STELLAR MODELS

9.1 Introduction

Stars are complex physical systems, but not too complex to be modelled numerically and, with some simplifying assumptions, analytically. This is not always the case in astrophysics. For example, the interstellar medium in galaxies is a very complex environment and we are still a long way from being able to satisfactorily model it in our computers. On the other hand, the evolution of the whole Universe can be described with the Friedmann equations and once the fundamental cosmological parameters are known with sufficient accuracy, its past history can be deduced and its future evolution predicted.

The aim of stellar models is to construct a representation of a star which is physically accurate to a sufficient degree to reproduce the observed properties of stars, and thereby obtain a physical understanding of what determines those properties. For example, in the first few lectures of the course we saw that most stars fall in narrow strip on the colour-magnitude diagram (Figure 2.8); with the aid of stellar models we now interpret this Main Sequence as the locus of stars during their hydrogen burning phase. Stellar models can help us understand how stars move onto the Main Sequence when they are born out of an interstellar cloud, and how they evolve off the Main Sequence—that is the tracks they follow in the colour-magnitude plane during the late stages of their evolution (see Figure 9.1). In Lecture 4 we also saw that stars obey a mass-luminosity relation (Figure 4.8) and that there is a strong dependence of stellar lifetimes on stellar mass (Figure 4.9); again, a successful stellar model should be able to reproduce these relationships and thereby provide physical insight into their origin.

9.2 Static Models

In this lecture we are going to consider the simplest class of stellar models. We assume that stars:

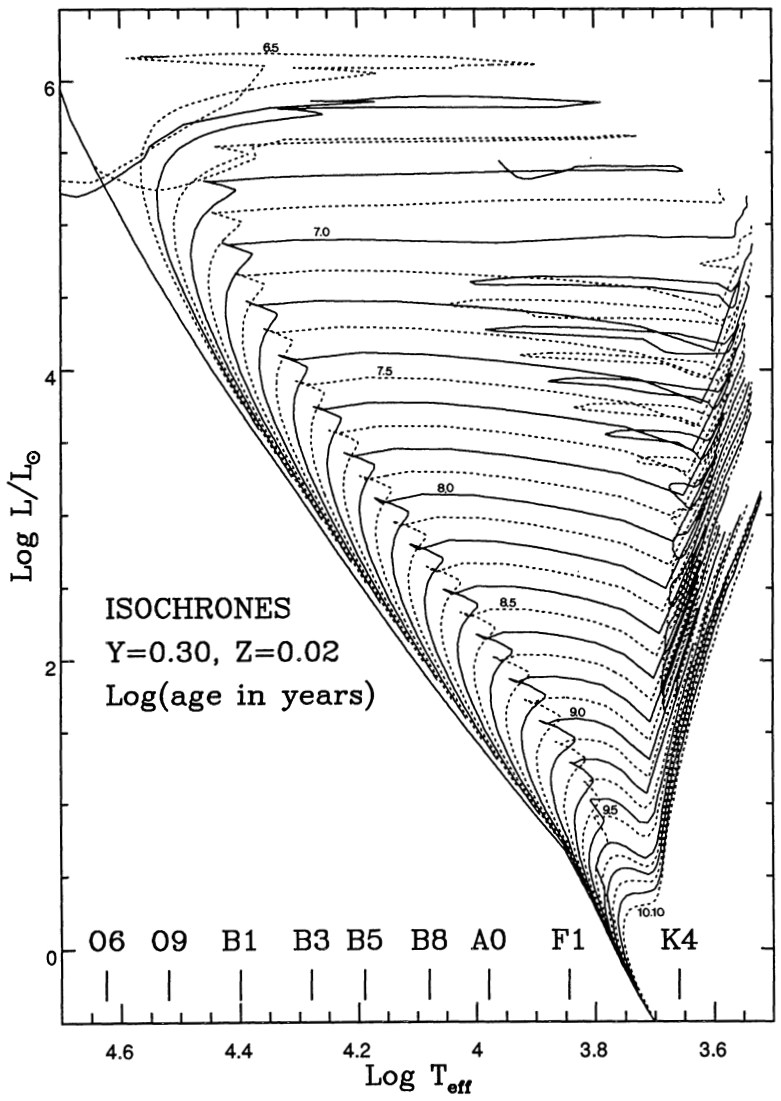

Figure 9.1: Theoretical H-R diagram computed with the Geneva stellar models which include moderate overshooting. (Reproduced from Meynet et al. (1993), A&A Supp., 98, 477).

- are spherically symmetric,
- are static,
- have no magnetic field.

None of these conditions apply to real stars. Real stars rotate, and as a consequence they are slightly 'flattened' at the poles compared to the equator. In neglecting this slight asymmetry, we make the assumption that the additional centripetal force acting on a test mass at the equator of the star is small compared to the gravitational force. This inequality can be also expressed in terms of timescales: departures from spherical

symmetry can be neglected when the period of rotation is much longer than the dynamical time, i.e. $P \gg \tau_{\rm dyn}$. This condition is certainly satisfied in the Sun, where $P \sim 30\,$days, while $\tau_{\rm dyn} \sim 2000\,$s. (Incidentally, in the Sun the flattening at the poles is only $1 \times 10^{-5} R_\odot$). However, some stars rotate much faster than the Sun and in some cases rotational effects must be included in the structure equations—and can have a big effect on the output of the models.

Similar arguments can be used to argue for static models, by which we mean models which do not include time as a variable. So long as the timescale for evolutionary changes is much longer than the dynamical timescale, time-dependent effects can be neglected. This is certainly the case while stars are on the Main Sequence, but the approximation may no longer be valid during periods when the star expands or contracts (or pulsates). The inclusion of magnetic fields in the stellar models is beyond the scope of these lectures.

To build a stellar model for a star in hydrostatic and thermal equilibrium[1], we begin with four differential equations. We have already encountered three of these, but we shall recapitulate now.

1. **The equation of Mass Continuity**:

$$\boxed{\frac{dm}{dr} = 4\pi r^2 \rho} \tag{9.1}$$

   (see Figure 7.1).

2. **The equation of Hydrostatic Equilibrium**:

$$\boxed{\frac{dP}{dr} = -G\frac{M_r\,\rho}{r^2}} \tag{9.2}$$

   (see Figure 8.7).

3. **The equation of Thermal Equilibrium**:

$$\boxed{\frac{dL_r}{dr} = 4\pi r^2 \rho\,\mathcal{E}} \tag{9.3}$$

   to which we shall return in a moment, and

[1]That is, a star for which $\tau_{\rm dyn}$ (Lecture 8.4.2) $\ll \tau_{\rm KH}$ (Lecture 7.1) $\ll \tau_{\rm nuclear}$ (Lecture 7.2).

4. **The equations of Energy Transport**:

$$\boxed{\frac{dT}{dr} = -\frac{3}{4} \cdot \frac{1}{ac} \cdot \frac{\kappa\rho}{T^3} \cdot \frac{L_r}{4\pi r^2}} \tag{9.4}$$

i.e. the Eddington equation for radiative transport, or

$$\boxed{\frac{dT}{dr} = -\frac{\gamma - 1}{\gamma} \frac{\mu m_{\rm H}}{k} \frac{GM_r}{r^2}} \tag{9.5}$$

for energy transport by adiabatic convection.

Eq. 9.3 follows directly from 9.1. We can write the contribution to the total luminosity of a star by an infinitesimal mass dm as:

$$dL = \mathcal{E}\, dm$$

where $\mathcal{E}$ is the rate of energy release per unit mass (erg s^{-1} g^{-1}) by all processes. In a static model, $\mathcal{E} = \mathcal{E}_{\rm nuclear}$ but, more generally, if the layer of the star is expanding or contracting, $\mathcal{E} = \mathcal{E}_{\rm nuclear} + \mathcal{E}_{\rm gravity}$, where the second term is negative if the star is expanding. Rigorously, even in the static case, one should also include the energy lost from escaping neutrinos, i.e. $\mathcal{E} = \mathcal{E}_{\rm nuclear} - \mathcal{E}_{\nu}$. In any case, using eq. 9.1, we have:

$$\frac{dL_r}{dr} = 4\pi r^2 \rho\, \mathcal{E}\,.$$

So, we have five coupled differential equations describing the run of mass, pressure, luminosity and temperature within a star with r, the distance from the centre of the star, as the independent variable. Note that the first two equations describe the *mechanical* structure of the star, while the last three describe the *energy and thermal* structure. The equations are coupled to each other through the fact that, for a general equation of state, P is a function of both ρ and T.

More specifically, in order to tackle these five equations, we need expressions for the pressure P, the opacity κ, and the energy generation rate $\mathcal{E}$, in terms of the fundamental physical characteristics of the plasma, that is density ρ, temperature T, and chemical composition. Thus we have three additional relations, which are sometimes referred to as the 'Constitutive Relations':

1. **The nuclear energy production rate**:

$$\mathcal{E} \propto \mathcal{E}_0 \, \rho^{\alpha} \, T^{\beta} \tag{9.6}$$

where the exponents α and β and the constant of proportionality depend on the specific nuclear reaction rate (for hydrogen burning stars, the p-p chain or the CNO cycle), as we saw in Lecture 7. Thus, implicit in eq. 9.6 is a dependence on the chemical composition of the star.

2. As we saw in Lecture 5, **the opacity** κ is the sum of several processes: $\langle \kappa \rangle = \langle \kappa_{\rm bb} + \kappa_{\rm bf} + \kappa_{\rm ff} + \kappa_{\rm es} + \kappa_{\rm H^-} \rangle$. Of these, two of the main sources of opacity, $\kappa_{\rm bf}$ and $\kappa_{\rm ff}$ have functional form:

$$\langle \kappa_{\rm bf} \rangle = \kappa_{0,bf} \, \rho \, T^{-3.5} \tag{9.7}$$

where the constant of proportionality $\kappa_{0,bf}$ depends on the composition of the gas.

3. **The Pressure** has two components: gas pressure and radiation pressure:

$$P = P_g \, + \, P_{\rm rad} \, = \, \frac{\rho k T}{\mu m_{\rm H}} \, + \, \frac{1}{3} a T^4 \tag{9.8}$$

We discuss each component in the following two subsections.

9.2.1 Mean Molecular Weight

The P_g term in eq. 9.8 is simply the ideal gas law, $PV = NkT$, expressed in terms of the density $\rho = N \langle m \rangle / V$, where N is the number of particles, and the average mass of a particle, $\langle m \rangle = \mu \, m_{\rm H}$. μ is the *mean molecular weight* which we have already encountered several times. Let us look at it more closely.

In the interiors of stars, the gas is a mixture of atoms and ions of different elements, in a range of ionisation stages. We want to know what is the mean mass of a particle in this gas mixture.

$$\langle m \rangle = \frac{\sum_j n_{j,\rm I} m_{j,\rm I} + n_{\rm e} m_{\rm e}}{\sum_j n_{j,\rm I} + n_{\rm e}} \approx \frac{\sum_j n_{j,\rm I} m_{j,\rm I}}{\sum_j n_{j,\rm I} + n_{\rm e}} \tag{9.9}$$

where $n_{j,\mathrm{I}}$ is the number density of ion j, $m_{j,\mathrm{I}}$ is its mass, and n_e and m_e are, respectively, the number density of electrons and the electron mass. Since $m_\mathrm{e} \ll m_{j,\mathrm{I}}$ for all atoms of interest, we set $m_\mathrm{e} = 0$ for simplicity.

Further, the mass of the jth ion is approximately its number of protons and neutrons (i.e. its atomic mass number A_j) times the atomic mass unit, or $m_{j,\mathrm{I}} = A_j m_\mathrm{u}$. So we can define the mean molecular weight, μ as:

$$\mu = \frac{\langle m \rangle}{m_\mathrm{u}} = \frac{\sum_j n_{j,\mathrm{I}} A_j}{\sum_j n_{j,\mathrm{I}} + n_\mathrm{e}} \tag{9.10}$$

Thus, the mean molecular weight of the gas can be interpreted as the average mass per particle in units of the amu. For simplicity, and in order to avoid using too many symbols, sometimes we use the mass of the hydrogen atom, m_H as a proxy for m_u. This 'sloppiness' is inconsequential for our present purposes.

As an example, let us calculate the value of μ for fully ionised gas (a reasonably good assumption in the *interiors* of stars, where the most abundant elements, H and He, are fully ionised. However, the assumption is no longer valid in cool stellar atmospheres).

Recall our definition (Lecture 1) of X, Y, Z as the mass fractions, respectively, of H, He and 'everything else'. Thus, in a unit volume of density ρ, the mass of H is $\mathrm{X}\rho$, that of He is $\mathrm{Y}\rho$ and so on. In a fully ionised plasma, H will contribute two particles (one proton and one electron) per m_H. He will contribute 3/4 particles per m_H: two electrons and one alpha particle (the He nucleus) which weighs $4\,m_\mathrm{H}$. Heavier elements will in general give 1/2 particles per m_H: fully ionised C, 7/12, fully ionised O, 9/16 and so on. Thus, the total number of particles per unit volume is:

$$n = \frac{2\mathrm{X}\rho}{m_\mathrm{H}} + \frac{3\mathrm{Y}\rho}{4m_\mathrm{H}} + \frac{\mathrm{Z}\rho}{2m_\mathrm{H}} \tag{9.11}$$

or

$$n = \frac{\rho}{4m_\mathrm{H}} \cdot (8\mathrm{X} + 3\mathrm{Y} + 2\mathrm{Z})\,. \tag{9.12}$$

Recalling that $\mathrm{X} + \mathrm{Y} + \mathrm{Z} = 1$,

$$n = \frac{\rho}{m_\mathrm{H}} \cdot \frac{6\mathrm{X} + \mathrm{Y} + 2}{4}\,. \tag{9.13}$$

and that the density is just $\rho = n \langle m \rangle = n\, m_{\rm H}\, \mu$, we have:

$$\mu = \frac{4}{6{\rm X} + {\rm Y} + 2} \simeq 0.6 \tag{9.14}$$

for a fully ionised plasma of solar composition, where X = 0.75, Y = 0.235 and Z = 0.015.

9.2.2 Radiation Pressure

Einstein's relativistic energy equation

$$E^2 = p^2 c^2 + m^2 c^4 \tag{9.15}$$

expresses the total energy of a particle as the sum of its momentum p and rest-mass mc^2 contributions. Even though photons are massless particles, they do have a momentum $p = E/c = h\nu/c$ associated with their frequency $\nu = c/\lambda$. This momentum can be transferred to particles during absorption and scattering, that is photons can exert *radiation pressure*. In many astrophysical situations, the second term on the right-hand side of eq. 9.8 can be greater than the first term. In some cases, the force exerted by photons can even exceed the gravitational force, resulting in an overall expansion of the system. Radiation pressure is, for example, believed to be the force that triggers mass-loss through stellar winds in the most luminous stars.

For an isotropic radiation field of intensity I_λ, the radiation pressure per unit wavelength interval is simply:

$$P_{{\rm rad},\lambda}\, d\lambda = \frac{1}{c} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} I_\lambda\, d\lambda\, \cos^2\theta\, \sin\theta\, d\theta\, d\phi \;=\; \frac{4\pi}{3c} I_\lambda\, d\lambda \tag{9.16}$$

and, for blackbody radiation:

$$P_{\rm rad} = \frac{4\pi}{3c} \int_0^\infty B_\lambda(T)\, d\lambda \tag{9.17}$$

or

$$P_{\rm rad} = \frac{4\sigma T^4}{3c} \;=\; \frac{1}{3} a T^4 \equiv \frac{1}{3} u\,. \tag{9.18}$$

Thus, the blackbody radiation pressure is one third of the energy density (see Lecture 5). For comparison, the pressure of an ideal monoatomic gas is two thirds of its energy density.

9.2.3 The Equations of Stellar Structure in Terms of Mass

As we saw earlier, eqs. 9.1–9.5 describe the run of mass, pressure, luminosity and temperature within a star with r, the distance from the centre of the star, as the independent variable. This is sometimes referred to as the formulation of the stellar structure equations in Euler coordinates.

Alternatively, we could express these equations in terms of the mass $m = M(r)$, sometimes referred to as Lagrange coordinates. A formulation in terms of mass makes more sense when we consider that the mass of a star remains approximately constant during most of the star's lifetime, whereas its radius can change by orders of magnitude.

The Lagrangian equations can be derived from the Euler equations by using:

$$\frac{dx}{dm} = \frac{dx}{dr} \cdot \frac{dr}{dm}$$

and

$$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}$$

Thus we have:

1a. **Mass Continuity**:

$$\boxed{\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}} \tag{9.19}$$

2a. **Hydrostatic Equilibrium**:

$$\boxed{\frac{dP}{dm} = -G\frac{M_r}{4\pi r^4}} \tag{9.20}$$

3a. **Thermal Equilibrium**:

$$\boxed{\frac{dL}{dm} = \mathcal{E}} \tag{9.21}$$

4a. **Energy Transport**:

$$\boxed{\frac{dT}{dm} = -\frac{3}{4ac} \cdot \frac{\kappa}{T^3} \cdot \frac{L_r}{(4\pi r^2)^2}} \tag{9.22}$$

for radiative transport, and

$$\boxed{\frac{dT}{dm} = -\frac{\gamma - 1}{\gamma} \frac{\mu m_{\rm H}}{k} \frac{GM_r}{4\pi r^4 \rho}} \tag{9.23}$$

for convection.

9.3 Solving the Equations of Stellar Structure

9.3.1 Boundary Conditions

In order to solve these four differential equations (for either radiative or convective energy transport), we need to specify four boundary conditions. Boundary conditions are physical constraints to the mathematical equations defining the limits of integration. Two of the boundary conditions are obvious:

$$\left.\begin{aligned} M_r &= 0 \\ L_r &= 0 \end{aligned}\right\} \quad \text{at } r = 0 \tag{9.24}$$

Physically, this is simply saying that there isn't a core of infinite density or negative luminosity at the centre of a spherically symmetric star.

Unfortunately, we do not have similar boundary conditions for pressure and temperature in the stellar core. However, we can place some boundary conditions at the stellar surface. If stars had clear-cut surfaces, then appropriate boundary conditions might be:

$$\left.\begin{aligned} P &= 0 \\ T &= 0 \end{aligned}\right\} \quad \text{at } r = R_* \tag{9.25}$$

where R_* is the stellar radius, and indeed this is what is often adopted. The justification for this simplification is that $P(r = R_*) \ll \langle P \rangle$ and similarly, $T(r = R_*) \ll \langle T \rangle$. A more appropriate boundary condition for the temperature might be (eq. 2.15):

$$T(r = R_*) = \left[\frac{L}{4\pi\sigma R_*^2}\right]^{1/4} \tag{9.26}$$

An even better treatment would include a proper model atmosphere for the outer layers of the star.

Irrespectively of whether they are formulated in Eulerian or Lagrangian coordinates, the four independent equations of stellar structure cannot be solve analytically without making some simplifying assumptions. The reasons are that:

(i) the equations are very non linear. The energy generation rate $\mathcal{E}$ in the equation of thermal equilibrium (9.3 and 9.21) depends very steeply on temperature, $\mathcal{E} \propto \rho^{\alpha} T^{\beta}$ with $\beta \gg 1$, as we saw in Lecture 7.4. The opacity κ in the equation of radiative energy transport (9.4 and 9.22) is also a complicated function of ρ and T, as we saw in Lecture 5.3.

(ii) The four differential equations are coupled and have to be solved simultaneously.

(iii) There are two boundary conditions at $r = 0$ and two boundary conditions at $r = R_*$.

Thus, unless one makes some simplifying assumption,[2] the equations of stellar structure are normally integrated numerically. This is accomplished by replacing the differential equations with difference equations and approximating the internal structure of a star by a series of concentric shells separated by a small, but finite, radial distance δr. For example, if the pressure in shell i is P_i, the pressure in the next shell is $P_{i+1} = P_i + (\Delta P/\Delta r)\, \delta r$.

The numerical integration is carried out from the centre to the surface using some initial estimates of $P(r = 0)$ and $T(r = 0)$ and, simultaneously, from the surface towards the centre adopting some initial values of L and R_*. Out of a large number of possible models, the condition for a satisfactory model is a smooth transition in the values of all the quantities, P, T, L and their derivatives at some transition radius r'—see Figure 9.2. This is the method developed by Martin Schwarzschild in 1958.

With modern computing facilities, the equations are solved iteratively with multidimensional matrices. Figures 9.3, 9.4 and 9.5 are examples of stellar

[2] One such family of approximate solutions is known as polytropes, obtained by assuming that pressure and density are simply related through an equation of the form $P = K\rho^{\gamma}$, appropriate to an adiabatic gas.

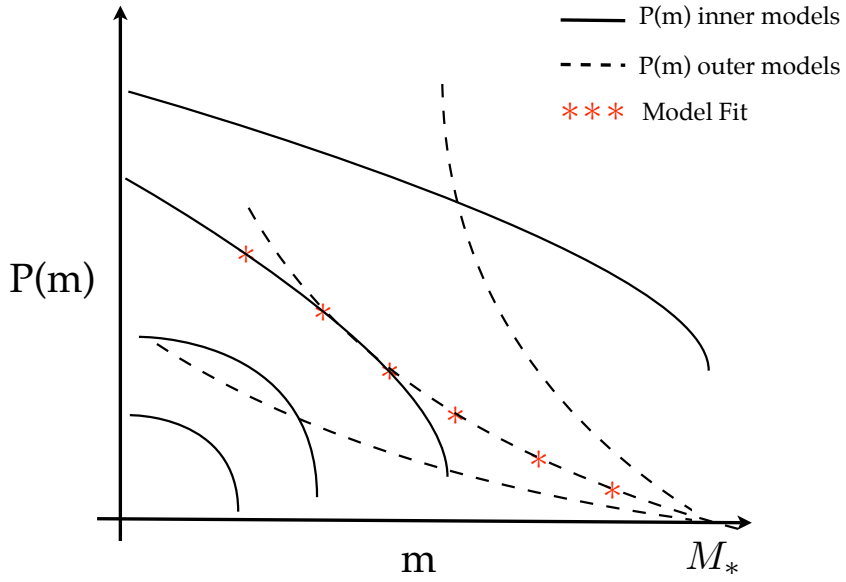

Figure 9.2: Illustration of M. Schwarzschild's method to solve iteratively the equations of stellar structure.

models computed with modern numerical methods for stars on the Main Sequence of masses, respectively 1, 3, and $15M_\odot$. Comparison between the different curves gives us insight into the internal structure of stars of different masses.

Figure 9.6 is an example of an application of some of the best current stellar models, showing a fully theoretical Main Sequence, covering a wide range of stellar masses from $0.4M_\odot$ to $120M_\odot$, as indicated.

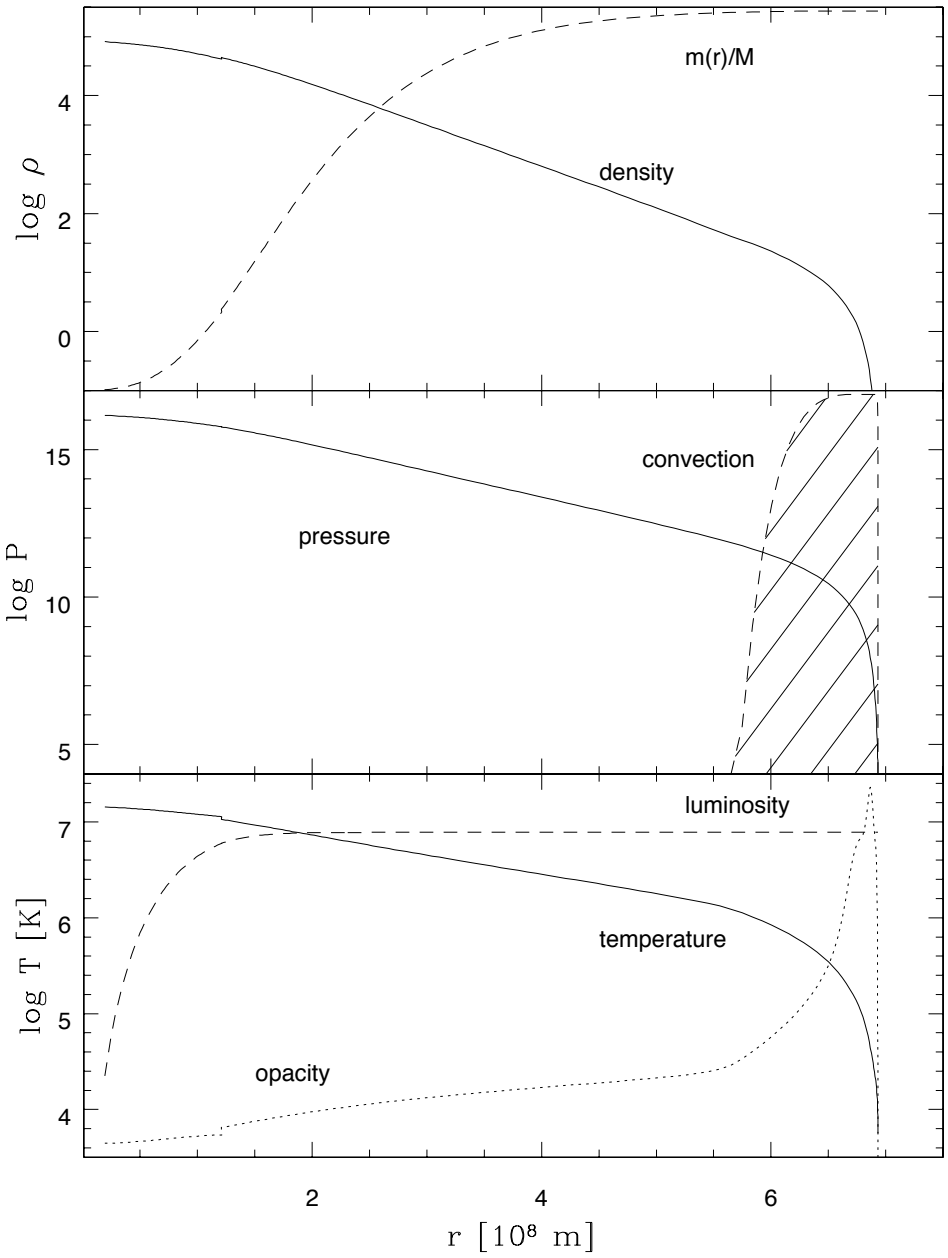

Figure 9.3: Internal structure of a $1M_\odot$ star computed with modern stellar models. (A. Zijlstra, private communication).

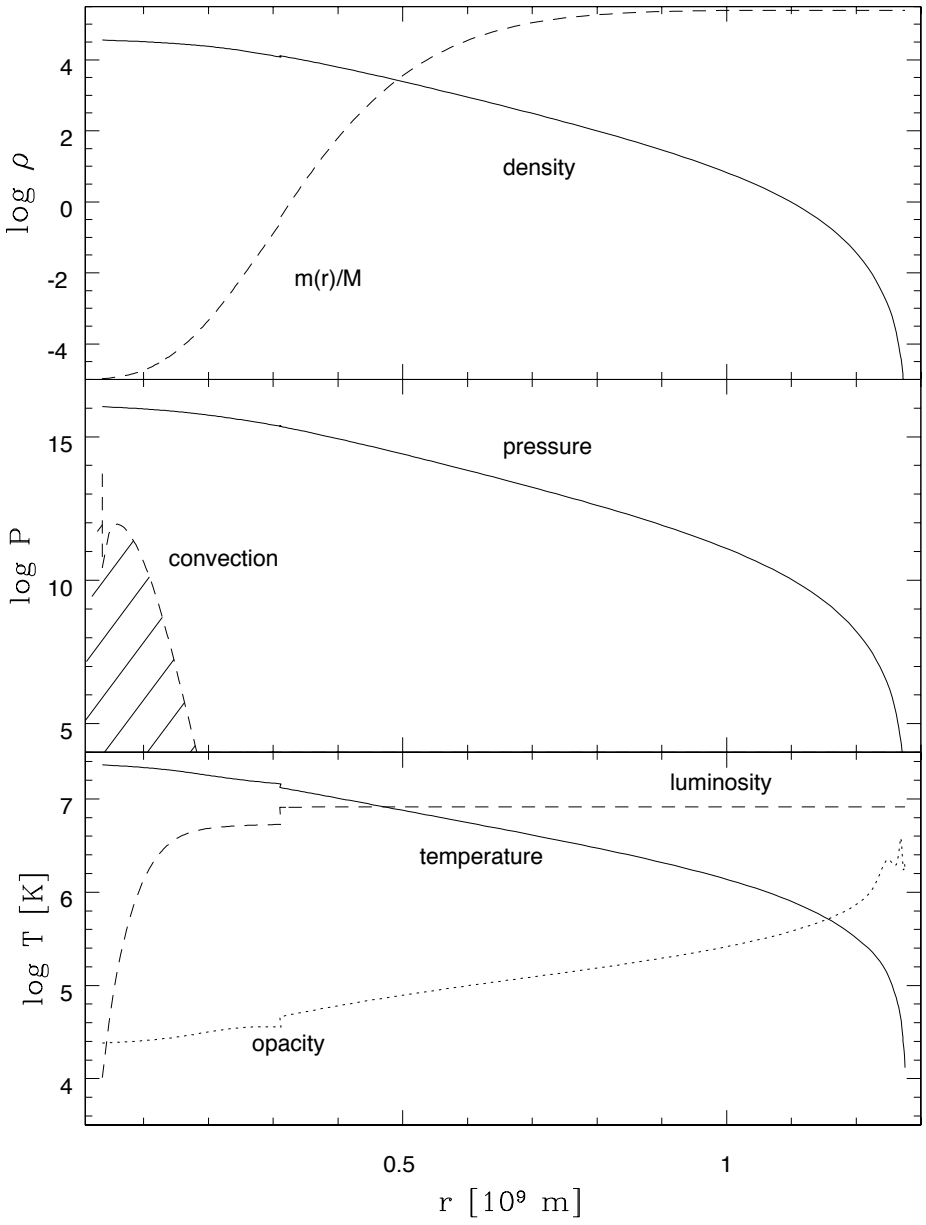

Figure 9.4: Internal structure of a $3M_\odot$ star computed with modern stellar models. (A. Zijlstra, private communication).

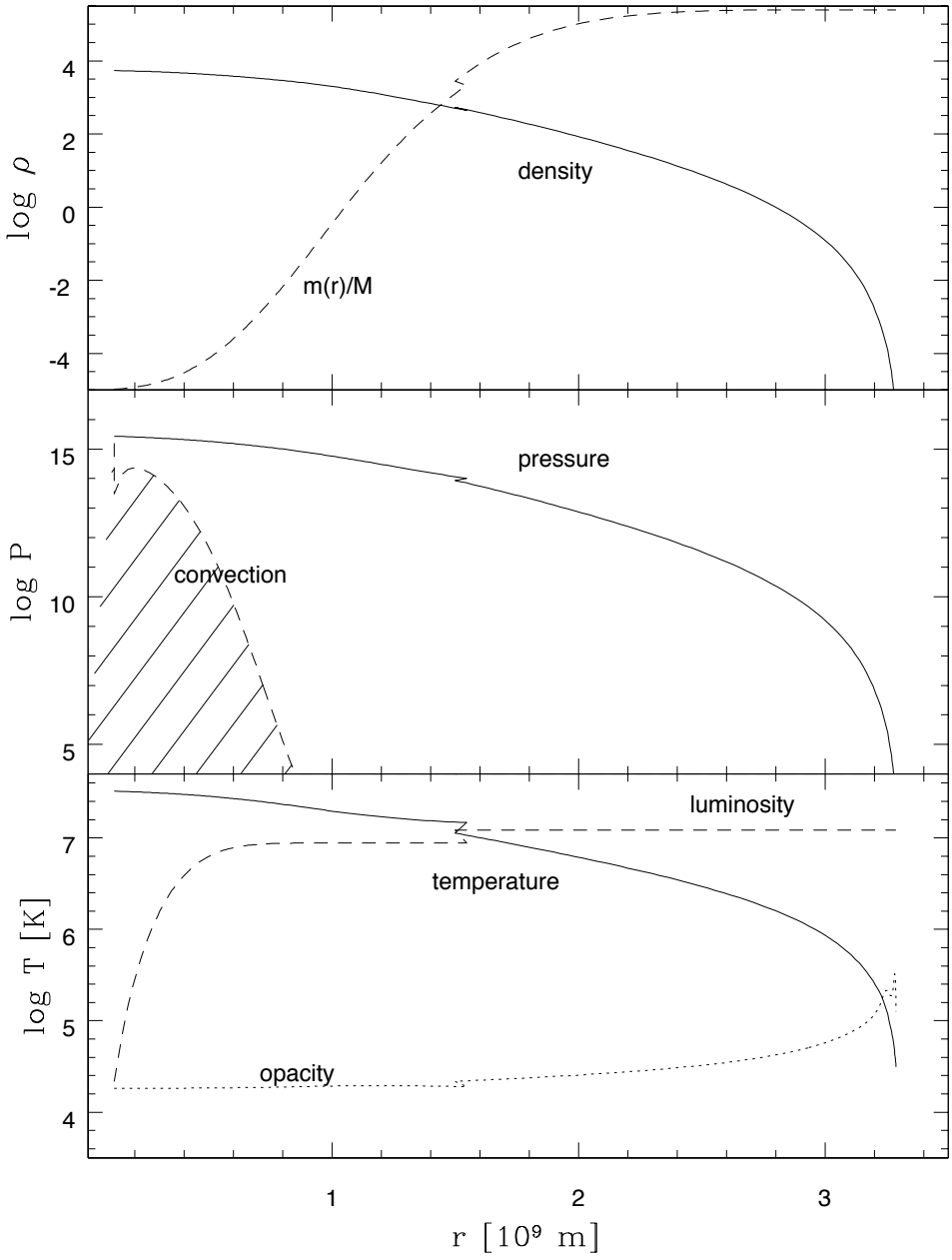

Figure 9.5: Internal structure of a $15 M_\odot$ star computed with modern stellar models. (A. Zijlstra, private communication).

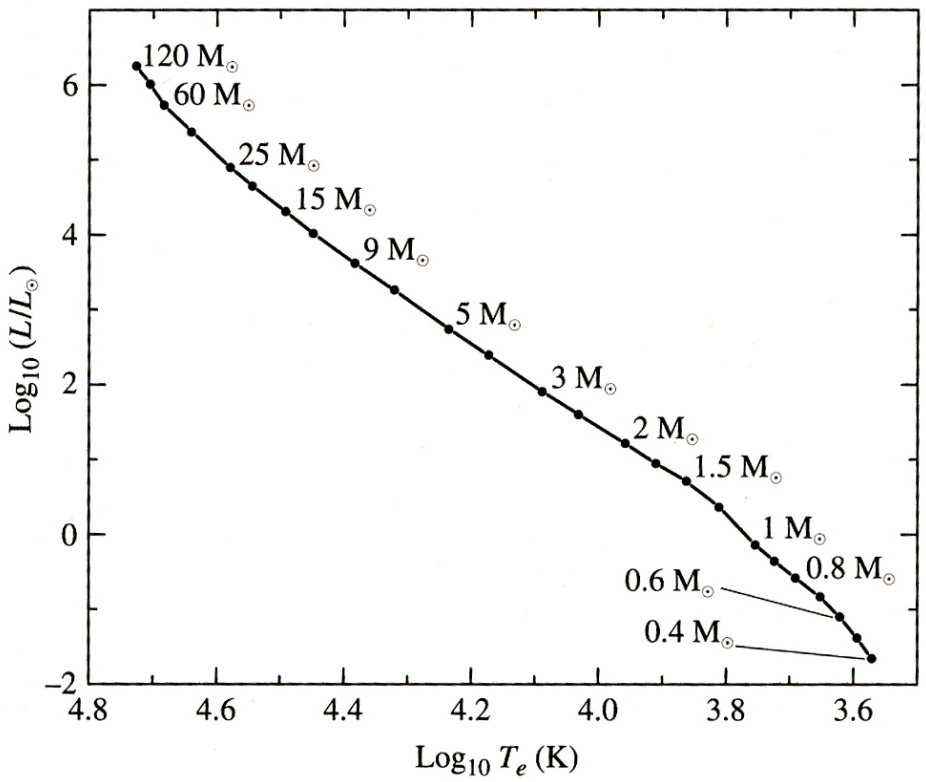

Figure 9.6: Theoretical Main Sequence computed with modern stellar models (Reproduced from Carroll & Ostlie's *Modern Astrophysics*).