75 ± 0.08	1.76 ± 0.03
2	He	$[10.93 \pm 0.01]$	1.29	45	Rh	0.91 ± 0.10	1.06 ± 0.04
3	Li	1.05 ± 0.10	3.26 ± 0.05	46	Pd	1.57 ± 0.10	1.65 ± 0.02
4	Be	1.38 ± 0.09	1.30 ± 0.03	47	Ag	0.94 ± 0.10	1.20 ± 0.02
5	B	2.70 ± 0.20	2.79 ± 0.04	48	Cd		1.71 ± 0.03
6	C	8.43 ± 0.05	7.39 ± 0.04	49	In	0.80 ± 0.20	0.76 ± 0.03
7	N	7.83 ± 0.05	6.26 ± 0.06	50	Sn	2.04 ± 0.10	2.07 ± 0.06
8	O	8.69 ± 0.05	8.40 ± 0.04	51	Sb		1.01 ± 0.06
9	F	4.56 ± 0.30	4.42 ± 0.06	52	Te		2.18 ± 0.03
10	Ne	$[7.93 \pm 0.10]$	-1.12	53	I		1.55 ± 0.08
11	Na	6.24 ± 0.04	6.27 ± 0.02	54	Xe	$[2.24 \pm 0.06]$	-1.95
12	Mg	7.60 ± 0.04	7.53 ± 0.01	55	Cs		1.08 ± 0.02
13	Al	6.45 ± 0.03	6.43 ± 0.01	56	Ba	2.18 ± 0.09	2.18 ± 0.03
14	Si	7.51 ± 0.03	7.51 ± 0.01	57	La	1.10 ± 0.04	1.17 ± 0.02
15	P	5.41 ± 0.03	5.43 ± 0.04	58	Ce	1.58 ± 0.04	1.58 ± 0.02
16	S	7.12 ± 0.03	7.15 ± 0.02	59	Pr	0.72 ± 0.04	0.76 ± 0.03
17	Cl	5.50 ± 0.30	5.23 ± 0.06	60	Nd	1.42 ± 0.04	1.45 ± 0.02
18	Ar	$[6.40 \pm 0.13]$	-0.50	62	Sm	0.96 ± 0.04	0.94 ± 0.02
19	K	5.03 ± 0.09	5.08 ± 0.02	63	Eu	0.52 ± 0.04	0.51 ± 0.02
20	Ca	6.34 ± 0.04	6.29 ± 0.02	64	Gd	1.07 ± 0.04	1.05 ± 0.02
21	Sc	3.15 ± 0.04	3.05 ± 0.02	65	Tb	0.30 ± 0.10	0.32 ± 0.03
22	Ti	4.95 ± 0.05	4.91 ± 0.03	66	Dy	1.10 ± 0.04	1.13 ± 0.02
23	V	3.93 ± 0.08	3.96 ± 0.02	67	Ho	0.48 ± 0.11	0.47 ± 0.03
24	Cr	5.64 ± 0.04	5.64 ± 0.01	68	Er	0.92 ± 0.05	0.92 ± 0.02
25	Mn	5.43 ± 0.05	5.48 ± 0.01	69	Tm	0.10 ± 0.04	0.12 ± 0.03
26	Fe	7.50 ± 0.04	7.45 ± 0.01	70	Yb	0.84 ± 0.11	0.92 ± 0.02
27	Co	4.99 ± 0.07	4.87 ± 0.01	71	Lu	0.10 ± 0.09	0.09 ± 0.02
28	Ni	6.22 ± 0.04	6.20 ± 0.01	72	Hf	0.85 ± 0.04	0.71 ± 0.02
29	Cu	4.19 ± 0.04	4.25 ± 0.04	73	Ta		-0.12 ± 0.04
30	Zn	4.56 ± 0.05	4.63 ± 0.04	74	W	0.85 ± 0.12	0.65 ± 0.04
31	Ga	3.04 ± 0.09	3.08 ± 0.02	75	Re		0.26 ± 0.04
32	Ge	3.65 ± 0.10	3.58 ± 0.04	76	Os	1.40 ± 0.08	1.35 ± 0.03
33	As		2.30 ± 0.04	77	Ir	1.38 ± 0.07	1.32 ± 0.02
34	Se		3.34 ± 0.03	78	Pt		1.62 ± 0.03
35	Br		2.54 ± 0.06	79	Au	0.92 ± 0.10	0.80 ± 0.04
36	Kr	$[3.25 \pm 0.06]$	-2.27	80	Hg		1.17 ± 0.08
37	Rb	2.52 ± 0.10	2.36 ± 0.03	81	Tl	0.90 ± 0.20	0.77 ± 0.03
38	Sr	2.87 ± 0.07	2.88 ± 0.03	82	Pb	1.75 ± 0.10	2.04 ± 0.03
39	Y	2.21 ± 0.05	2.17 ± 0.04	83	Bi		0.65 ± 0.04
40	Zr	2.58 ± 0.04	2.53 ± 0.04	90	Th	0.02 ± 0.10	0.06 ± 0.03
41	Nb	1.46 ± 0.04	1.41 ± 0.04	92	U		-0.54 ± 0.03
42	Mo	1.88 ± 0.08	1.94 ± 0.04				

Figure 6.13: The most recent compilation of solar element abundances by Asplund, Grevesse, Sauval, & Scott, P. 2009, ARAA, 47, 481.