CLUSTERS OF GALAXIES: SPHERICAL COLLAPSE AND VIRIALIZATION

Figure 15.1: *Hubble Space Telescope* image of the galaxy cluster Abell 2218 at a redshift $z = 0.18$. The cluster acts as a powerful lens, magnifying all galaxies lying behind the cluster core. The lensed galaxies are all stretched along the cluster's center and some of them are multiply imaged. Those multiply imaged usually appear as a pair of images with a third—generally fainter—counter image. The color of the lensed galaxies is a function of their distances and types. The orange arc is an elliptical galaxy at moderate redshift ($z = 0.7$). The blue arcs are star-forming galaxies at intermediate redshift ($z = 1 - 2.5$). A pair of faint red images near the bottom of the picture is a star-forming galaxy at redshift $z \sim 7$ recently discovered by a team of astronomers from Caltech, the University of Cambridge and the Observatoire Midi-Pyrenees in France. The lensed galaxies are particularly numerous, as we are looking in between two mass clumps, in a saddle region where the magnification is large.

15.1 Clusters of Galaxies

Clusters of galaxies are among the largest—and most spectacular—structures in the universe, marking the sites of the greatest overdensities of matter. In nearby clusters one can discern 100s to 1000s of bright ($L \geq L^*$) galaxies—mostly ellipticals— concentrated within volumes comparable to that of the Local Group of galaxies, with radii $r \sim 1\,\mathrm{Mpc}$.

Most clusters are the source of intense X-ray emission, with typical temperatures of $2 - 10\,\mathrm{keV}$. Such high temperatures are indicative of gas in hydrostatic equilibrium within a deep gravitational potential well. The deep concentration of matter makes clusters act as gravitational lenses, magnifying and distorting the images of background galaxies; 'gravity's telescopes' provide a tool which is much exploited in searches for the highest redshift galaxies.

In the local universe, clusters of galaxies are gravitationally bound, virialised systems. The three-dimensional velocity dispersion of the galaxies within the cluster, typically $\sigma_{3\mathrm{D}} \simeq 1500\,\mathrm{km\ s^{-1}}$, is one order of magnitude greater than the Hubble expansion over the Mpc-scale of the cluster. The relevant timescale for virialization is the dynamical timescale t_{dyn}—you can think of this as the time it takes for the cluster to communicate with itself through its own potential. We can obtain an estimate of t_{dyn} from the cluster crossing time $t_{\mathrm{cr}} \propto r/\sigma_{3\mathrm{D}}$:

$$t_{\mathrm{dyn}} \approx \frac{r}{\sigma_{3\mathrm{D}}} = 1 \times 10^9 \left(\frac{r}{\mathrm{Mpc}}\right) \left(\frac{1000\,\mathrm{km\ s^{-1}}}{\sigma_{3\mathrm{D}}}\right) \text{ years}$$

which is approximately 1/10 of the Hubble time. Virial equilibrium cannot be established on timescales shorter than t_{dyn}.

15.2 Evolution of a Density Perturbation: Linear Theory

In this lecture we follow the evolution of a density perturbation, from early times through to gravitational collapse and virialization. For convenience we will develop our formalism for the simplest case of a flat, matter dominated, universe. Consider the idealised case of a spherical volume where the density is infinitesimally higher than the cosmic mean; in Fig. 15.2 this has been achieved by 'emptying out' a thin shell surrounding our spherical region and placing the material which was inside this shell within the region of radius r. Thus, the density inside the volume of radius r is $\rho_{\rm crit}+\delta\rho$, while the density of the background universe remains $\rho_{\rm crit}$.

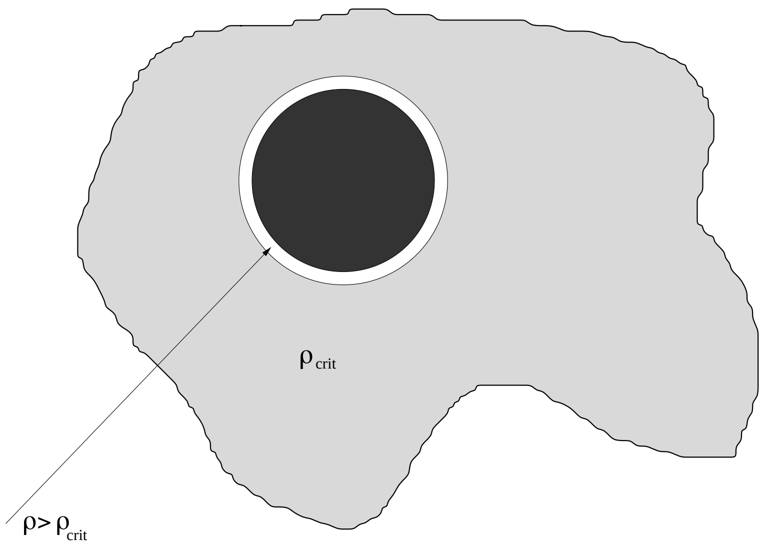

Figure 15.2: Sketch of a spherical over-density, with decoupled evolution from the background universe

Our density perturbation will then evolve like a closed universe with $\Omega_{\rm m} = 1 + \delta$. As we saw in lecture 4 (Figure 4.2), the scale factor $a(t)$ of such a universe reaches a maximum value $a_{\rm max}$ and then decreases again—in other words, our perturbation will grow to a maximum size $r = r_{\rm max}$ at time $t = t_{\rm max}$ and then collapse in a finite time.

It is convenient to express the evolution of the scale factor in terms of the 'development angle' θ, because the radius r and time t have simple functional forms in terms of θ . It can be shown that the Friedmann equation for a closed universe:

$$\frac{1}{a}\frac{da}{dt} = H_0\left(\Omega_{\rm m,0}a^{-3} + (1-\Omega_{\rm m,0})a^{-2}\right)^{1/2} \tag{15.1}$$

has a parametric solution in terms of the development angle:

$$\theta = H_0\,\eta\,(\Omega_{\rm m,0}-1)^{1/2} \tag{15.2}$$

whereby:

$$r(\theta) = A(1 - \cos\theta) \tag{15.3}$$

and

$$t(\theta) = B(\theta - \sin\theta) \tag{15.4}$$

with:

$$A = r_0 \frac{\Omega_{\rm m,0}}{2(\Omega_{\rm m,0} - 1)}; \quad B = \frac{1}{H_0} \frac{\Omega_{\rm m,0}}{2(\Omega_{\rm m,0} - 1)^{3/2}}. \tag{15.5}$$

The development angle θ is a scaled form of the 'conformal time' $\eta(t)$, also called the "arc-parameter measure of time". During the interval of time dt, a photon travelling on a hypersphere of radius $a(t)$ covers an arc measured in radians equal to

$$d\eta = \frac{dt}{a(t)}.$$

The 'arc parameter" is defined by the integral of $d\eta$ from the start of the expansion:

$$\eta = \int_0^t \frac{dt}{a(t)}.$$

Thus, small values of the "arc parameter time", η, mean early times and larger values mean later times.[1]

We can find the maximum size which the perturbation will grow by considering (eq. 15.3):

$$\frac{dr}{d\theta} = A\sin\theta = 0 \tag{15.6}$$

which is satisfied at $\theta = 0, \pi, 2\pi$. The solution $\theta = 0$ corresponds to time $t = 0$, but $\theta = \pi$ corresponds to the time of turn-around, when the overdensity reaches its maximum size before collapsing. At this time, $t = t_{\rm max}$, we have

$$r_{\rm max} = 2A = r_0 \frac{\Omega_{\rm m,0}}{\Omega_{\rm m,0} - 1} \tag{15.7}$$

and, more generally,

$$\frac{r}{r_{\rm max}} = \frac{1}{2}(1 - \cos\theta). \tag{15.8}$$

[1]When the model universe is not closed, the same parameter can be defined—only the words 'hypersphere' and 'arc' have to be replaced by the corresponding words for a flat hypersurface of homogeneity ($k = 0$), or a hyperboloidal hypersurface ($k = -1$).

From eq. (15.4), we have:

$$t_{\rm max} = t(\pi) = \pi B; \quad H_0 t_{\rm max} = \frac{\pi}{2} \frac{\Omega_{\rm m,0}}{(\Omega_{\rm m,0} - 1)^{3/2}} \tag{15.9}$$

and

$$\frac{t}{t_{\rm max}} = \frac{1}{\pi}(\theta - \sin\theta)\,. \tag{15.10}$$

The solution $\theta = 2\pi$ corresponds to the time when the structure has completely recollapsed (but see later), when

$$r \to 0 \quad \text{and} \quad t = 2\pi B \quad (\theta = 2\pi)\,. \tag{15.11}$$

The constants A and B are related through the enclosed mass:

$$M = \frac{4\pi}{3} r_0^3 \Omega_{\rm m,0} \rho_{\rm crit} = \frac{4\pi}{3} r_0^3 \Omega_{\rm m,0} \frac{3H_0^2}{8\pi G} \tag{15.12}$$

by the simple relation:

$$A^3 = GMB^2\,. \tag{15.13}$$

In the linear regime, we can follow the growth of the perturbation by using the Maclaurin expansions for $\cos\theta$ and $\sin\theta$ in (15.3) and (15.4):

$$\lim_{\theta\to 0} r(\theta) = A\left(\frac{1}{2}\theta^2 - \frac{1}{24}\theta^4\right) \tag{15.14}$$

and

$$\lim_{\theta\to 0} t(\theta) = B\left(\frac{1}{6}\theta^3 - \frac{1}{120}\theta^5\right)\,. \tag{15.15}$$

The leading order, $r = A\theta^2/2$ and $t = B\theta^3/6$, just gives the expansion of the background (i.e. outside the volume including the overdensity) universe where:

$$r = a = \frac{A}{2}\left(\frac{6t}{B}\right)^{2/3} \tag{15.16}$$

that is $a \propto t^{2/3}$ (matter-dominated universe).

Our overdensity will grow according to the equations:

$$\frac{r}{r_{\rm max}} \simeq \frac{\theta^2}{4} - \frac{\theta^4}{48}\,, \qquad \frac{t}{t_{\rm max}} \simeq \frac{1}{\pi}\left(\frac{\theta^3}{6} - \frac{\theta^5}{120}\right)$$

which can be combined to give the linearised scale factor of our closed universe:

$$\frac{a_{\rm lin}}{a_{\rm max}} \simeq \frac{1}{4}\left(6\pi\frac{t}{t_{\rm max}}\right)^{2/3}\left[1-\frac{1}{20}\left(6\pi\frac{t}{t_{\rm max}}\right)^{2/3}\right]. \tag{15.17}$$

Again, the first term is just the expansion of the background in a flat matter dominated universe. Including both terms in the square brackets gives the *linear theory* expression for the growth of a perturbation.

In both cases (inside and outside the volume containing the overdensity), we are dealing with matter dominated universes where the mass-energy density varies as a^{-3}. Hence, throughout the evolution (expansion, turn-around, and collapse) of the perturbation, the relationship:

$$1+\delta_{\rm lin} = \left(\frac{a_{\rm back}}{a_{\rm lin}}\right)^3 \tag{15.18}$$

remains valid. Substituting (15.18) into eq. (15.17) where $a_{\rm back}$ is given by the leading order term, and with the substitution $(1+\delta)^{-1/3} \simeq 1-\frac{1}{3}\delta$ valid for $\delta \ll 1$, we have:

$$\delta_{\rm lin} = \frac{3}{20}\left(6\pi\frac{t}{t_{\rm max}}\right)^{2/3} \tag{15.19}$$

so that at turnaround ($t = t_{\rm max}$) we have:

$$\delta_{\rm lin}^{\rm turn} = \frac{3}{20}(6\pi)^{2/3} = 1.06\,.$$

Of course, turnaround also represents the breakdown of linear theory, in that it represents the time when our volume containing the perturbation breaks away from the background expansion (but has not yet collapsed to form a gravitationally bound structure). The actual nonlinear density contrast at turnaround is

$$1+\delta_{\rm nonlin}^{\rm turn} = \left(\frac{a_{\rm back}}{a_{\rm max}}\right)^3 = \left[\frac{1}{4}\left(6\pi\frac{t}{t_{\rm max}}\right)^{2/3}\right]^3 = \frac{(6\pi)^2}{4^3} = 5.55 \tag{15.20}$$

obtained by considering just the leading order term of eq. (15.17).

After turnaround, the evolution of the overdensity mirrors the expansion phase (see Figure 4.2) until the object collapses at $t = 2t_{\rm max}$. At this time the linear density contrast has become

$$\delta_{\rm lin}^{\rm coll} = \delta_{\rm c} = \frac{3}{20}(12\pi)^{2/3} = 1.686\,. \qquad (15.21)$$

Thus, a *linear* density contrast $\delta_{\rm c} \simeq 1.7$ corresponds to the epoch of complete gravitational collapse of a spherically symmetric perturbation. This value of $\delta_{\rm c} \simeq 1.7$ is used in analytical treatments of the growth of structure in the universe, such as the Press-Schechter formalism (Press & Schechter 1974, ApJ, 187, 425 — this is one of the most influential papers in the field of structure formation, with over 1300 citations).

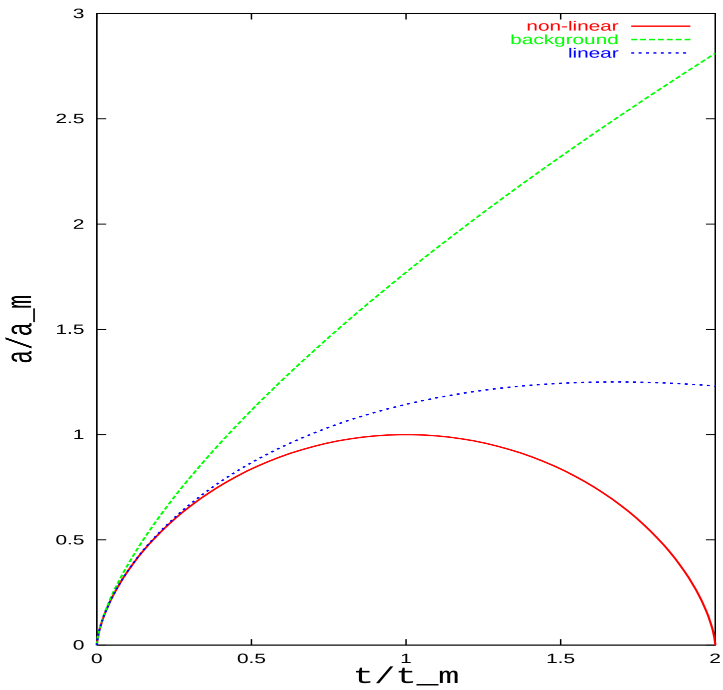

Figure 15.3: The evolution of the background scale factor, the linear scale factor and the non-linear, collapsing scale factor.

15.3 Virialization

A real density perturbation is neither spherical nor homogeneous. Thus, the collapse does not proceed to a point of infinite density, but reaches virial equilibrium at a radius $r_{\rm vir} = 1/2 r_{\rm max}$.[2] This condition is achieved when $\theta = 2\pi$. By this time, the density within our volume has increased by a factor of 2^3, while that of the background universe has decreased by a factor of 2^2, since $\rho \propto a^{-3}$ and $a \propto t^{2/3}$ in a matter-dominated universe. Thus, at virialization, the overdensity within our volume has grown from $1 + \delta^{\rm turn}_{\rm nonlin} = 5.55$ (eq. 15.20) to $5.55 \times 8 \times 4$, or

$$\boxed{1 + \delta^{\rm vir}_{\rm nonlin} \simeq 178 \qquad (15.22)}$$

a value which is confirmed by simulations.

So far we have considered the simplest case of a universe with $\Omega_{\rm m,0} = 1$. In the more general case of a low density universe, the linear density contrast at collapse, $\delta^{\rm coll}_{\rm lin}$, is still close to the value $\delta_{\rm c} \simeq 1.7$ (eq. 15.22), but the true density contrast at virialization is increased to $1 + \delta^{\rm vir}_{\rm nonlin} = 178\,\Omega^{-0.6}_{\rm m,0}$, or about a factor of two for $\Omega_{\rm m,0} = 0.3$ (see Figure 15.4). The threshold $\delta^{\rm vir}_{\rm nonlin} \simeq 200$ is often used to define a collapsed object. The 'virial' radius is the radius around a structure (a galaxy, a cluster) within which the density is $\simeq 200$ times higher than the average background density.

In terms of the initial comoving radius, $r_{\rm i,com}$, we have:

$$r^3_{\rm max} = \frac{1}{5.55}\,\frac{1}{(1+z_{\rm max})^3}\,r^3_{\rm i,com} \qquad (15.23)$$

and

$$r^3_{\rm vir} = \frac{1}{178}\,\frac{1}{(1+z_{\rm vir})^3}\,r^3_{\rm i,com}\,. \qquad (15.24)$$

The virial theorem for bound objects tells us that:

$$v^2 = \frac{GM}{r_{\rm g}} \qquad (15.25)$$

[2]This can be appreciated by considering the virial theorem $U_{\rm vir} = -2T_{\rm vir}$, where as usual the symbols U and T represent the potential and kinetic energies respectively. At turnaround, the kinetic energy of the collapsing sphere is zero. From conservation of energy, we have: $U_{\rm r_{max}} = U_{\rm vir} + T_{\rm vir}$. Thus, $U_{\rm r_{max}} = U_{\rm vir} - \frac{1}{2}U_{\rm vir} = \frac{1}{2}U_{\rm vir}$. Since the gravitational energy of a mass M within a spherical volume of radius R is $U \propto 1/R$, it follows that $R_{\rm turn} = 2R_{\rm vir}$.

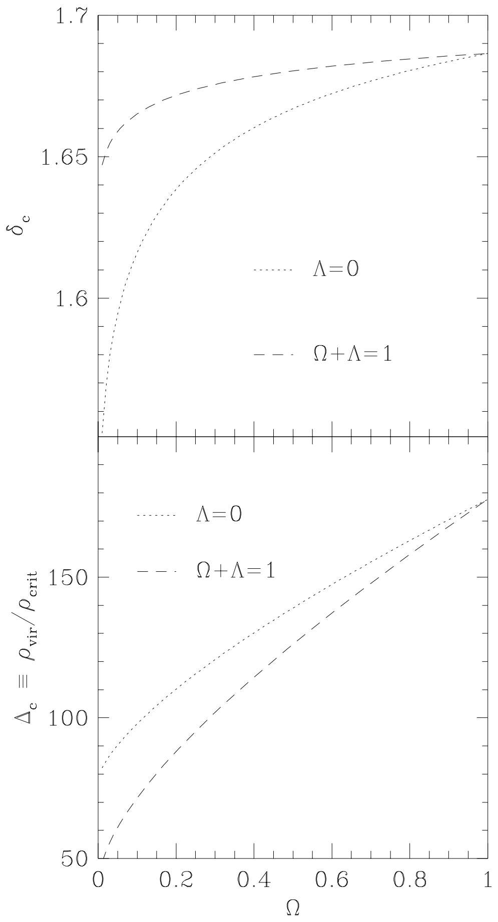

Figure 15.4: (Reproduced from Eke et al. 1996, MNRAS, 282, 263). Upper panel: critical threshold for collapse, δ_c, as a function of $\Omega_{m,0}$ in the spherical collapse model. Lower panel: the virial density of collapsed objects in units of the critical density. Since we have defined $\delta \equiv (\rho_m - \langle\rho_m\rangle)/\langle\rho_m\rangle$, $\Delta_c = \delta^{vir}_{nonlin}\,\Omega_{m,0}$.

where M is the mass of the system and r_g is the radius within which the gravitational energy is $U = -GM^2/r_g$. The mass within an initial comoving radius $r_{i,com}$ is:

$$M = \frac{4\pi}{3}\rho_{m,0}\, r^3_{i,com}\,. \tag{15.26}$$

Combining the above equations, we find that the velocity dispersion and the mass of a collapsed object are related by:

$$\left(\frac{v}{127\ \mathrm{km\ s^{-1}}}\right)^2 = \left(\frac{M}{10^{12}\,h^{-1}M_\odot}\right)^{2/3}(1+z_{vir})\ . \tag{15.27}$$

Note the factor $(1+z_{vir})$ involved in the scaling of v^2 with $M^{2/3}$. This means that perturbations which collapse at earlier times have higher velocity dispersions for the same enclosed mass. The epoch when an initial density perturbation collapses is related to its overdensity via eqs. (15.11) and (15.2), that is:

$$\theta = H_0\,\eta\,(\Omega_{\mathrm{m},0} - 1)^{1/2} = 2\pi\,. \tag{15.28}$$

Thus higher overdensities turn around and collapse at the earlier times, when the background universe was smaller and denser, and when virialised have proportionally higher velocity dispersions (15.27) and are more compact (15.25) than larger regions of lower overdensity enclosing the same mass.

The scaling of eq. (15.27) has been confirmed by simulations, although the normalisation turns out to be somewhat different from that obtained with the simple analytical analysis we have considered here.

Finally, if the matter within the volume is in hydrostatic equilibrium, we can associate a temperature to the velocity dispersion, $T \propto v^2$, and hence obtain the scaling:

$$\frac{kT}{7\ \mathrm{keV}} = \left(\frac{M}{10^{15}h^{-1}M_\odot}\right)^{2/3}(1+z_{\mathrm{vir}})\ . \tag{15.29}$$

Gas at such high temperatures gives rise to X-ray emission through thermal *bremsstrahlung* radiation (see Figure 15.5).

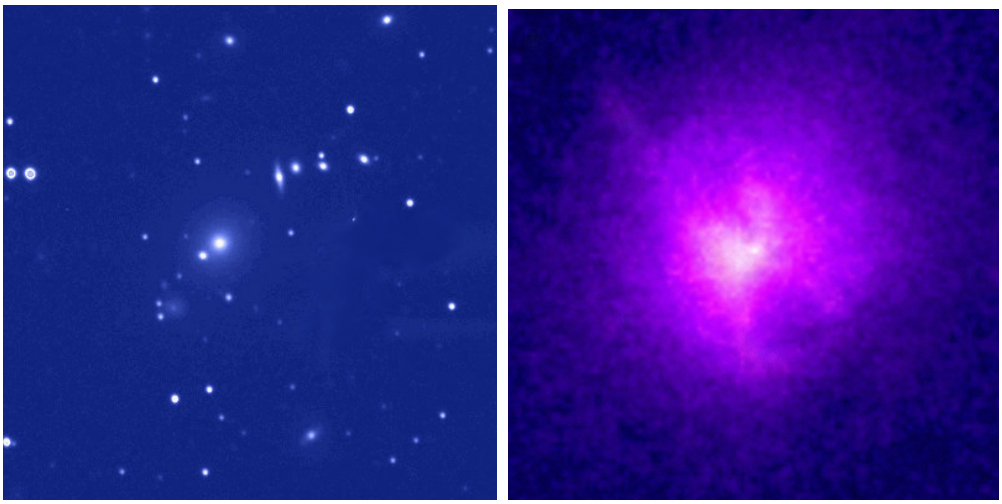

Figure 15.5: *Left:* the Hydra cluster of galaxies in optical light. The cluster consists of several hundred galaxies at a mean redshift $z = 0.054$. *Right:* the same region observed at X-ray wavelengths with the *Chandra* satellite. Hot gas at $T \simeq 4 \times 10^7\,\mathrm{K}$ extends throughout the cluster, indicative of the deep gravitational potential within which galaxies and intracluster gas move.