

STELLAR REMNANTS. I: WHITE DWARFS

14.1 Introduction

We do not have to look very far to find a white dwarf. Sirius (α CMa) is the brightest star in the sky, and the fifth closest to the Sun, at a distance of only 2.6 pc. In 1862 the telescope maker Alvan Clark, while testing an 18-inch refractor lens, discovered that Sirius has a very faint companion: Sirius B. Sirius B always appears very close to Sirius A but is 10 magnitudes (10,000 times!) fainter. This makes it a very difficult object to recognise visually (see Figure 14.1). Sirius B has the same mass as our Sun but only one millionth its volume. That's because Sirius B is a White Dwarf.

Its existence had been predicted nearly twenty years earlier by the German astronomer Bessel (of Bessel functions fame), who correctly deduced the orbital period of the binary star system to be ~ 50 yr from observations of the motion of Sirius A relative to fixed stars over a period of ten years (Figure 14.2).

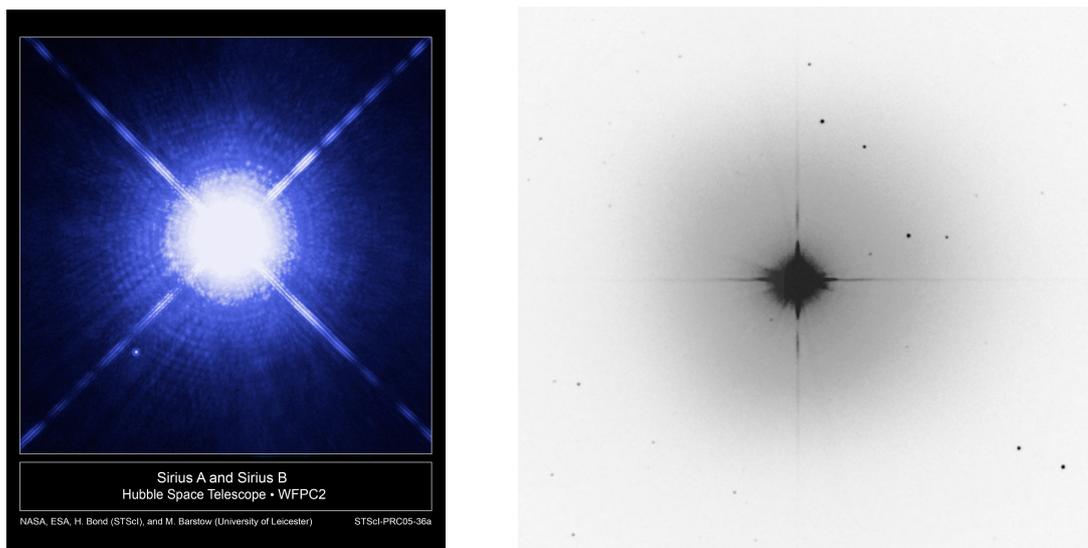


Figure 14.1: Sirius A and B, as seen by the 2.5 m *Hubble Space Telescope* (left) and a ground-based 18-in Celestron (right).

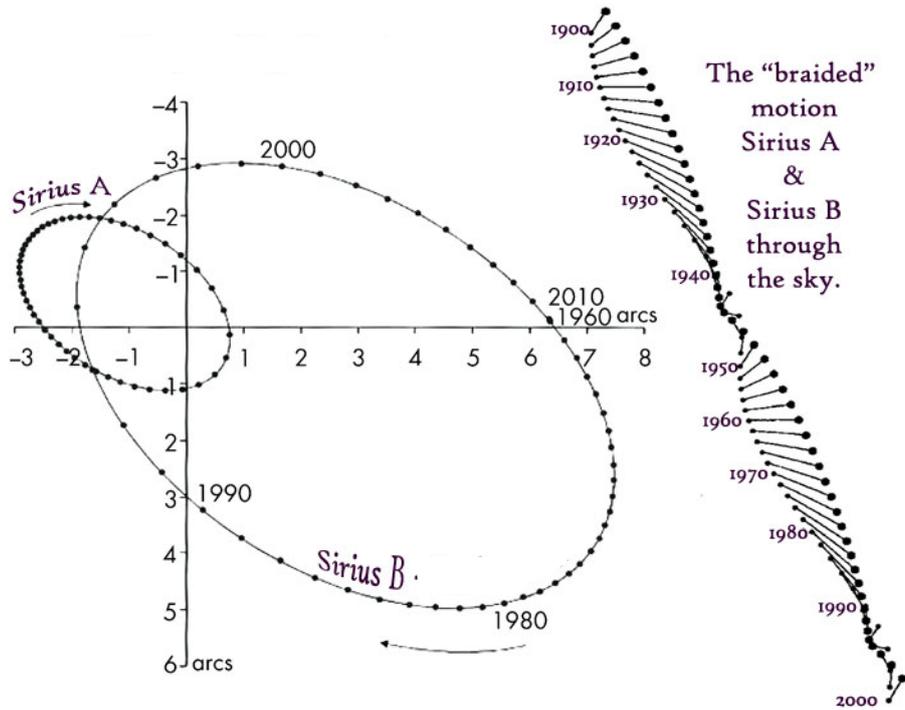


Figure 14.2: The orbits of Sirius A and B about the common centre of mass of the binary system, and their projection on the sky.

The physical parameters of this binary system are collected in Table 14.1. Sirius B has the mass of the Sun confined within a volume smaller than the Earth! The acceleration due to gravity at the surface of Sirius B is $\sim 4 \times 10^5$ times greater than on Earth (where $\log g \simeq 3$). Such strong gravity is reflected in highly pressure-broadened (Lecture 6.5.2) H I Balmer lines superposed on otherwise featureless blue continuum (see Figure 14.4). White dwarf ‘material’ is so dense that a teaspoon would weight 16 tons on Earth, and ~ 6.4 million tons on the surface of the white dwarf. These are extreme physical conditions indeed!

Table 14.1 Physical Parameters of the Sirius A B Binary System.

Property	Sirius A	Sirius B
Spectral type	A1V	DA2
M_V (mag)	1.4	11.2
Mass (M_\odot)	2.0	0.98
Radius (R_\odot)	1.7	0.0084
Surface gravity ($\log g$)	4.3	8.57
Luminosity (L_\odot)	25	0.026
Temperature (K)	9940	25 200

As we have already discussed in Lecture 13.7, white dwarfs are the stellar remnants of low and intermediate mass stars. Their name is related to the fact that they are hot and compact, occupying a narrow sliver in the H-R diagram that is roughly parallel to and well below the Main Sequence (Figure 14.3). The name itself is, however, something of a misnomer since white dwarfs come in all colours, with surface temperatures ranging from $T_{\text{eff}} < 5\,000\text{ K}$ to $T_{\text{eff}} > 80\,000\text{ K}$.

The core of a white dwarf is mostly He, or C and O. Stars with $M \lesssim 0.5M_{\odot}$ do not have sufficient gravitational energy to heat up their core to the temperature required to ignite He fusion, and they will end up as He white dwarfs. However, since the lifetime of such stars is greater than the current age of the Universe, such He white dwarfs should not exist yet! Another channel for the creation of a He-core WD may be through binary evolution, whereby the outer layers of a star in the process of becoming a red giant may be stripped by the gravitational pull of its binary companion. If the mass transfer happens before He ignition, further evolution of the star is halted, leaving a white dwarf made up mostly of He.

Stars with masses $M \simeq 1\text{--}8M_{\odot}$ evolve through the stages outlined in Lecture 13, leaving a white dwarf with a CO core of mass $M \simeq 0.6M_{\odot}$ (Fig-

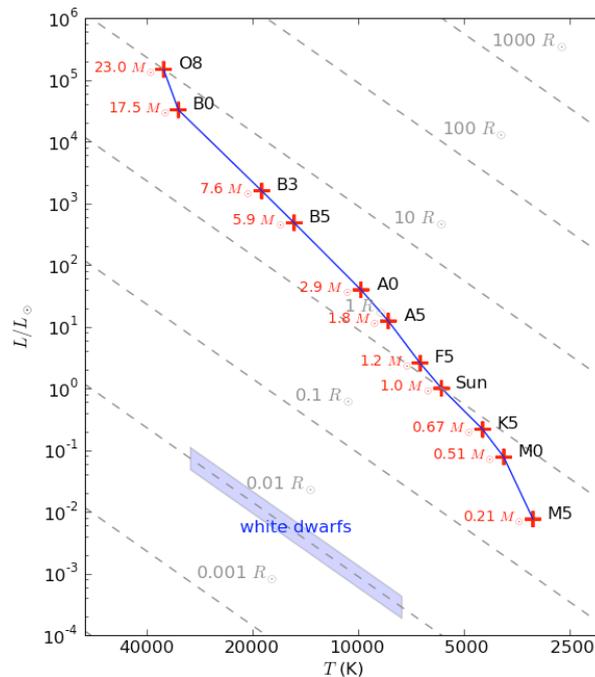


Figure 14.3: Location of white dwarf stars in the H-R diagram.

ure 13.11) following the ejection of the star's outer layers in the planetary nebula stage. In either case, without an internal source of energy, white dwarfs simply cool off at an essentially constant radius as they slowly deplete their supply of thermal energy.

14.1.1 Spectral Classification of White Dwarfs

Although most white dwarfs are thought to be composed of carbon and oxygen, spectroscopy typically shows that their emitted light comes from an atmosphere which is observed to be either H-dominated or He-dominated. (see Figure 14.4). The dominant element is usually at least 1000 times more abundant than all other elements. The cause for this 'purity' is thought to be the high surface gravity which leads to a stratification of the atmosphere, with heavy elements on the bottom and lighter ones on top. This vertical stratification takes only ~ 100 years to become established.

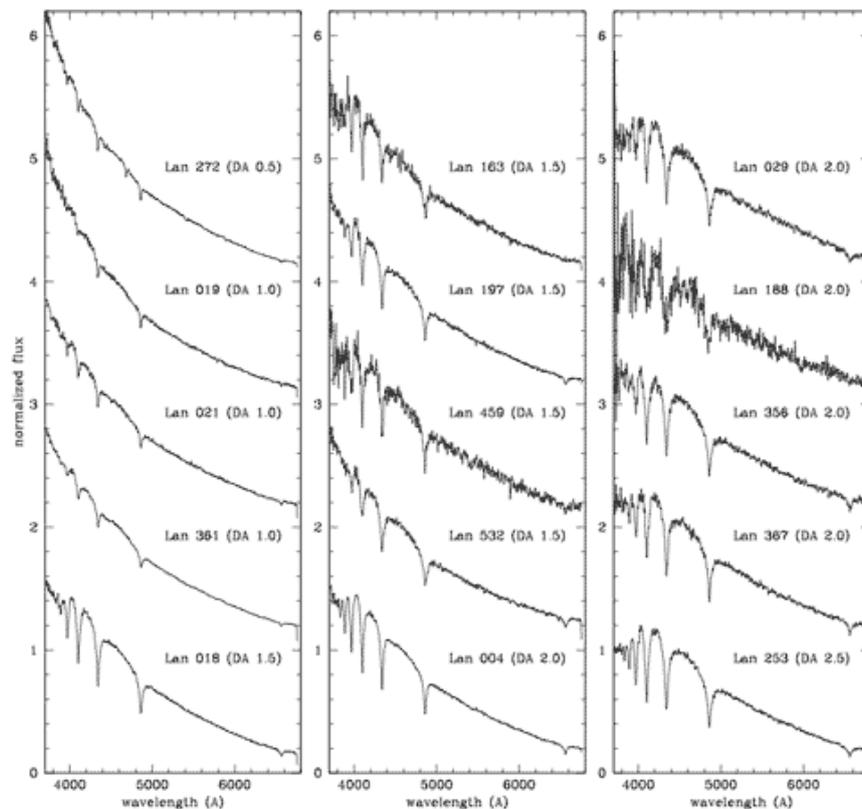


Figure 14.4: Optical spectra of white dwarfs of spectral type DA. Note the blue continuum, indicative of high effective temperatures ($T_{\text{eff}} > 10\,000$ K), and the pressure-broadened (Lecture 6.5.2) H I Balmer absorption lines.

The atmosphere, which is the only part of the white dwarf visible to us, is thought to be the top of an envelope which is a residue of the star's envelope in the AGB phase and may also contain material accreted from the interstellar medium. The envelope is believed to consist of a He-rich layer with mass no more than 1/100 of the star's total mass which, if the atmosphere is H-dominated, is overlaid by a H-rich layer with mass approximately 10^{-4} of the star's total mass.

The current classification scheme uses the letter D, followed by another letter to describe the primary spectral feature. The main classes are DA (H-dominated spectrum) and DB (He-dominated spectrum), with DAs making up $\sim 80\%$ of all observed white dwarfs and DB $\sim 16\%$. The remainder includes DC white dwarfs (continuum spectrum, no lines), DQ (carbon lines present) and DZ (metal lines present).

14.2 Electron Degeneracy Pressure

Without an internal supply of energy, the pressure to support white dwarfs against the pull of gravity is provided by electron degeneracy. We have already introduced the general principle of electron degeneracy in Lecture 12.6; we will now calculate in detail the pressure that the electrons can provide, P_e , as a function of their density ρ_e . We shall use some of the concepts that you have already encountered in the *Principles of Quantum Mechanics* and in the *Statistical Mechanics* Part II Astrophysics courses.

14.2.1 Density of States

To derive $P_e = f(\rho_e)$, we start by considering the density of states for free electrons: how many free-electron states fit into a box of volume $V = L^3$. We should think of this box as filling all space, via periodic replication. This means that the wave vectors of the free-electron quantum states can only take certain values. If the electron wave function is $\psi \propto \exp(i\mathbf{k} \cdot \mathbf{x})$, where $\mathbf{k} = (k_x, k_y, k_z)$, then the requirement of periodicity implies:

$$k_x = n_x \frac{2\pi}{L} \quad \text{where } n_x = 1, 2, \dots \quad (14.1)$$

The allowed states therefore lie on a lattice with spacing $2\pi/L$, and the density of states in k space is

$$dN = g \frac{L^3}{(2\pi)^3} d^3k \quad (d^3k \equiv dk_x dk_y dk_z), \quad (14.2)$$

where g is a degeneracy factor for spin.

The de Broglie relation between the electron's momentum to its wave vector: $\mathbf{p} = \hbar\mathbf{k}$ allows us to convert the density of states in k space to that in momentum space:

$$dN = g \frac{L^3}{(2\pi\hbar)^3} d^3p. \quad (14.3)$$

The number density of particles (per unit volume) with momentum states in the range of d^3p is then:

$$dn = g \frac{1}{(2\pi\hbar)^3} f(p) d^3p \quad (14.4)$$

where $f(p)$ is the occupation number of the mode, i.e. the number of particles in the box with that particular wave function. For bosons (e.g. photons), $f(p)$ is unrestricted. But fermions (such as electrons with spin angular momentum $\hbar/2$) obey Pauli exclusion principle, which states that:

$$f(p) \leq 1.$$

This criterion immediately imposes a restriction on how dense an electron gas can be before it has to be treated in a manner very different from the classical one. Normally, the distribution of momenta of the particles would be treated as a Maxwellian distribution, with each component of velocity having a Gaussian distribution with standard deviation:

$$\Psi(v) = \frac{1}{(2\pi\sigma^2)^{3/2}} \exp[-v^2/2\sigma^2] d^3v, \quad (14.5)$$

where v is the particle velocity. The dispersion in velocities, σ , is related to the temperature by equipartition of energy: $\sigma^2 = kT/m$. We can convert this to a number density of particles in a given range of momentum space, by multiplying by the total number density of particles, n , and using $p = mv$:

$$dn = \frac{n}{(2\pi mkT)^{3/2}} \exp[-p^2/2mkT] d^3p. \quad (14.6)$$

Comparing the Maxwellian distribution of particle momenta in eq. 14.6 with the general expression for the number density of particles in momentum space given by eq. 14.4, we deduce that there is a *critical density* at which the classical law would yield $f(p) > 1$ (at $p = 0$ which is where eq. 14.6 peaks):

$$n_{\text{crit}} = \frac{g}{(2\pi)^{3/2}} \frac{(mkT)^{3/2}}{\hbar^3} \quad (14.7)$$

At a fixed density, the gas will be in the classical regime at high values of T , but quantum effects become important as $T \rightarrow 0$.

Integrating eq. 14.4 over all momentum states gives the total number density of particles:

$$n = g \frac{1}{(2\pi\hbar)^3} \int f(p) d^3p. \quad (14.8)$$

In the limit of zero temperature, states are occupied only up to the *Fermi momentum*, p_F . Hence:

$$n = g \frac{1}{(2\pi\hbar)^3} \int_0^{p_F} d^3p = g \frac{1}{(2\pi\hbar)^3} \frac{4\pi}{3} p_F^3. \quad (14.9)$$

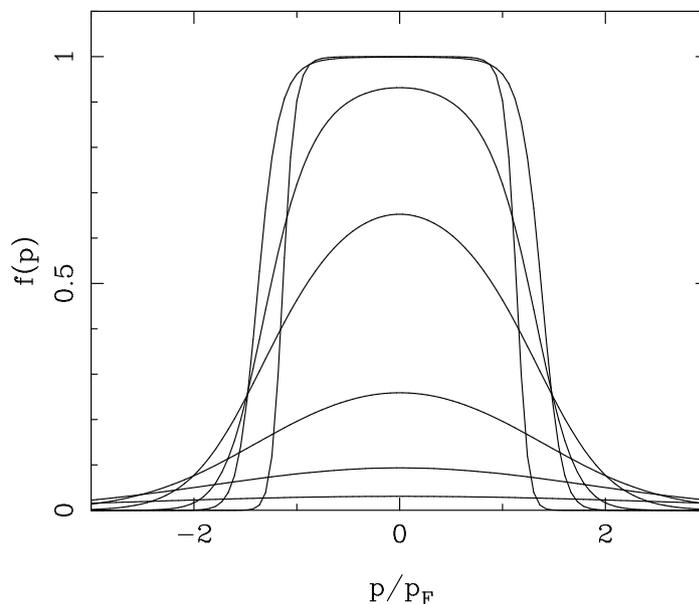


Figure 14.5: The occupation number for a gas of fermions as a function of their density relative to the critical density, ranging from $n/n_{\text{crit}} = 0.03$ to $n/n_{\text{crit}} = 30$. For the lowest densities (or highest temperatures), we have almost exactly the classical Maxwellian velocity distribution. For densities well above critical, the occupation number tends to a ‘top-hat’ distribution: unity for momenta less than the Fermi momentum, and zero otherwise.

Thus, the Fermi momentum is related to the particle density by:

$$p_F = 2\pi\hbar \left(\frac{3}{4\pi g} \right)^{1/3} n^{1/3} \quad (14.10)$$

As the density of the gas goes up, the Fermi momentum increases: the additional particles have to fill higher momentum states because the lower momentum states are fully occupied (see Figure 14.5).

14.2.2 Degeneracy Pressure

We can use the above treatment to work out an expression for the pressure exerted by degenerate electrons. Recall that pressure can be thought of as a flux of momentum. If the number density of electrons is n_e , then the flux of electrons in the x -direction is just the number of electrons crossing a unit area per unit time, or $n_e v_x$. The pressure is then approximately:

$$P_e \simeq p_x n_e v_x. \quad (14.11)$$

The contribution to the total pressure in the x -direction from all electrons with momentum p_x is then just:

$$dP_x = p_x v_x dn_{e,x} \quad (14.12)$$

where $dn_{e,x}$ is the number density of electrons with x -momentum in the range p_x to $p_x + dp_x$. Using eq. 14.4, we obtain for the total pressure in the x -direction (which by isotropy must be equal to the pressure in any direction):

$$P = P_x = g \frac{1}{(2\pi\hbar)^3} \int p_x v_x f(p) d^3p. \quad (14.13)$$

Using spherical polar coordinates in momentum space:

$$\int p_x v_x dp_x dp_y dp_z = \frac{1}{3} \int (p_x v_x + p_y v_y + p_z v_z) dp_x dp_y dp_z = \frac{1}{3} \int p \cdot v 4\pi p^2 dp, \quad (14.14)$$

so that:

$$P = \frac{g}{3} \frac{1}{(2\pi\hbar)^3} \int_0^\infty p \cdot v f(p) 4\pi p^2 dp. \quad (14.15)$$

In the zero-temperature limit (eq. 14.9), if the electron speeds are **non-relativistic** so that $p \cdot v = p^2/m_e$, we have:

$$P_e = \frac{4\pi g}{3(2\pi\hbar)^3} \int_0^{p_F} \left(\frac{p^2}{m_e}\right) p^2 dp = \frac{g}{30\pi^2\hbar^3 m_e} p_F^5. \quad (14.16)$$

What we want is the pressure as a function of density $\rho_e = m_e n_e$. Rewriting eq: 14.10 in terms of ρ_e :

$$p_F = \left(\frac{6\pi^2\hbar^3 \rho_e}{gm_e}\right)^{1/3} \quad (14.17)$$

and substituting into (14.16), we finally arrive at the result stated in Lecture 12.6.1:

$$P_e = \frac{g}{30\pi^2\hbar^3} \left(\frac{6\pi^2\hbar^3}{g}\right)^{5/3} \rho_e^{5/3} m_e^{-8/3} \equiv K_1 \rho_e^{5/3} \quad (14.18)$$

with

$$K_1 = \frac{\pi^2\hbar^2}{5m_e^{8/3}} \left(\frac{6}{g\pi}\right)^{2/3}.$$

Equation 14.10 implies that, at sufficiently high densities, the Fermi momentum can reach relativistic values, with some particles forced into momentum states with velocities approaching the speed of light. This leads to the **relativistic** expression for $P_e = f(\rho_e)$. With $v = c$ and $p \cdot v = pc$, we have:

$$P_e = \frac{4\pi g}{3(2\pi\hbar)^3} \int_0^{p_F} pc p^2 dp = \frac{gc}{24\pi^2\hbar^3} p_F^4 \quad (14.19)$$

or

$$P_e = \frac{gc}{24\pi^2\hbar^3} \left(\frac{6\pi^2\hbar^3 \rho_e}{gm_e}\right)^{4/3} \equiv K_2 \rho_e^{4/3} \quad (14.20)$$

with

$$K_2 = \frac{\pi\hbar c}{4m_e^{4/3}} \left(\frac{6}{g\pi}\right)^{1/3}.$$

What eqs. 14.18 and 14.20 show us is a result that has been stated before: in degenerate gas, the pressure depends only on density and is independent of temperature. Of course, in partially degenerate gas there will be a residual temperature dependence. Note also that the mass of the particle appears on the denominator in these equations. Thus, while electrons may

be degenerate, protons—with a mass $m_p/m_e = 1836$ times larger—exert negligible degeneracy pressure, justifying the fact that we have neglected it in the above treatment.

14.3 The Chandrasekhar Limit to the Mass of a White Dwarf

In July 1930, the bright, not-yet-twenty-year-old, indian student Subrahmanyan Chandrasekhar who had obtained his B.Sc. degree in Physics the previous month, was awarded a Government of India scholarship to pursue graduate studies at the University of Cambridge, where he was admitted to Trinity College and became a research student of R. H. Fowler. On his journey from India to Cambridge, Chandrasekhar worked out that there is a maximum mass for a white dwarf, now generally referred to as the Chandrasekhar limit and estimated to be $1.44M_\odot$. To see how this limit arises, consider the following.

We can estimate the energy density of degenerate gas as we did the pressure: by integrating over momentum space and including a term, $\epsilon(p)$, to denote the energy per mode:

$$U = g \frac{1}{(2\pi\hbar)^3} \int_0^\infty \epsilon(p) f(p) 4\pi p^2 dp. \quad (14.21)$$

In the zero-temperature limit, $f(p) = 1$ up to the Fermi momentum and $f(p) = 0$ at all other values. In the relativistic case, with $\epsilon(p) = pc$, integration of (14.21) gives:

$$U_e = g \frac{1}{(2\pi\hbar)^3} \frac{1}{4} 4\pi c p_F^4. \quad (14.22)$$

Using (14.10) to express U_e in terms of the number density of electrons, we have:

$$U_e = \frac{3}{4} \left(\frac{6\pi^2}{g} \right)^{1/3} \hbar c n_e^{4/3}. \quad (14.23)$$

In the non-relativistic case, $\epsilon(p) = p^2/2m_e$, which results in:

$$U_e = \frac{3\hbar^2}{10m_e} \left(\frac{6\pi^2}{g} \right)^{2/3} n_e^{5/3}. \quad (14.24)$$

The total kinetic energy supplied by the degenerate electrons is proportional to their energy density times the volume. In the relativistic case, $E_K \propto U_e V \propto n_e^{4/3} V \propto M^{4/3}/R$, where M is the mass and R is the radius. On the other hand, the gravitational energy is proportional to $-M^2/R$. The total energy is the sum of the two terms:

$$E_{\text{tot}} = \frac{(AM^{4/3} - BM^2)}{R} \quad (14.25)$$

where A and B are constants.

Now we see that there is a critical mass for which the two terms in the bracket are equal. If the mass is smaller than this limit, then the total energy is positive and will be reduced by making the star expand until the electrons reach the mildly relativistic regime and the star can exist as a stable white dwarf (see next section). But, if the mass exceeds the critical value, the binding energy increases without limit as the star shrinks: gravitational collapse has become unstoppable. This is believed to be the mechanism responsible for producing Type Ia supernovae which will be discussed in a later lecture.

To find the exact limiting mass, we need to find the coefficients A and B above, where the argument has implicitly assumed the star to be of constant density. In this approximation, the kinetic and potential energies are:

$$E_K = \left(\frac{243\pi}{128g}\right)^{1/3} \frac{\hbar c}{R} \left(\frac{M}{\mu m_p}\right)^{4/3} \quad (14.26)$$

where the mass per electron is μm_p , and

$$E_V = -\frac{3}{5} \frac{GM^2}{R} \quad (14.27)$$

Equating the two terms, we have:

$$M_{\text{crit}} = \frac{3.7}{\mu^2} \left(\frac{2}{g}\right)^{1/2} \left(\frac{\hbar c}{G}\right)^{3/2} m_p^{-2} \simeq \frac{7}{\mu^2} M_{\odot} \quad (14.28)$$

for $g = 2$. In a star which has burnt most of its initial fuel into elements heavier than Helium $\mu \simeq 2$, giving $M_{\text{crit}} = 1.75M_{\odot}$. More precise calculations, which take into account the density profile within the white dwarf, give $M_{\text{crit}} = 1.44M_{\odot}$.

14.4 The Mass-Volume Relation of White Dwarfs

As explained in the introduction, white dwarfs are extremely compact objects. To find their radius, we consider the non-relativistic case in which the energy density is $U_e \propto n_e^{5/3}$ (eq. 14.24), so that the total kinetic energy is $E_K = CM^{5/3}/R^2$, where C is a constant. As before, the gravitational potential energy is $E_V = -BM^2/R$, and the total energy is the sum of the two terms. The equilibrium radius is found by imposing the condition $dE_{\text{tot}}/dr = 0$, which gives:

$$R = \frac{2C}{B}M^{-1/3} \quad (14.29)$$

This is a remarkable relation. Since $V \propto R^3$, we have:

$$\boxed{M_{\text{wd}}V_{\text{wd}} = \text{constant}} \quad (14.30)$$

and more massive white dwarfs are actually smaller. This surprising result is a consequence of the star deriving its support from electron degeneracy pressure. The electrons must be more closely confined to generate the larger degeneracy pressure required to support a more massive star. Thus, the mass-volume relation implies that $\rho \propto M_{\text{wd}}^2$.

Note also that eq. 14.30 seemingly implies that the volume of a white dwarf can get infinitely small for an arbitrarily large mass. This is another way of recognising the existence of a critical mass; the former statement is incorrect because it ignores relativistic effects.

We can put numerical values into eq. 14.29 considering the following:

$$n_e = \frac{M}{\mu m_p} \left(\frac{4\pi R^3}{3} \right)^{-1} \quad (14.31)$$

which, when combined with (14.24), gives:

$$C = \frac{3\hbar^2}{10m_e} \left(\frac{6\pi^2}{g} \right)^{2/3} \left(\frac{1}{\mu m_p} \right)^{5/3} \left(\frac{4\pi}{3} \right)^{-2/3}. \quad (14.32)$$

Combined with our previous $B = 3G/5$, we get:

$$R = \frac{3}{2} \left(\frac{6\pi^2}{g^2} \right)^{1/3} \frac{\hbar^2}{Gm_e(\mu m_p)^{5/3}} M^{-1/3}. \quad (14.33)$$

Expressed in terms of the Chandrasekhar mass, this is:

$$R = 3\sqrt{\pi/5} \frac{\hbar}{\mu m_p m_e} \left(\frac{2\hbar}{gcG} \right)^{1/2} \left(\frac{M}{M_{\text{crit}}} \right)^{-1/3} \simeq 5975 \left(\frac{M}{M_{\text{crit}}} \right)^{-1/3} \text{ km}, \quad (14.34)$$

where again we have taken $g = 2$ and $\mu \simeq 2$. In other words, a Chandrasekhar-mass white dwarf is about the size of the Earth. This is an extremely high density, which can be evaluated to be:

$$\rho = \frac{M}{4\pi R^3/3} \simeq 10^{6.6} \left(\frac{M}{M_{\text{crit}}} \right)^2 \text{ g cm}^{-3} \quad (14.35)$$

This are only approximate expressions because they are based on the constant density approximation. If we wanted to do better, we would need to solve for the internal structure of a white dwarf.

In summary, non-relativistic white dwarfs are stable over a range of masses, but as their mass increases, they shrink. This must force the electrons closer and closer to becoming relativistic, and the limit of this is a star in which the electrons are all relativistic, which has a unique mass. It appears that electron degeneracy pressure has no way of supporting more massive bodies.

14.5 Cooling and the Ages of White Dwarfs

With nuclear reactions over, a white dwarf will slowly cool over time, radiating away its stored heat.¹ Much effort has been directed at understanding the process and timescale of white dwarf cooling, as it potentially provides a chronology of the past history of star formation in our Galaxy.

Recalling the discussion in Lecture 8, in normal stars the mean free path for photons is much greater than that of electrons or atoms; consequently energy transport is mainly by radiative diffusion. This is not the case in white dwarfs, where the degenerate electrons can travel long distances before losing energy in a collision with a nucleus, since the vast majority of lower-energy electron states are already occupied. Thus, in a white dwarf

¹This idea was developed in 1952 by Leon Mestel, who had recently graduated from Cambridge (he was at Trinity College) and was on his first postdoctoral appointment at the University of Leeds. He returned to Cambridge as a lecturer between 1955 and 1966.

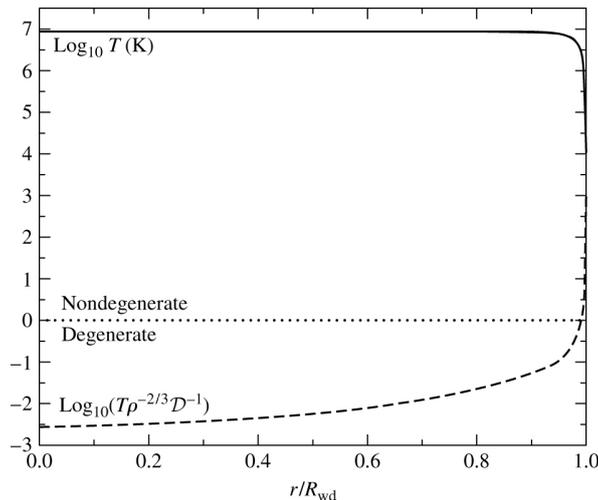


Figure 14.6: Temperature and degree of degeneracy in the interior of a model white dwarf. The condition for degeneracy is defined as $T/\rho^{2/3} < \mathcal{D}$, where $\mathcal{D} = (\hbar^2/3m_e k)(3\pi^2/\mu m_p)^{2/3}$. The smaller $T\rho^{-2/3}$, the more degenerate the gas.

energy is carried by electron conduction (similar to conduction in metals), rather than by radiation.

Electron conduction is so efficient that the interior of a white dwarf is nearly isothermal, with the temperature dropping significantly only in the non-degenerate surface layers (Figure 14.6). The thin ($\sim 1\%$ of the white dwarf radius), non-degenerate envelope transfers heat less efficiently, and acts as an insulating blanket, causing energy to leak out slowly. The steep temperature gradient near the surface creates convection zones that can alter the appearance of a white dwarf's spectrum as it cools (Section 14.1.1).

We can calculate the initial temperature of a white dwarf by recalling that it forms from the contraction of a thermally unsupported stellar core, of mass M , down to the radius at which degeneracy pressure stops the contraction. The virial theorem tells us that, just before reaching the final point of equilibrium, the thermal energy will equal half of the potential energy:

$$E_{\text{th}} \sim \frac{1}{2} \frac{GM^2}{R}. \quad (14.36)$$

For a pure He composition, the number of nuclei in the core is $M/4m_p$, and the number of electrons is $M/2m_p$. The total thermal energy (which, once degeneracy sets in, will no longer play a role in supporting the star against gravity) is therefore:

$$E_{\text{th}} = \frac{3}{2} N k T = \frac{3}{2} \frac{M}{m_p} \left(\frac{1}{2} + \frac{1}{4} \right) k T = \frac{9}{8} \frac{M}{m_p} k T, \quad (14.37)$$

so that:

$$kT \sim \frac{4}{9} \frac{GMm_p}{R}. \quad (14.38)$$

Substituting the equilibrium radius of a white dwarf from eq. 14.33, we find:

$$kT \propto M^{4/3} \simeq 1.1 \times 10^{-7} \text{ erg} \sim 70 \text{ keV} \quad (14.39)$$

for a $0.5M_\odot$ white dwarf. With $k = 1.38 \times 10^{-16} \text{ erg K}^{-1}$, the corresponding temperature is $T \sim 8 \times 10^8 \text{ K}$. Clearly, a just-formed degenerate core is a very hot object, with thermal emission that peaks at X-ray wavelengths. This is why, once the core becomes an exposed white dwarf, its radiation ionises the layers of gas that were blown off during the AGB phase, giving rise to a planetary nebula.

The energy radiated away from the surface of a white dwarf is the thermal energy stored in the still classical gas of *nuclei* within the star's volume. (The degeneracy of the electron gas limits almost completely the ability of the electrons to lose their kinetic energies). We can obtain an *upper limit* to the cooling rate by neglecting the envelope and assuming a uniform temperature throughout. The radiative energy loss is obtained by equating the luminosity of the blackbody, given by the Stefan-Boltzmann law, to the rate of change of thermal energy:

$$L = 4\pi R_{\text{WD}}^2 \sigma T^4 \sim \frac{dE_{\text{th}}}{dt} = \frac{3}{8} \frac{Mk}{m_p} \frac{dT}{dt} \quad (14.40)$$

where we have included only the contribution of nuclei to the thermal energy from eq. 14.37.

Separating the variables T and t and integrating, the cooling time to a temperature T is:

$$\tau_{\text{cool}} \sim \frac{3}{8} \frac{Mk}{m_p} \frac{1}{4\pi R_{\text{WD}}^2 \sigma} \frac{1}{3} \frac{1}{T^3}. \quad (14.41)$$

Using as before eq. 14.33 for R_{WD} and entering the numerical values, we find:

$$\tau_{\text{cool}} = 3 \times 10^9 \text{ yr} \left(\frac{T}{10^3 \text{ K}} \right)^{-3} \quad (14.42)$$

for a $M = 0.5M_\odot$ white dwarf. Writing the temperature as a function of time:

$$\frac{T}{10^3 \text{ K}} \sim \left(\frac{t}{3 \times 10^9 \text{ yr}} \right)^{-1/3} \quad (14.43)$$

Thus, even with the unrealistically efficient cooling we have assumed, it would take a $0.5M_{\odot}$ white dwarf several Gyr to cool to 10^3 K. In reality, the insulation provided by the non-degenerate envelope results in an effective temperature that is significantly lower than the interior temperature (Figure 14.6), and lowers the cooling rate. Furthermore, as a white dwarf cools, it *crystallises* in a gradual process that starts at the centre and moves outwards. The regular crystal structure is maintained by the mutual electrostatic repulsion of the nuclei: it minimises their energy as they vibrate about their average position in the lattice. As the nuclei undergo this phase change, the latent heat that is released is added to the thermal balance, further slowing down the decline in temperature.

Detailed models that include these and other processes for various masses and chemical compositions of white dwarfs show that over a period of 10 Gyr, comparable to the age of the Universe, white dwarfs cannot cool below ~ 3000 – 4000 K. This explains why most white dwarfs are observed to have high temperatures, and hence their blue to white colours. The coolest white dwarfs known have effective temperatures $T_{\text{eff}} \simeq 3500$ K.

14.5.1 White Dwarfs as Fossil Records of Star Formation

The interest in an accurate calculation of white dwarf cooling stems in part from the fact that, with cooling times comparable to the age of the Universe, it may be possible to use these fossil stars to uncover the history of star formation in our Galaxy by modelling their luminosity function as in Figure 14.7. The model luminosity function shown in the Figure was calculated using the best available models of white dwarf cooling and reproduces well the sudden drop in the population of white dwarfs with luminosities $\log L_{\text{wd}}/L_{\odot} < -4.5$. This decline can best be explained if the first white dwarfs were formed and started cooling 9.0 ± 1.8 Gyr ago. Including the pre-white dwarf evolutionary time, one would arrive at an age of 9.3 ± 2.0 Gyr for the Galactic disk. The model shown in Figure 14.7 assumes that star formation proceeded at the same rate throughout this time. This is an oversimplification, and more realistic models of the past history of star formation in the Milky Way may result in better agreement between calculated and observed luminosity functions.

As a final comment, it is interesting to note that the paper which first

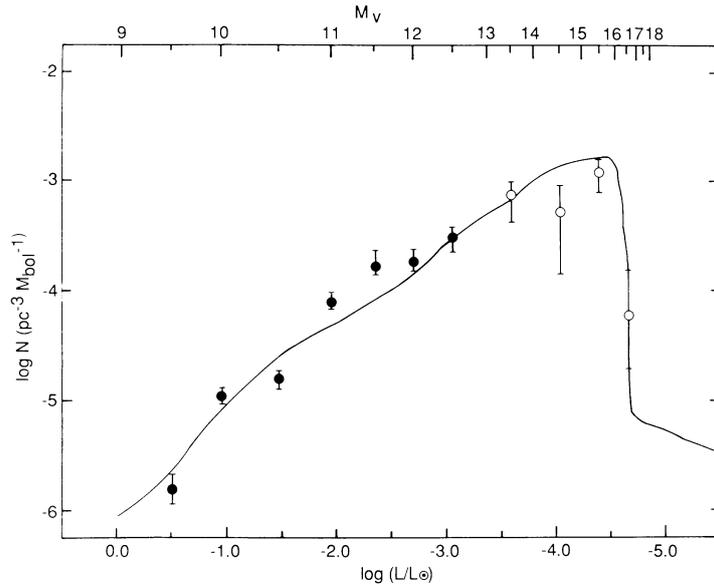


Figure 14.7: Observed (circles) and calculated (continuous line) luminosity function of Galactic disk white dwarfs, reproduced from Winget et al. (1987).

made use of white dwarfs as ‘cosmic clocks’, by Winget et al. (1987, ApJ, 315, L77), was entitled: “*An Independent Method for Determining the Age of the Universe*”. This reflects the fact that the paper predates the current era of ‘precision cosmology’, achieved primarily (but not exclusively) through precise measurement of the spectrum of temperature anisotropies of the cosmic microwave background, which encodes most fundamental cosmological parameters. Today we know the age of the Universe to be 13.80 ± 0.02 Gyr, a precision far superior than any that could possibly be achieved by modelling the luminosity function of white dwarfs. But, by turning the argument around, we can now use this well determined age, together with models of galaxy formation, to refine our understanding of white dwarf cooling.