WHAT MAKES A STAR SHINE?

In this lecture, we are going to look in more detail at some of the ideas outlined in Lecture 1, section 1.3. What are the sources of energy available to a star?

7.1   Gravitational Potential Energy

A potential source of energy to a newly born star is the gravitational energy released when the interstellar clouds from which the star formed began contracting. The virial theorem for a system in equilibrium:

$$-2\langle K \rangle = \langle U \rangle \tag{7.1}$$

where K and U are the kinetic and potential energy respectively and the brackets denote time averages, tells us that:

$$\langle E \rangle = \langle K \rangle + \langle U \rangle = \frac{1}{2}\langle U \rangle \tag{7.2}$$

only half the change in gravitational potential energy is available to be radiated away as the protostar contracts; the remaining potential energy supplies the thermal energy that heats the gas.

Referring to Figure 7.1, the gravitational force acting on a test particle of mass dm_i at a distance r from the centre of a spherically symmetric mass distribution is

$$dF_{g,i} = G\frac{M_r\,dm_i}{r^2} \tag{7.3}$$

where M_r is the mass contained within r, which acts as a point mass M located at the centre of the sphere, and the force is directed towards the centre of the sphere. The corresponding gravitational potential energy of the test mass is:

$$dU_{g,i} = -G\frac{M_r\,dm_i}{r} \tag{7.4}$$

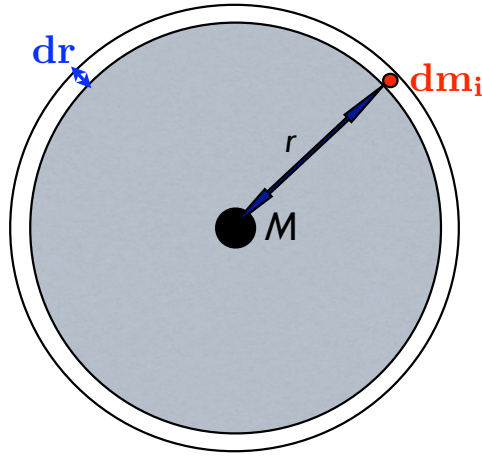Summing all the mass within a thin shell of thickness dr, with total mass

Figure 7.1: Gravitational potential energy of a star.

$dm = 4\pi r^2\, dr\, \rho$ where ρ is the density (mass per unit volume, e.g. g cm^{-3}) and $4\pi r^2\, dr$ is the volume of the shell, we can rewrite eq. 7.4 as:

$$dU_g = -G\frac{M_r\, 4\pi r^2 \rho}{r}\, dr \tag{7.5}$$

To obtain the total gravitational potential energy of a star, we integrate 7.5 over all mass shells from the centre of the star to its surface:

$$U_g = -4\pi G \int_0^R M_r \rho\, r\, dr \tag{7.6}$$

To properly evaluate the integral, we need to know $M_r = f(r)$ and $\rho = f(r)$. As an approximation, let us assume that an average density $\langle \rho \rangle$ obtains everywhere within the star; then:

$$M_r \sim \frac{4}{3}\pi r^3 \langle \rho \rangle\,, \tag{7.7}$$

and the mass of a star of radius R is $M = 4/3\,\pi R^3 \langle \rho \rangle$. Substituting 7.7 into 7.6 and integrating:

$$U_g \sim -\frac{16\pi^2}{15}\, G\, \langle \rho \rangle^2\, R^5 = -\frac{3}{5}\frac{GM^2}{R}\,. \tag{7.8}$$

As we saw, applying the virial theorem, tells us that half of this energy could have been radiated away as a protostar contracted from R_{initial} to R_{final}, where $R_{\text{initial}} \gg R_{\text{final}}$.

For the Sun:

$$\Delta E_g = \frac{3}{10}\frac{GM_\odot^2}{R_\odot} = \frac{3}{10}G\,\frac{(2\times 10^{33})^2}{7\times 10^{10}}\qquad \frac{\text{cm}^3\,\text{g}^{-1}\,\text{s}^{-2}\text{g}^2}{\text{cm}}$$

2

$$\Delta E_g = \frac{3}{10} 6.7 \times 10^{-8} \frac{4 \times 10^{66}}{7 \times 10^{10}} \qquad \mathrm{g\,cm^2\,s^{-2}}$$

$$\Delta E_g = 1.2 \times 10^{48} \ \mathrm{erg}$$

If the Sun radiated this energy on average at its present luminosity, the corresponding timescale, known as the *Kelvin-Helmholtz timescale*, would be:

$$\tau_{\mathrm{KH}} = \frac{\Delta E_g}{L_\odot} = \frac{1.2 \times 10^{48}}{3.8 \times 10^{33}} \qquad \frac{\mathrm{erg}}{\mathrm{erg\ s^{-1}}}$$

$$\tau_{\mathrm{KH}} \sim 3 \times 10^{14} \ \mathrm{s} \ \simeq 1 \times 10^7 \,\mathrm{yr}$$

The Kelvin-Helmholtz timescale is two orders of magnitude smaller than the age of the solar system, as measured by radioactive dating techniques applied to Moon rocks, for instance. Thus, the Sun must now be shining by a different mechanism than converting gravitational potential energy into radiation. Nevertheless, during its initial contraction from a protostellar cloud, the release of gravitational energy must have played an important role.

7.1.1 Average Temperature of the Sun

We can also use the virial theorem to roughly estimate the average temperature of a star. From eqs. 7.1 and 7.8, we have:

$$\langle K \rangle = \frac{3}{10} \frac{GM^2}{R} \ . \tag{7.9}$$

We now want to know the average temperature $\langle T \rangle$ which corresponds to the average kinetic energy $\langle K \rangle$. We obtain this by recalling that the distribution of particle velocities as a function of T is given by the Maxwell-Boltzmann distribution:

$$f(v) = \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{m}{kT}\right)^{3/2} v^2 \exp\left(-\frac{mv^2}{2kT}\right) \ . \tag{7.10}$$

We obtain the most probable speed, v_p, by imposing the condition $df(v)/dv = 0$, which gives:

$$v_p = \left(\frac{2kT}{m}\right)^{1/2} \ ,$$

and the rms speed:

$$v_{\text{rms}} = \left(\int_0^\infty v^2 f(v) dv \right)^{1/2} = \left(\frac{3kT}{m} \right)^{1/2} = \sqrt{\frac{3}{2}} \, v_p$$

Thus, the average kinetic energy per particle is:

$$\frac{1}{2} m_p v_{\text{rms}}^2 = \frac{3}{2} k \langle T \rangle \tag{7.11}$$

The total number of particles in a star of mass M is M/m_p; thus we have:

$$\frac{3}{2} k \langle T \rangle \frac{M}{m_p} = \frac{3}{10} \frac{GM^2}{R} \, . \tag{7.12}$$

Solving for the average temperature:

$$\langle T \rangle = \frac{1}{5} \frac{GM m_p}{kR} \, . \tag{7.13}$$

Note the dependence on the stellar mass and radius. Let us work out $\langle T_\odot \rangle$:

$$\langle T_\odot \rangle = \frac{1}{5} \frac{6.7 \times 10^{-8} \cdot 2 \times 10^{33} \cdot 1.7 \times 10^{-24}}{1.4 \times 10^{-16} \cdot 7 \times 10^{10}} \qquad \frac{\text{cm}^3 \, \text{g}^{-1} \, \text{s}^{-2} \, \text{g} \, \text{g}}{\text{erg} \, \text{K}^{-1} \, \text{cm}}$$

$$\langle T_\odot \rangle \simeq \frac{0.5 \times 10^1}{1 \times 10^{-6}} \qquad \frac{\text{cm}^3 \, \text{g}^{-1} \, \text{s}^{-2} \, \text{g} \, \text{g}}{\text{g} \, \text{cm}^2 \, \text{s}^{-2} \, \text{K}^{-1} \, \text{cm}}$$

$$\langle T_\odot \rangle \simeq 5 \times 10^6 \, \text{K} \, .$$

7.2 Nuclear Fusion

Returning to the question of the main source of radiant energy in stars, another possibility which is readily discounted is chemical reactions. Reactions between ions and atoms involve exchanges of electrons between different energy levels. Since the energies of most electronic levels are of order 1–10 eV, again they provide insufficient energy to power a star like the Sun for $\sim 10^{10}$ years.

On the other hand, the energies that bind protons and neutrons within atomic nuclei are a million times larger, of order MeV. For example, if we wanted to break up the He nucleus (also sometime referred to as an alpha particle) into its constituents two protons and two neutrons, we would need

to supply $\sim 27\,\mathrm{MeV}$. It follows that the inverse reaction, whereby four hydrogen nuclei fuse together to form a He nucleus (plus a number of low mass remnants) will release $\sim 27\,\mathrm{MeV}$. This energy is the binding energy of the He nucleus, and manifests itself as the mass difference between four H nuclei (4.03 unified atomic mass units or $4.03\,\mathrm{u}$) and one He nucleus ($4.00\,\mathrm{u}$). This mass difference $\Delta m = 0.029\,\mathrm{u}$, or $\sim 0.7\%$ of the rest mass of four H nuclei, corresponds to an energy $E = \Delta m\,c^2 = 26.7\,\mathrm{MeV}$, which is released by the nuclear fusion process.

Even if only 10% of the Sun's mass is converted from H into He, the energy generated would be:

$$E = 0.1 M_\odot \cdot 0.007 \cdot c^2 \;=\; 7 \times 10^{-4} \cdot 2 \times 10^{33} \cdot \left(3 \times 10^{10}\right)^2 \qquad \mathrm{g \cdot cm^2\,s^{-2}}$$

$$E = 1.26 \times 10^{2\,-4\,+33\,+20} \;=\; 1.26 \times 10^{51}\ \mathrm{erg}$$

Thus the nuclear timescale $\tau_{\mathrm{nuclear}} \sim 10^3 \times \tau_{\mathrm{KH}}$, or $\tau_{\mathrm{nuclear}} \sim 10^{10}\,\mathrm{years}$, consistent with the age of the solar system.

The process of nuclear fusion is not limited to $4\,{}^1_1\mathrm{H} \to 1\,{}^4_2\mathrm{He}$ (using the standard notation ${}^A_Z\mathrm{X}$, where A is the mass number (total number of protons + neutrons), Z is the atomic number (i.e. the number of protons, giving the total positive charge of the nucleus in units of e), and X is the chemical symbol for the element under consideration). Three ${}^4_2\mathrm{He}$ nuclei can fuse to give ${}^{12}_6\mathrm{C}$, in what is known as the triple alpha process. The addition of an alpha particle to ${}^{12}_6\mathrm{C}$ produces ${}^{16}_8\mathrm{O}$, and the subsequent incorporation of additional alpha particles into the nucleus produces the so called alpha-capture elements, Ne, Mg, Si, S and so on.

The process can continue so long as it is exothermic, that is so long as the mean mass per nucleon of the final fusion product is lower than that of the fusing nuclei. The binding energy per nucleon curve (see Figure 7.2) reaches a peak near Fe (hence the term Fe-peak). Fusion of Fe-peak elements to form elements of higher mass is an endothermic process requiring an additional supply of energy (and conversely, energy can be released by the fission of these heavier nuclei into lighter ones, as in a nuclear power station). The astrophysical production of these heavier elements is thought to occur during late stages of stellar evolution.
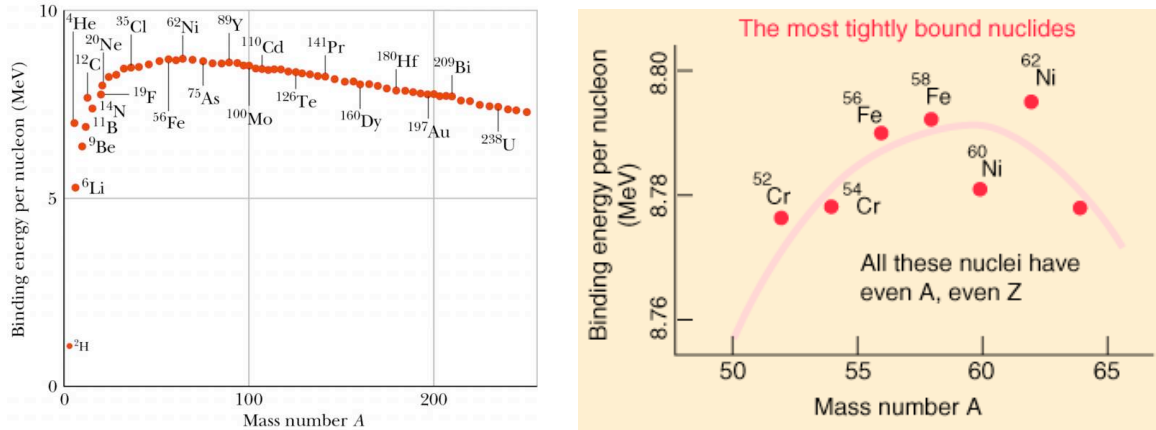
Figure 7.2: Left: Binding energy per nucleon as a function of mass number A. Right: Close-up near the iron peak. $^{62}_{28}$Ni has the highest binding energy per nucleon of any isotope of any element.

7.2.1 Quantum Mechanical Tunnelling

The nucleosynthesis of the common elements of the Periodic Table in stars is a good example of the fundamental relevance of quantum mechanics to our everyday world. In order for two nuclei to fuse, they must come sufficiently close together for the *short-range* strong nuclear force to overcome the Coulomb repulsion between the two positively charged particles. In the classical description, this would require kinetic energies corresponding to temperatures much higher than those found even in the interiors of stars. Only in the quantum-mechanical description of the process, can the reaction take place at stellar temperatures.

We can estimate the temperature required in the classical treatment by equating the mean velocity between two nuclei to the potential energy of the Coulomb barrier at the turn-around point (see Figure 7.3):

$$\frac{1}{2}\mu_m v_{\text{rms}}^2 = \frac{3}{2}kT_{\text{class}} = \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{r} \tag{7.14}$$

where ϵ_0 is the permittivity of free space [the constant that relates electric charge to mechanical quantities, defined by Coulombs law which gives the force between two electric charges separated by a distance r as $F_C = (1/4\pi\epsilon_0)(q_1 q_2/r^2)$], $Z_i\, e$ is the charge of each nucleus, and μ_m is the reduced
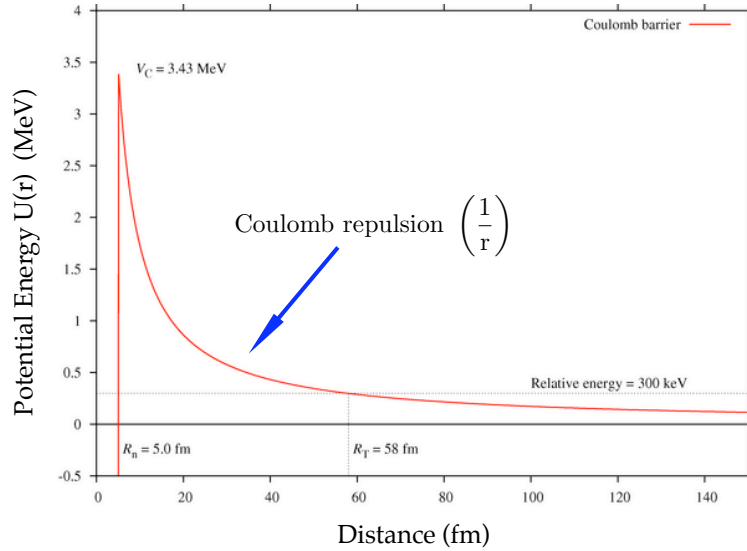
Figure 7.3: The Coulomb barrier classically prevents low-energy particles from approaching each other. The nuclear potential is represented here as a square well. In this example, the values shown are those appropriate to the $^{12}_{6}\text{C} + ^{4}_{2}\text{He} \rightarrow ^{16}_{8}\text{O}$ fusion reaction. The relative energy of $\sim 300\,\text{keV}$ corresponds to the Gamow-peak energy described below.

mass of the two colliding particles.[1] Thus,

$$T_{\text{class}} = \frac{Z_1 Z_2 e^2}{6\pi\epsilon_0 \, k \, r} \sim 10^{10}\,\text{K} \tag{7.15}$$

if $r \sim 10^{-13}\,\text{cm}$ (1 fm), the typical nuclear radius. The value of $\langle T_{\text{class}}\rangle$ is three orders of magnitude higher the central temperature of the Sun; even allowing for the high velocity tail of a Maxwell-Boltzmann distribution, an insufficient number of particles would penetrate the Coulomb barrier in classical physics to power the Sun's luminosity.

Quantum-mechanically, however, the two nuclei may find themselves within range of the strong nuclear force even if their kinetic energy is insufficient to overcome the Coulomb barrier, due to the inherent uncertainties in their positions and momenta given by Heisenberg uncertainty principle:

$$\Delta p_x \, \Delta x \geq \frac{\hbar}{2}.$$

The distance of interest here is the de Broglie wavelength associated with a particle, $\lambda = h/p$. Rewriting the mean kinetic energy in terms of the momentum:

$$\frac{1}{2}\mu_m v_{\text{rms}}^2 = \frac{p^2}{2\mu_m}$$

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[1]We have already encountered the concept of reduced mass for a two-body problem in Lecture 4, where it was defined as $\mu = m_1 \cdot m_2/(m_1 + m_2)$.

we can equate potential and kinetic energies at $r = \lambda$:

$$\frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 \lambda} = \frac{p^2}{2\mu_m} = \frac{(h/\lambda)^2}{2\mu_m} \tag{7.16}$$

or

$$\lambda = \frac{2\pi\epsilon_0 h^2}{\mu_m Z_1 Z_2 e^2} \, . \tag{7.17}$$

Substituting λ for r in eq. 7.15, we find:

$$T_{\text{quantum}} = \frac{Z_1^2 Z_2^2 e^4 \, \mu_m}{12\pi^2 \epsilon_0^2 h^2 k} \, . \tag{7.18}$$

In the case of two protons, $T_{\text{quantum}} \simeq 10^7$ K, comparable to the temperature in the core of the Sun.

7.3   Nuclear Reaction Rates

While nuclear fusion reactions can in principle supply the energy that makes the stars shine, we still need to consider quantitatively the rate at which such reactions occur in stellar interiors. In general terms, the rate at which a given nuclear reaction will proceed (number of reactions per unit volume per unit time) will depend on: (i) the volume density of the particles involved, (ii) their energy distribution (which will determine how closely two particles can approach each other), and (iii) the cross-section, or probability, of interaction.

With the Maxwell-Boltzmann distribution in energy terms:

$$n_E \, dE = \frac{2n}{\pi^{1/2}} \frac{1}{(kT)^{3/2}} \, E^{1/2} \, \exp[-\frac{E}{kT}] \, dE \, , \tag{7.19}$$

the reaction rate defined as above can be written as:

$$r_{it} = \int_0^\infty n_i n_t \, \sigma(E) \, v(E) \, \frac{n_E}{n} \, dE \, , \tag{7.20}$$

where n_i and n_t are the volume densities of incident and target particles respectively, $v(E)$ is the particle velocity, and $\sigma(E)$ is the cross-section for the interaction.

In order to evaluate this integral, we need to know the functional form of $\sigma(E)$. A great deal of effort is devoted to the calculation and laboratory

measurement of nuclear cross-sections, which can have quite complicated functional forms. But we can still make some general considerations about the overall behaviour of $\sigma(E)$ and r_{it}.

We saw that the quantum-mechanical size of a particle, its de Broglie wavelength, is inversely proportional to its momentum, $\lambda = h/p$. Thus, the cross-sectional *area* for interaction may reasonably be expected to have a $\sigma(E) \propto 1/E$ dependence. But the probability of quantum tunneling is also related to the ratio of the Coulomb barrier potential energy to the particle kinetic energy. This second factor gives rise to an exponential dependence of the cross-section on the energy. Taken together, these effects give rise to the traditional form of the cross-section:

$$\sigma(E) = \frac{S(E)}{E} \exp[-bE^{-1/2}] \,. \tag{7.21}$$

where $S(E)$ is a slowly varying function of E. Combining 7.21 and 7.20, we find:

$$r_{it} = \left(\frac{2}{kT}\right)^{3/2} \frac{n_i n_t}{(\mu_m \pi)^{1/2}} \int_0^\infty S(E) \exp[-bE^{-1/2}] \, \exp[-\frac{E}{kT}] \, dE \,, \tag{7.22}$$

where b is a constant that includes the charge and the mass of the particles:

$$b = \frac{\pi \, \mu_m^{1/2} \, Z_1 Z_2 \, e^2}{\sqrt{2} \, \epsilon_0 \, h}$$

The important thing to appreciate is that there are two competing energy dependencies in the integral at 7.22. The $\exp[-E/kT]$ is the decreasing number of particles with high energies in the tail of the Maxwellian distribution (eq. 7.19). On the other hand, the probability of tunnelling through the Coulomb barrier increases at high energies as indicated by the term $\exp[-bE^{-1/2}]$. The net result of combining these two functions is a strongly peaked dependence of the reaction rate on energy, as shown in Figure 7.4. Thus, the greatest contribution to the reaction rate integral is from a relatively narrow energy range that depends on the temperature of the gas, and the charges and masses of the nuclei involved. By setting $dr_{it}/dE = 0$, we find that the peak of the Gamow curve (so named from the Russian physicist George Gamow who for a period in the late 1920s and early1930s worked with Rutherford at the Cavendish Laboratory) occurs at energy:

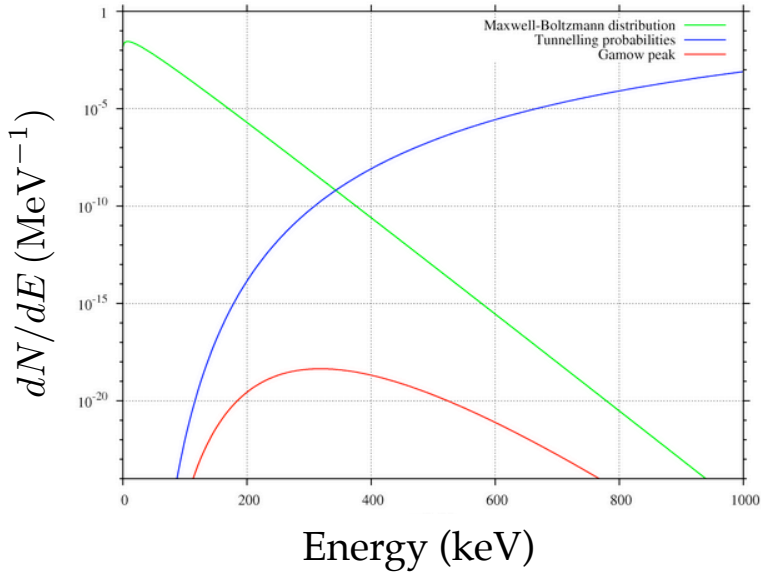$$E_0 = \left(\frac{bkT}{2}\right)^{2/3}$$

Figure 7.4: The red curve is the product of the Maxwell-Boltzmann distribution (number of particles per unit energy interval, shown in green) with the tunnelling probability of the nuclei through their Coulomb barrier (dimensionless, shown in blue). The curve is strongly peaked: this is the energy range over which the nuclear reaction is most likely to take place. At higher energies the number of particles becomes insignificant, while at lower energies the tunnelling through the Coulomb barrier becomes improbable. This Figure refers to the $^{12}_{6}\text{C} + ^{4}_{2}\text{He} \rightarrow ^{16}_{8}\text{O}$ fusion reaction at $T = 2 \times 10^8$ K; the Gamow peak is at an energy $E \simeq 300\,\text{keV}$, much larger than $\langle E \rangle = 3kT/2 = 26\,\text{keV}$.

Before moving on to consider the nuclear reaction rates which are most important for stellar nucleosynthesis, we mention two effects which add to the above treatment of the reaction rates. First, the cross sections of some nuclear reactions exhibit resonances—small energy intervals where the likelihood of interaction is boosted significantly by energy levels within the nucleus. Second, at the high temperatures of stellar interiors, the high densities of free electrons can partially shield the positive nuclear charge. The net effect of this electron screening is to reduce the Coulomb barrier, thereby enhancing the reaction rates.

7.4 Stellar Nucleosynthesis

We now consider the main pathways for energy generation by nuclear reactions in stellar interiors. For each nucleosynthetic reaction, we are interested in the amount of energy released per second per unit mass of nuclear fuel. For this calculation, it is most convenient to express the reaction rate

in power-law form:

$$r_{it} \simeq r_0 X_i X_t \, \rho^\gamma \, T^\beta \qquad (7.23)$$

where r_0 is a constant, $X_{i,t}$ are the mass fractions of the two particles, the exponent of the density dependence is normally $\gamma = 2$ for two-body collisions, whereas the power-law dependence on the temperature β can range from ~ 1 to $\gtrsim 40$. If $\mathcal{E}_0$ is the energy released per reaction, then the rate of energy release per unit mass of nuclear fuel is:

$$\mathcal{E}_{it} = \left(\frac{\mathcal{E}_0}{\rho}\right) r_{it} \qquad \frac{\mathrm{erg\,s^{-1}\,cm^{-3}}}{\mathrm{g\,cm^{-3}}} \Rightarrow \mathrm{erg\,s^{-1}\,g^{-1}} \qquad (7.24)$$
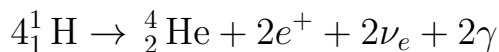
or

$$\mathcal{E}_{it} = \mathcal{E}_0 \, r_0 \, X_i X_t \, \rho^\alpha \, T^\beta \qquad (7.25)$$

where $\alpha = \gamma - 1$.

7.4.1   The Proton-Proton Chain

We begin with the main reaction chain which powers stars on the Main Sequence, hydrogen burning:

$$4\,^1_1\mathrm{H} \rightarrow \,^4_2\mathrm{He} + 2e^+ + 2\nu_e + 2\gamma$$

with $\mathcal{E}_0 = 26.73\,\mathrm{MeV}$. The two positrons subsequently annihilate with free electrons: $e^+ + e^- = 2\gamma$. The reaction can proceed through three channels (it is quite a common situation in nuclear reactions that different pathways can lead to the same end result), as shown in Figure 7.5. The balance between PP I and PP II varies with temperature, with the former preferred at $T \lesssim 1.5 \times 10^7\,\mathrm{K}$; the values indicated in Figure 7.5 are those appropriate to the central temperature of the Sun, $T = 1.57 \times 10^7\,\mathrm{K}$. PP III is never very important, but it is a source of high energy neutrinos.

Each step in the PP chain (and indeed in any chain) has its own reaction rate, determined by the Coulomb barriers and cross-sections involved. The slowest step in the PP chain is the first one because it is necessary for one of the protons to undergo a β^+ decay: $p^+ \rightarrow n + e^+ + \nu_e$ via the weak nuclear force. On average, a proton in the Sun will undergo one such decay only once in the lifetime of the Sun ($\sim 10^{10}$ years)! All the subsequent steps are much quicker. For the pp-chain:

$$\mathcal{E}_{pp} \propto X^2 \, \rho \, T^4 \qquad \mathrm{erg\,s^{-1}\,g^{-1}},$$

with the neutrinos carrying off $\sim 1\%$ of the energy generated.

$$\ce{^1_1H + ^1_1H -> ^2_1H + e^+ + \nu_e}$$

$$\ce{^2_1H + ^1_1H -> ^3_2He + \gamma}$$

69%   31%

$$\ce{^3_2He + ^3_2He -> ^4_2He + 2\,^1_1H}$$ $$\ce{^3_2He + ^4_2He -> ^7_4Be + \gamma}$$

(PP I)

99.7%   0.3%

$$\ce{^7_4Be + e^- -> ^7_3Li + \nu_e}$$ $$\ce{^7_4Be + ^1_1H -> ^8_5B + \gamma}$$

$$\ce{^7_3Li + ^1_1H -> 2\,^4_2He}$$ $$\ce{^8_5B -> ^8_4Be + e^+ + \nu_e}$$

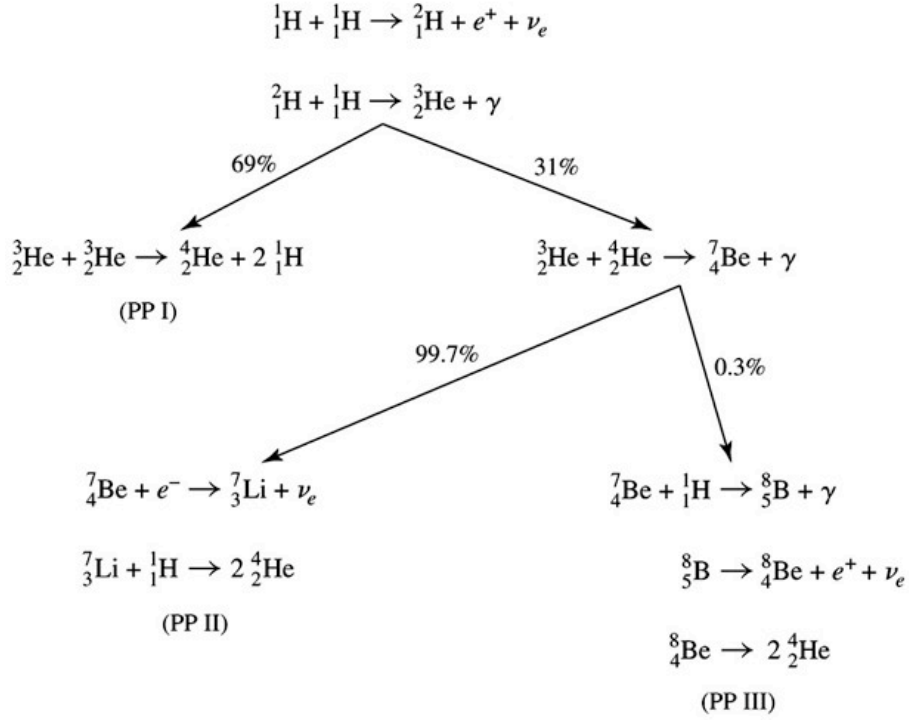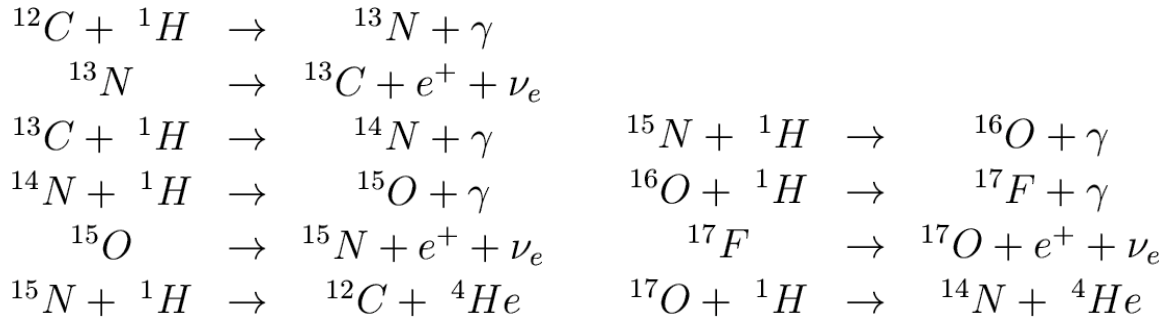(PP II) $$\ce{^8_4Be -> 2\,^4_2He}$$

(PP III)

Figure 7.5: The three proton-proton chains. The branching ratios are appropriate to the temperature in the core of the Sun.

7.4.2 The CNO Cycle

When C, N and O are present and the temperature is sufficiently high, He can be synthesised from four H nuclei through a series of pathways known as the CNO cycle. C, N and O act as catalysts: they make He fusion possible through a series of reactions, but their number is conserved in the cycle. The two main pathways of the CNO cycle are as follows:

$$
\begin{aligned}
{}^{12}C + {}^{1}H &\rightarrow {}^{13}N + \gamma \\
{}^{13}N &\rightarrow {}^{13}C + e^+ + \nu_e \\
{}^{13}C + {}^{1}H &\rightarrow {}^{14}N + \gamma & {}^{15}N + {}^{1}H &\rightarrow {}^{16}O + \gamma \\
{}^{14}N + {}^{1}H &\rightarrow {}^{15}O + \gamma & {}^{16}O + {}^{1}H &\rightarrow {}^{17}F + \gamma \\
{}^{15}O &\rightarrow {}^{15}N + e^+ + \nu_e & {}^{17}F &\rightarrow {}^{17}O + e^+ + \nu_e \\
{}^{15}N + {}^{1}H &\rightarrow {}^{12}C + {}^{4}He & {}^{17}O + {}^{1}H &\rightarrow {}^{14}N + {}^{4}He
\end{aligned}
$$

with the pathway on the right occurring only $\sim 0.04\%$ of the time. For the CNO cycle:

$$\mathcal{E}_{\mathrm{CNO}} \propto X\, X_{\mathrm{CNO}}\, \rho\, T^{17} \qquad \mathrm{erg\, s^{-1}\, g^{-1}}.$$

Note the much steeper dependence on T than the p-p chain. The two

functions $\mathcal{E} = f(T)$ are shown in Figure 7.6 from where it can be seen that He fusion proceeds mostly via the CNO cycle in stars with $M \gtrsim 2M_\odot$, and via the p-p chain in cooler stars.
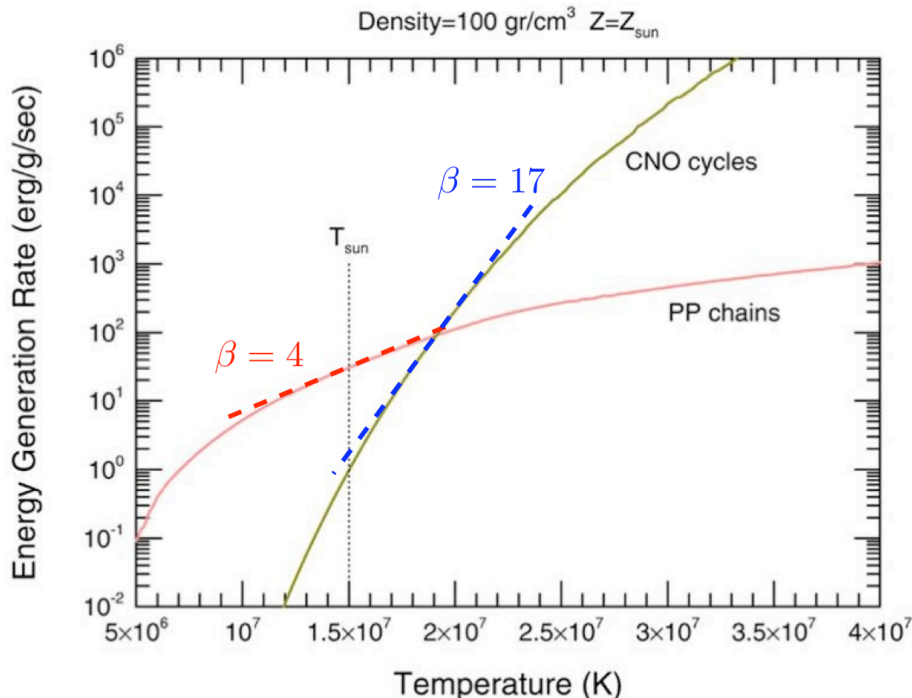

Figure 7.6: The temperature dependence of the energy generation rates of the p-p chain and CNO cycle.

Of course, in stars of lower metallicity, the cross-over point moves to higher temperatures (X_{CNO} is lower). In the first stars, which presumably consisted just of H and He synthesised in the Big Bang, only the p-p chain was operative.

A consequence of the steep T dependence of $\mathcal{E}_{\mathrm{CNO}}$ is that in massive stars H fusion must be more concentrated in the inner core of the star than is the case for lower mass stars where the p-p chain is the main channel. A second point of note is the following. As the name implies, the CNO cycle is a cyclic process that quickly reaches equilibrium. This means that: (a) the total number of C+N+O ions is conserved, and (b) each step has to proceed at the same rate. Given that rate for each step is $r_{i \to j} \propto n_i \cdot \sigma_{i \to j}$, the requirement $r = $ constant implies that $n_i \propto 1/\sigma_{i \to j}$. In other words, steps with smaller cross-sections require proportionally higher concentrations of the ions involved to keep the cycle in equilibrium. In particular, $\sigma(^{14}\mathrm{N}) \simeq 1/30\,\sigma(^{12}\mathrm{C})$, so that in equilibrium $n_{^{14}\mathrm{N}} \simeq 30 \times n_{^{12}\mathrm{C}}$. In other words, the CNO cycle piles up $^{14}\mathrm{N}$ at the expense of $^{12}\mathrm{C}$ and $^{16}\mathrm{O}$,

and is in fact the main channel for the nucleosynthesis of N in stars.

7.4.3   Helium Burning

The fusion of four H nuclei to form He, via either process, increases the mean molecular weight of the gas μ (defined so that the mean particle mass in a gas is $\langle m \rangle = \mu\, m_H$, where m_H is the mass of the hydrogen atom). From the ideal gas law:
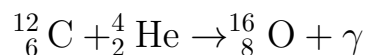
$$P = \frac{\rho k T}{\mu m_H}$$

the pressure decreases and is no longer able to support the star against the pull of gravity. As a result, the stellar core contracts, raising both the temperature and the density of the gas (recall the virial theorem at the beginning of this lecture). When the temperature and density become sufficiently high, He nuclei can overcome their Coulomb barrier and combine to form ^{12}C by capturing three ^{4}He nuclei, as follows:

$$^4\text{He} + {}^4\text{He} \rightarrow {}^8\text{Be} + \gamma$$
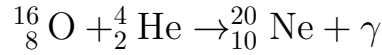$$^4\text{He} + {}^8\text{Be} \rightarrow {}^{12}\text{C} + \gamma$$
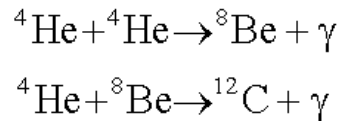
The lifetime of ^{8}Be is very short, $\sim 3 \times 10^{-16}$ s; consequently, this is really a three-body interaction, with a reaction rate $r \propto (\rho Y)^3$, where Y is the mass fraction of He. This triple alpha reaction was predicted by Fred Hoyle (a previous director of the Institute of Astronomy) in 1954, based on the high abundance of C in the Sun and H II regions like the Orion nebula. It produces C from He nuclei bypassing completely the intermediate elements Li, Be, B, and it explains why C is 10^5–10^7 times more abundant that Li, Be and B in the Universe. The energy generation rate of the triple alpha reaction has an extraordinarily steep dependence on T:

$$\mathcal{E}_{3\alpha} \propto Y^3\, \rho^2\, T^{40} \qquad \text{erg s}^{-1}\, \text{g}^{-1}.$$

Once sufficient quantities of ^{12}C have been synthesised via the triple alpha reaction, heavier elements can be formed from ^{12}C via the capture of additional ^{4}He nuclei:

$$^{12}_{6}\text{C} + {}^{4}_{2}\text{He} \rightarrow {}^{16}_{8}\text{O} + \gamma$$

and

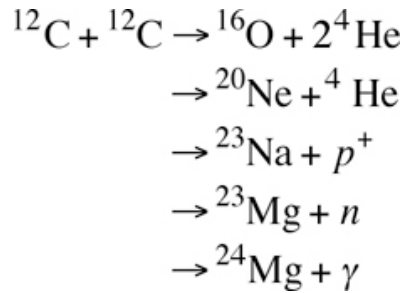$$^{16}_{8}\text{O} + ^{4}_{2}\text{He} \rightarrow ^{20}_{10}\text{Ne} + \gamma$$
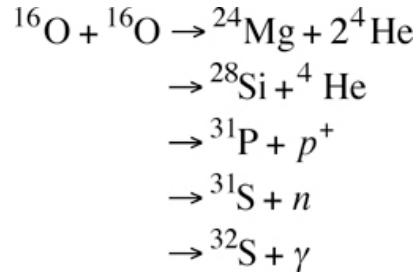
7.4.4 Carbon and Oxygen Burning

Higher and higher temperatures are required to overcome the increasing Coulomb barriers as heavier elements are synthesised. At temperatures $T \geq 6 \times 10^8$ K, as found in the cores of stars with mass $M \geq 8 M_\odot$, carbon burning can proceed via several reactions:

$$
\begin{aligned}
^{12}\text{C} + ^{12}\text{C} &\rightarrow ^{16}\text{O} + 2\,^4\text{He} \\
&\rightarrow ^{20}\text{Ne} + ^4\text{He} \\
&\rightarrow ^{23}\text{Na} + p^+ \\
&\rightarrow ^{23}\text{Mg} + n \\
&\rightarrow ^{24}\text{Mg} + \gamma
\end{aligned}
$$

When $T \geq 1 \times 10^9$ K, oxygen can burn:

$$
\begin{aligned}
^{16}\text{O} + ^{16}\text{O} &\rightarrow ^{24}\text{Mg} + 2\,^4\text{He} \\
&\rightarrow ^{28}\text{Si} + ^4\text{He} \\
&\rightarrow ^{31}\text{P} + p^+ \\
&\rightarrow ^{31}\text{S} + n \\
&\rightarrow ^{32}\text{S} + \gamma
\end{aligned}
$$

When $T > 1.5 \times 10^9$ K, we find from Wien's law (eq. 2.11), the maximum emission occurs at $\lambda = 2 \times 10^{-2}$ Å. The corresponding photon energy, 0.6 MeV, is sufficient to photo-disintegrate heavy nuclei, creating a mixture of massive nuclei, p, n, He nuclei, and photons. These can facilitate reactions (silicon burning) that ultimately lead to the formation of stable elements of the iron group, at the peak of the binding energy per nucleon curve (Figure 7.2).

Each of the steps in the nucleosynthesis chain we have described, starting from H and ending at the Fe-peak, requires progressively higher temperatures, because of the increasing height of the corresponding Coulomb barrier. Thus, the core has to contract before the next step can start. Each successive reaction also has a steeper temperature dependence. Thus, it
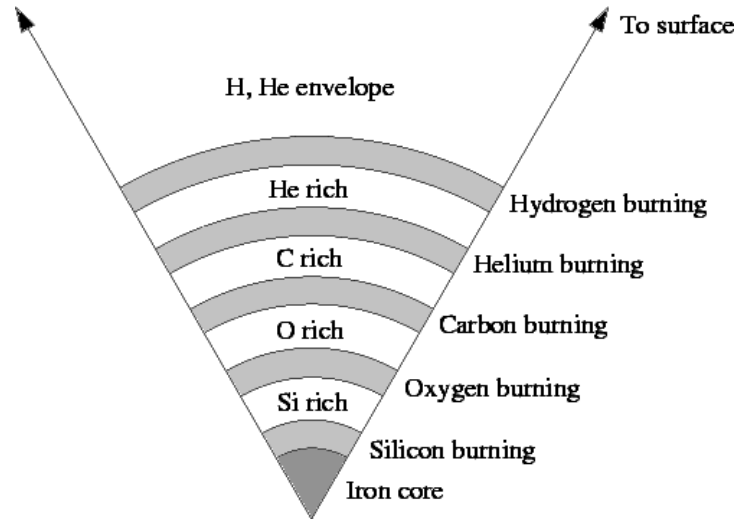
Figure 7.7: The stratified structure of the core of a massive star.

will take place in a more concentrated region within the core of the star, and will involve less mass. Therefore, at the end of the process, the interior of the star is stratified, as shown in Figure 7.7; this structure is sometimes referred to as an onion-skin structure.

Furthemore, each step in the nucleosynthesis of elements from H to Fe extracts less energy from the system (notice that the biggest individual 'jump' in Figure 7.2 occurs when He is synthesised). Consequently, the reaction rate has to be higher and fusion faster to provide the luminosity. Another factor is that at $T > 10^9$ K, neutrinos carry away larger and larger fractions of the energy produced. Neutrinos travel right through the star without interacting with the plasma; this has the effect of reducing the net energy production by nuclear burning. Thus, each successive nuclear burning phase lasts a progressively shorter period of time.

Table 7.1 summarises the main nuclear burning processes.

Table 7.1 Main nuclear burning processes

Fuel	Process	T_{thresh}^1 (K)	Products	E/nucleon[2] (MeV)	Timescale[3] (yr)
H	p-p	$\sim 4 \times 10^6$	He	6.55	
H	CNO	1.5×10^7	He	6.25	1×10^7
He	triple-α	1×10^8	C, O	0.61	1×10^6
C	C + C	6×10^8	O, Ne, Na, Mg	0.54	300
O	O + O	1×10^9	Mg, S, P, Si	~ 0.3	0.5
Si	Nucl. equil.	3×10^9	Co, Fe, Ni	$\lesssim 0.2$	0.005 (2 days!)

Notes:

[1] Threshold temperature

[2] Energy released per nucleon

[3] Typical timescale for a $15 M_\odot$ star.

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7.5 Neutron Capture

We conclude our description of stellar nucleosynthesis with a brief mention of the mechanism whereby elements heavier than Fe are thought to be produced in stars.

As we have seen, at $T \gtrsim 1.5 \times 10^9$ K photodisintegration of nuclei becomes important and creates a mixture of neutrons, protons and other nuclei. Neutrons play an important role here. Since they do not experience a Coulomb barrier, they can easily penetrate the nuclei of even fully ionised heavy elements, such as Fe^{+26}. If they are captured by the nucleus, new neutron-rich isotopes can be produced. Such isotopes can be either stable or unstable.

It is important to distinguish between slow and rapid neutron capture (termed the s-process and the r-process), depending on the relative timescales of β-decay and neutron capture. In the example of the s-process shown in Figure 7.8, ^{56}Fe absorbs a neutron to form ^{57}Fe. Subsequent capture of two more neutrons leads to the formation of ^{59}Fe. Of the four Fe isotopes shown, the three lighter ones are stable, but ^{59}Fe is unstable, with a half-life of 44.5 days. Thus, if the flux of neutrons is not high and the interval between successive n-captures is longer than the half-life of ^{59}Fe, there is time for ^{59}Fe to decay to ^{59}Co by β-decay (n $\rightarrow$ p + e$^-$ + $\overline{\nu}_e$). The process can continue to form higher and higher mass elements, as shown in Figure 7.8.
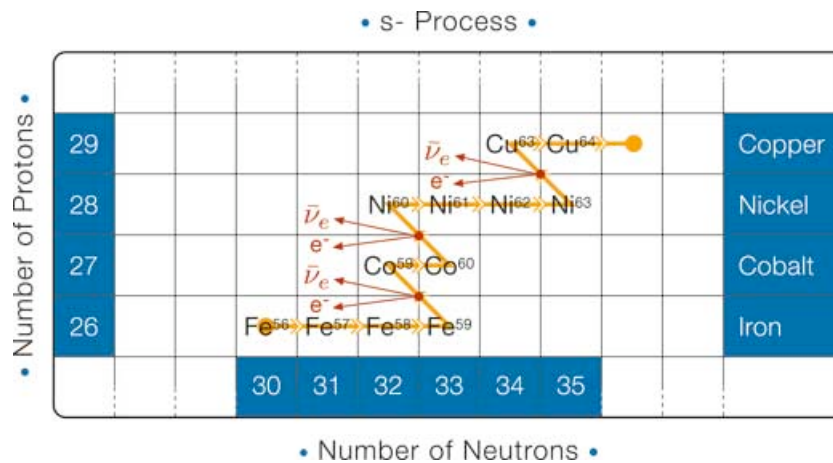

Figure 7.8: Example of the nucleosynthesis of ^{59}Co, ^{60}Ni and ^{63}Cu from ^{56}Fe via slow neutron capture.
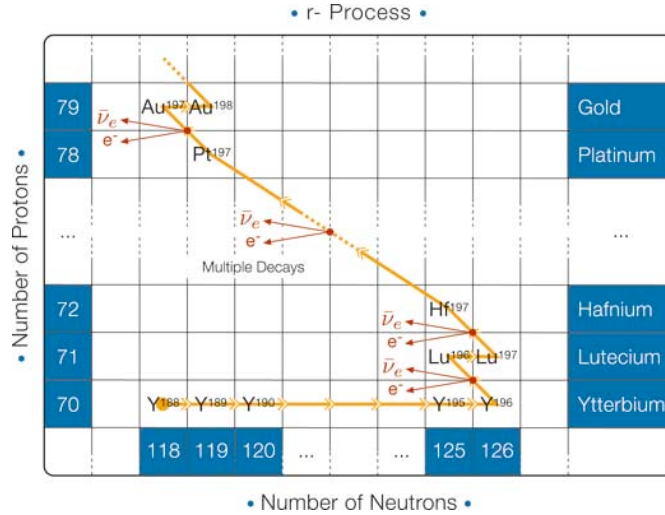
Figure 7.9: Example of the nucleosynthesis of ^{197}Au (stable gold) from ^{188}Yb via rapid neutron capture.

On the other hand, if the flux of neutrons is sufficiently high and the time interval between subsequent neutron captures is small compared to the half-life of the isotopes concerned, super-neutron-rich isotopes can be formed, as in the example of the r-process shown in Figure 7.9. When the neutron flux stops, these super-neutron-rich isotopes will undergo a series of β-decays until a stable isotope is reached.

Trans-Fe-peak elements can be formed by either s- or r-process nucleosynthesis, or both, depending on the stability of their neighbours in the Periodic Table. Typical s-process elements include Cu and Pb, while Eu is the prototypical signature of r-process nucleosynthesis in stellar spectra (see Figure 7.10). It is generally though that s-process nucleosynthesis takes place primarily in AGB stars, while the r-process occurs mainly in supernova explosions and, as now confirmed observationally, in the explosion accompanying the merger of two neutron stars.
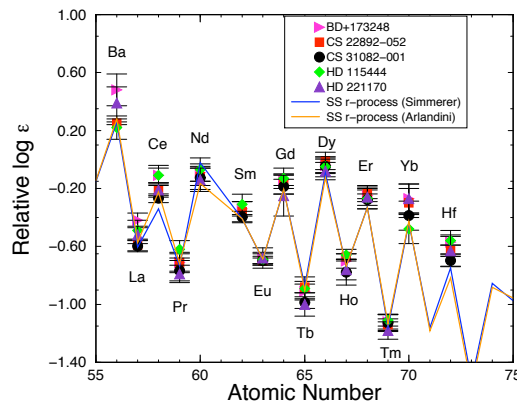

Figure 7.10: Relative abundances of several r-process elements in five Galactic halo stars (Cowan et al 2011).