REDSHIFTS AND DISTANCES IN COSMOLOGY

5.1 Introduction

As we have seen in the previous lecture, a wide variety of world models are conceivable, depending on the values of the parameters Ω_i, Ω_Λ, and Ω_k. Observational cosmologists are interested in assessing which, if any, of these models is a valid description of the Universe we live in. The measurements on which these tests are based generally involve the redshifts and radiant fluxes of distant objects—modern telescopes now have the capability of reaching to early epochs in the history of the Universe. To accomplish

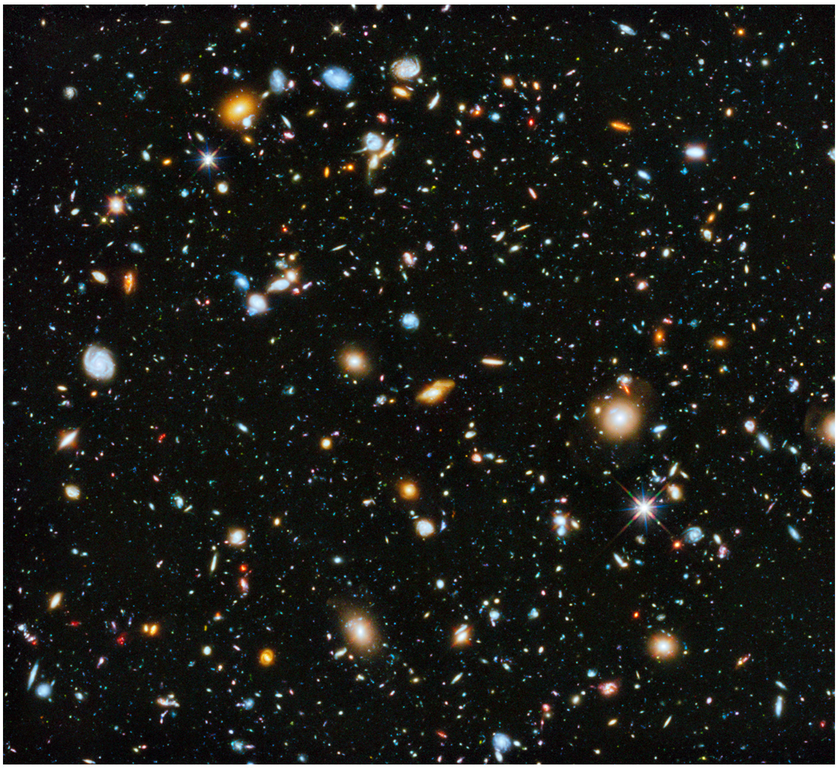

Figure 5.1: The *Hubble Ultra-Deep Field*—the deepest portrait of the visible Universe ever achieved by mankind—is a two-million-second-long exposure obtained by combining images taken with ultraviolet, optical and infrared cameras on board the *Hubble Space Telescope*. The area shown is equivalent to that subtended by a grain of sand (1 mm) held at arm's length and yet it contains an estimated 10 000 galaxies, at all distances—from nearby ones to some at redshifts $z > 7$.

this, a connection must be made between the models and the photons that arrive from distant astronomical sources.

5.2 Cosmological Redshifts

We first show that the redshift we measure for a distant galaxy (or any other source of photons) is directly related to the scale factor of the Universe at the time the photons were emitted from the source:

$$1 + z_{\rm e} = \frac{a_0}{a(t = e)}$$

We have already encountered this cosmological redshift in Lecture 1 (eq. 1.8), where we emphasised that it is intrinsically different from kinematic redshifts.

To uncover the origin of cosmological redshifts, we start with the Robertson-Walker metric:

$$(ds)^2 = (c\,dt)^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] \tag{5.1}$$

and exploit the fact that in general relativity the propagation of light is along a null geodesic ($ds = 0$). With the observer (i.e. us, on Earth) at the origin ($r = 0$), we choose a radial null geodesic so that $d\theta = d\phi = 0$, and eq. 5.1 reduces to:

$$\frac{c\,dt}{a(t)} = \pm\frac{dr}{(1-kr^2)^{1/2}} \tag{5.2}$$

where the + sign corresponds to an emitted light ray and the − sign to a received one.

Imagine now that one crest of the light wave was emitted at time t_e at distance r_e, and received at the origin $r_0 = 0$ at t_0 , and that the next wave crest was emitted at $t_e+\Delta t_e$ and received at $t_0+\Delta t_0$ (see Fig. 5.2, where the subscript '1' is used instead of subscript 'e' to indicate emission quantities).

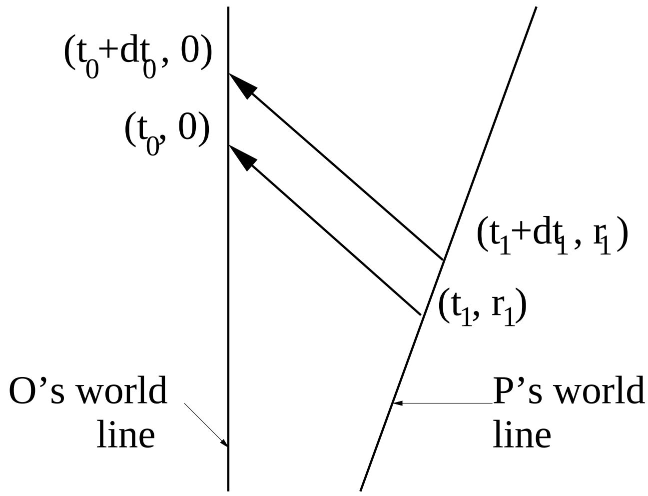

Figure 5.2: Propagation of light rays.

These times, which describe how long it takes for successive crests of the light wave to travel to Earth, satisfy the relations:

$$\int_{t_e}^{t_0} \frac{dt}{a(t)} = -\frac{1}{c} \int_{r_e}^{0} \frac{dr}{\sqrt{1-kr^2}} \tag{5.3}$$

for the first crest and:

$$\int_{t_e+\Delta t_e}^{t_0+\Delta t_0} \frac{dt}{a(t)} = -\frac{1}{c} \int_{r_e}^{0} \frac{dr}{\sqrt{1-kr^2}} \tag{5.4}$$

for the next crest. Subtracting these equations produces:

$$\int_{t_e+\Delta t_e}^{t_0+\Delta t_0} \frac{dt}{a(t)} - \int_{t_e}^{t_0} \frac{dt}{a(t)} = 0 \tag{5.5}$$

But

$$\int_{t_e+\Delta t_e}^{t_0+\Delta t_0} \frac{dt}{a(t)} = \int_{t_e}^{t_0} \frac{dt}{a(t)} + \int_{t_0}^{t_0+\Delta t_0} \frac{dt}{a(t)} - \int_{t_e}^{t_e+\Delta t_e} \frac{dt}{a(t)} \tag{5.6}$$

so that

$$\int_{t_0}^{t_0+\Delta t_0} \frac{dt}{a(t)} = \int_{t_e}^{t_e+\Delta t_e} \frac{dt}{a(t)} \tag{5.7}$$

Any change in $a(t)$ during the time intervals Δt_0 and Δt_e between successive wave crests can be safely neglected, so that we can treat $a(t)$ as a constant with respect to the time integration. Consequently

$$\frac{\Delta t_e}{a(t_e)} = \frac{\Delta t_0}{a(t_0)}; \qquad \frac{\Delta t_e}{\Delta t_0} = \frac{a(t_e)}{a(t_0)} \tag{5.8}$$

The time interval between successive wave crests is the inverse of the frequency ν of the light wave, related to its wavelength λ by the relation $c = \lambda \cdot \nu$ (or, if we had stuck to our natural units where $c = 1$, we could have said that Δt is the wavelength of the light), so that

$$\frac{\lambda_0}{\lambda_e} = 1 + z = \frac{a(t_0)}{a(t_e)} \tag{5.9}$$

Obviously, in a *contracting* Universe where $a(t_e) > a(t_0)$ [see Figures 4.1 and 4.2] we would measure blueshifts, rather than redshifts, for distant objects.

5.2.1 Time Evolution of the Hubble Parameter

Having established this correspondence between redshift and the scale factor of the Universe, we can now derive an expression for the Hubble parameter as a function of redshift, $H(z)$. Recalling the Friedmann equation in the notation of eq. 4.10:

$$\dot{a}^2 = H_0^2 \Omega_{\mathrm{m},0} a^{-1} + H_0^2 \Omega_{\Lambda,0} a^2 \tag{5.10}$$

and the definition of Ω_k (eq. 4.7):

$$\Omega_\mathrm{k} \equiv -k/(aH)^2 \tag{5.11}$$

it can be easily seen that:

$$\boxed{\left(\frac{H(z)}{H_0}\right)^2 = \Omega_{\mathrm{m},0} \cdot (1+z)^3 + \Omega_{\mathrm{k},0} \cdot (1+z)^2 + \Omega_{\Lambda,0}} \tag{5.12}$$

The right-hand side of eq. 5.12 is sometimes referred to as $E(z)$, so that:

$$H(z) = H_0 \cdot E(z)^{1/2}$$

In the simplest case of an Einstein-de Sitter cosmology ($\Omega_{\rm m,0} = 1,\ \Omega_{\rm k,0} = \Omega_{\Lambda,0} = 0$):

$$H(z) = H_0 \cdot (1+z)^{3/2}\ .$$

In today's 'consensus' cosmology $\Omega_{\rm m,0} = 0.31,\ \Omega_{\rm k,0} = 0,\ \Omega_{\Lambda,0} = 0.69$ (see Table 1.1):

$$H(z) = H_0 \cdot \sqrt{\Omega_{\rm m,0} \cdot (1+z)^3 + \Omega_{\Lambda,0}}\ . \tag{5.13}$$

A more general version of eq. 5.12 includes the energy density in radiation (i.e. relativistic components of the Universe) which, as we saw in Lecture 2, varies as $\rho_{\rm rad} \propto 1/a^4$ (eq. 2.23):

$$\frac{H(z)}{H_0} = \sqrt{\Omega_{\rm m,0}(1+z)^3 + \Omega_{\rm rad,0}(1+z)^4 + \Omega_{\rm k,0}(1+z)^2 + \Omega_{\Lambda,0}}\,, \tag{5.14}$$

from which it can be seen that at the highest redshifts it is the relativistic component of the Universe that drives the expansion, Conversely, from $z \simeq 0.3$, it is Λ that has progressively become the driving force of cosmic expansion.

5.2.2 Redshift vs. Time

We can derive a relationship between time t and redshift z by considering the following:

$$H(z) \equiv \frac{da}{dt}\frac{1}{a} = \frac{da}{dz}\frac{dz}{dt}\frac{(1+z)}{a_0}\ . \tag{5.15}$$

But, with:

$$a = \frac{a_0}{1+z}; \qquad da = -\frac{a_0}{(1+z)^2}\,dz\,, \tag{5.16}$$

and therefore:

$$dt = -\frac{dz}{H(z)\,(1+z)} \tag{5.17}$$

so that:

$$\int_{t1}^{t2} dt = -\frac{1}{H_0}\int_{z1}^{z2} \frac{dz}{(1+z)E(z)^{1/2}}\,, \tag{5.18}$$

from which we can calculate the age of the Universe:

$$t_0 = \int_0^{t_0} dt = \frac{1}{H_0}\int_0^{\infty} \frac{dz}{(1+z)E(z)^{1/2}}\,. \tag{5.19}$$

In the simplest case of an Einstein-de Sitter cosmology ($\Omega_{\mathrm{m},0} = 1,\ \Omega_{\mathrm{k},0} = \Omega_{\Lambda,0} = 0$) , we recover:

$$t_0 = \frac{1}{H_0}\int_0^{\infty} \frac{dz}{(1+z)^{5/2}} = \frac{2}{3}H_0^{-1}(1+z)^{-3/2}\Big|_{\infty}^{0} = \frac{2}{3}H_0^{-1} \tag{5.20}$$

5.3 Cosmological Distances

5.3.1 Proper Distance

We saw in Lecture 1 that the cosmological principle implies the existence of a universal time t. Since all fundamental observers see the same sequence of events in the Universe, they can synchronise their clocks by means of these events.

This allows us to define a *proper distance*, as the distance between two events, A and B, in a reference frame for which they occur simultaneously ($t_A = t_B$).

The proper distance of an object from Earth can be found by starting again from the Robertson-Walker metric:

$$(ds)^2 = (c\,dt)^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] \tag{5.1}$$

with, as before, the Earth at the origin of the radial coordinate, $d\theta = d\phi = 0$, but this time setting $dt = 0$, so that we have:

$$s(t) = \int_0^s ds' = a(t) \int_0^r \frac{dr}{(1 - kr^2)^{1/2}} \tag{5.21}$$

which has three solutions depending on whether k is +ve, 0, or −ve:

$$s(t) = a(t) \cdot \begin{cases} \frac{1}{\sqrt{k}} \sin^{-1}(r\sqrt{k}) & \text{for } k > 0 \\ r & \text{for } k = 0 \\ \frac{1}{\sqrt{|k|}} \sinh^{-1}(r\sqrt{|k|}) & \text{for } k < 0 \end{cases} \tag{5.22}$$

In a flat Universe, the proper distance to an object is just its coordinate distance, $s(t) = a(t) \cdot r$. However, because the r coordinate was introduced in the context of a flat Newtonian Universe, the coordinate distance will generally not agree with the proper distance. Because $\sin^{-1}(x) \geq x$ and $\sinh^{-1}(x) \leq x$, in a closed Universe ($k > 0$) the proper distance to an object is *greater* than its coordinate distance, while in an open Universe ($k < 0$) the proper distance to an object is *less* than its coordinate distance.

5.3.2 The Horizon

As the Universe expands and ages, an observer at any point is able to see increasingly distant objects as the light from them has time to arrive. This means that, as time progresses, increasingly larger regions of the Universe come into causal contact with the observer. The proper distance to the furthest observable point—*the particle horizon*— at time t is the horizon distance, $s_h(t)$. Two observers separated by a distance greater than $s_h(t)$ are not in causal contact. Thus, we can think of $s_h(t)$ as the size of the observable Universe.

Again we return to the Robertson-Walker metric, placing an observer at the origin ($r = 0$) and let the particle horizon for this observer at time t be located at radial coordinate distance $r_{\rm hor}$. This means that a photon emitted at $t = 0$ at $r_{\rm hor}$ will reach our observer at the origin at time t.

Since photons move along null geodesics, $ds = 0$. Considering only radially travelling photons ($d\theta = d\phi = 0$), we find:

$$\int_0^t \frac{dt}{a(t)} = \frac{1}{c} \int_0^{r_{\rm hor}} \frac{dr}{(1 - kr^2)^{1/2}}\,, \tag{5.23}$$

from which we find that:

$$r_{\rm hor} = \begin{cases} \sin\left(c \int_0^t \frac{dt}{a(t)}\right) & \text{for } k = 1 \\ c \int_0^t \frac{dt}{a(t)} & \text{for } k = 0 \\ \sinh\left(c \int_0^t \frac{dt}{a(t)}\right) & \text{for } k = -1 \end{cases} \tag{5.24}$$

Now, if the scale factor evolves with time as $a(t) \propto t^\alpha$, we can see that the above time integral diverges as we approach $t = 0$, if $\alpha \geq 1$. If this were the case, there would be no particle horizon and the whole Universe would be in causal contact.

But we have seen that in fact $a(t) \propto t^{2/3}$ and $a(t) \propto t^{1/2}$ in the matter- and radiation-dominated regimes respectively (eqs. 2.21 and 2.24). Thus, we do have a horizon.

We also recall (from eq. 5.21) that the proper distance from the origin to $r_{\rm hor}$ is given by:

$$s_{\rm hor}(t) = a(t) \int_0^{r_{\rm hor}} \frac{dr}{(1 - kr^2)^{1/2}} \tag{5.25}$$

Combining 5.23 and 5.25, the proper distance to the horizon is found to be:

$$s_{\rm hor}(t) = a(t) \int_0^t \frac{cdt}{a(t)} \tag{5.26}$$

Considering the case of zero curvature, we saw that in a radiation-dominated Universe the scale factor evolves as $a(t) \propto t^{1/2}$ (eq. 2.24). With this functional form, we find that in the radiation dominated era, the horizon scale was

$$s_{\mathrm{hor}}(t) = 2ct \tag{5.27}$$

while in a dust-dominated Universe, where $a(t) \propto t^{2/3}$ (eq. 2.21),

$$s_{\mathrm{hor}}(t) = 3ct \tag{5.28}$$

Notice that these distances are larger than ct, the distance travelled by a photon in time t. How could this be? The reason lies in our definition of proper distance, as the distance between two events measured in a frame of reference where those two events happen at the same time. In order to actually measure the size of the visible Universe at the present time (say), we would have to devise some contrived scenario—such as adding up distances measured by observers spread throughout the Universe all making measurements at the same time, t_0. The photon emitted at time $t = 0$ traversed these same regions at earlier epochs, when the Universe was smaller.

The proper distance (i.e. the familiar concept of distance) is of little practical use when we are dealing with cosmological scales, because its determination requires a *synchronous* measurement. Thus, the proper distance s is only of interest on scales where $s/c \ll 1/H$. When dealing with cosmological distances, astronomers use two different concepts of distance—one based on the comparison of the observed angular diameter of an object to its true (proper) diameter, and the other based on the comparison of the apparent luminosity of a source to its absolute luminosity. We now describe these two methods in turn.

5.3.3 Angular Diameter Distance

Consider, as in Fig. 5.3, a light source of size D at $r = r_1$ and $t = t_1$ subtending an angle $\delta\theta$ at the origin ($r = 0, t = t_0$). The proper distance between the two ends of the object is, according to our metric (eq. 5.1)

$$D = a(t_1) r_1 \delta\theta \tag{5.29}$$

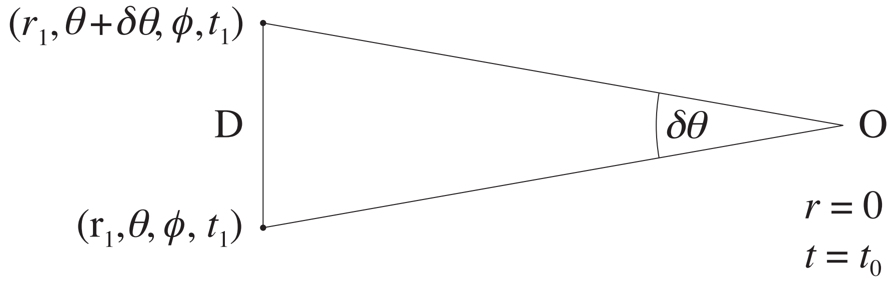

Figure 5.3: An object of size D subtending an angle $\delta\theta$ at the observer.

so that the angular diameter of the source is

$$\delta\theta = \frac{D}{a(t_1)\, r_1} \, . \tag{5.30}$$

In Euclidean geometry the angular diameter of a source of diameter D at a distance d is $\delta\theta = D/d$, so we define in *general* the angular diameter distance as:

$$d_{\mathrm{A}} \equiv \frac{D}{\delta\theta} \, , \tag{5.31}$$

and we now see that:

$$d_{\mathrm{A}} = a(t_1)\, r_1 = \frac{r_1}{1+z} \, . \tag{5.32}$$

Since we are studying the propagation of light, we can write as before (eq. 5.2 and following):

$$\int_{t_1}^{t_0} \frac{c\,dt}{a(t)} = c \int_0^z \frac{dz}{H(z)} = \frac{1}{|k|^{1/2}} S_k^{-1}(|k|^{1/2} r_1) \, , \tag{5.33}$$

where the first equation was obtained by substituting the time integration with a redshift integration and using:

$$\frac{dz}{dt} = -\frac{\dot{a}}{a^2} = -\frac{H}{a} \, , \tag{5.34}$$

and

$$S_k(x) = \begin{cases} \sin(x) & \text{for } k > 0 \\ x & \text{for } k = 0 \\ \sinh(x) & \text{for } k < 0 \end{cases} \tag{5.35}$$

Adopting, $|k|^{1/2} = \frac{H_0}{c}\sqrt{\Omega_{\rm k,0}}$, we arrive at:

$$d_{\rm A}(z) = \frac{c}{\sqrt{|\Omega_{\rm k,0}|}H_0(1+z)} \cdot S_k\left(H_0\sqrt{|\Omega_{\rm k,0}|}\int_0^z \frac{dz}{H(z)}\right). \tag{5.36}$$

Recalling eq. 5.12 for the Hubble parameter as a function of redshift, we now see that the angular diameter distance to an object at redshift z depends on the cosmological parameters $\Omega_{\Lambda,0}$, $\Omega_{\rm k,0}$, and $\Omega_{\rm m,0}$, as well as the Hubble constant H_0. In today's consensus cosmology with $\Omega_{\rm k,0} = 0$, eq. 5.36 reduces to the simpler form:

$$d_{\rm A}(z) = \frac{c}{H_0}\frac{1}{(1+z)}\int_0^z \frac{dz}{\left[\Omega_{\rm m,0}(1+z)^3 + \Omega_{\Lambda,0}\right]^{1/2}} \tag{5.37}$$

Interestingly, in some cosmologies $d_{\rm A}(z)$ is *not* a monotonically increasing function of z, but reaches a maximum value at some redshift $z_{\rm max}$ and then *decreases* with increasing redshift. Referring back to eq. 5.31, what this means is that in some cosmologies objects of a given proper size D will subtend a minimum angle $\delta\theta$ on the sky at $z = z_{\rm max}$. At redshifts $z > z_{\rm max}$ objects of a given proper size will appear bigger on the sky with increasing z. Fig. 5.4 shows the behaviour of $d_{\rm A}(z)$ in different cosmologies.

This counterintuitive behaviour of $d_{\rm A}(z)$ can be understood by referring to Fig. 5.5 which makes a distinction between 'emission distance' and 'reception distance'. In simple terms, the Universe was smaller when the light rays from the object under consideration were emitted. Back then, objects of a given proper size occupied a larger coordinate size than they do today, and so would have subtended a larger angle. Galaxies at $z = z_{\rm max}$ have an emission distance which is equal to the particle horizon at the time of emission. Galaxies at $z < z_{\rm max}$ were within the particle horizon at the time of emission, while galaxies which we now observe to be at $z > z_{\rm max}$ were

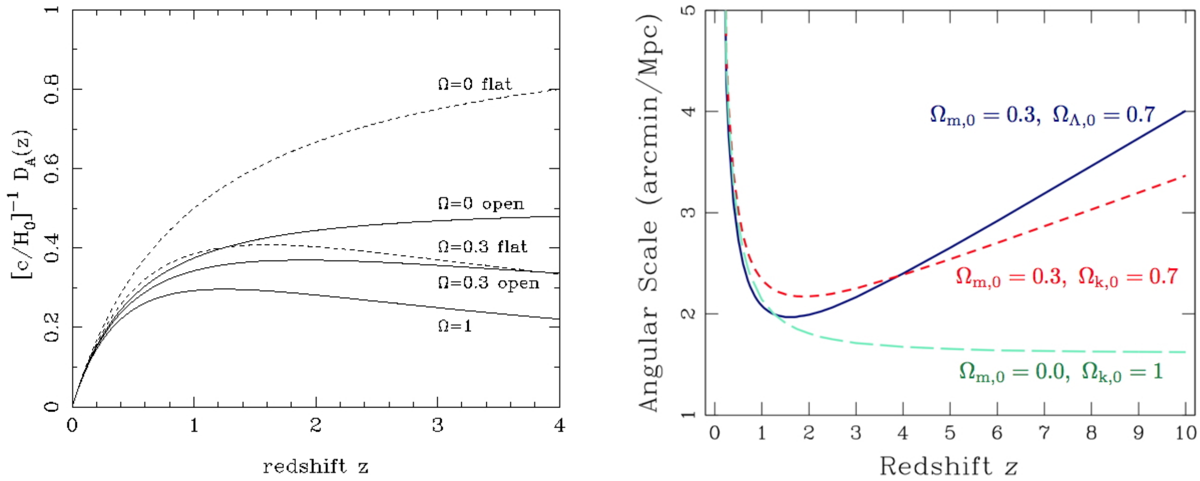

Figure 5.4: *Left:* Plot of the angular diameter distance, d_A, (in units of c/H_0) as a function of redshift z in different cosmologies (in this figure $\Omega \equiv \Omega_{m,0}$). *Right:* Angle subtended on the sky by an object of proper size $D = 1\,\text{Mpc}$ as a function of redshift for three different cosmologies, as indicated.

beyond our particle horizon at the time of emission. Notice from Fig. 5.5 that when Y emits light towards O, the lightcone is diverging away from O's worldline; the light rays leaving Y at first move away from O and only after reaching the maximum emission distance, converge on O.

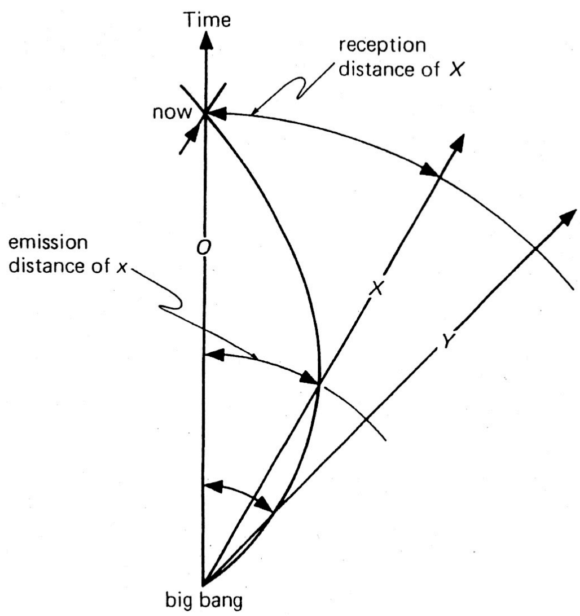

Figure 5.5: O's lightcone curves back into the Big Bang. The diagram shows the reception and emission distances of galaxies X and Y. Although galaxy Y has a greater reception distance, its emission distance is smaller than that of X. Thus Y, which is now further away than X, was closer to us than X at the time of the emission of the light which we now see (reproduced from E. R. Harrison's *Cosmology*).

Returning to eq. 5.37, it can be shown that for an Einstein-de Sitter cosmology ($\Omega_{\rm m,0} = 1$, $\Omega_{\Lambda,0} = \Omega_{\rm k,0} = 0$), $z_{\rm max} = 1.25$, where the angle subtended by an object is (5.31):

$$\delta\theta_{\rm min} = \delta\theta(z_{\rm m}) = 3.375\frac{H_0 D}{c} \tag{5.38}$$

For instance, a galaxy cluster of typical diameter $D = 1\,\text{Mpc}$ (see Figure 5.6), would never subtend on the sky an angle smaller than:

$$\delta\theta_{\rm min} = \frac{3.375 \cdot 100\,h \cdot 1}{3 \times 10^5} \simeq 1.13 \times 10^{-3}\,h\,\text{radians} \simeq 4\,h\,\text{arcmin}$$

regardless of how far away it is.

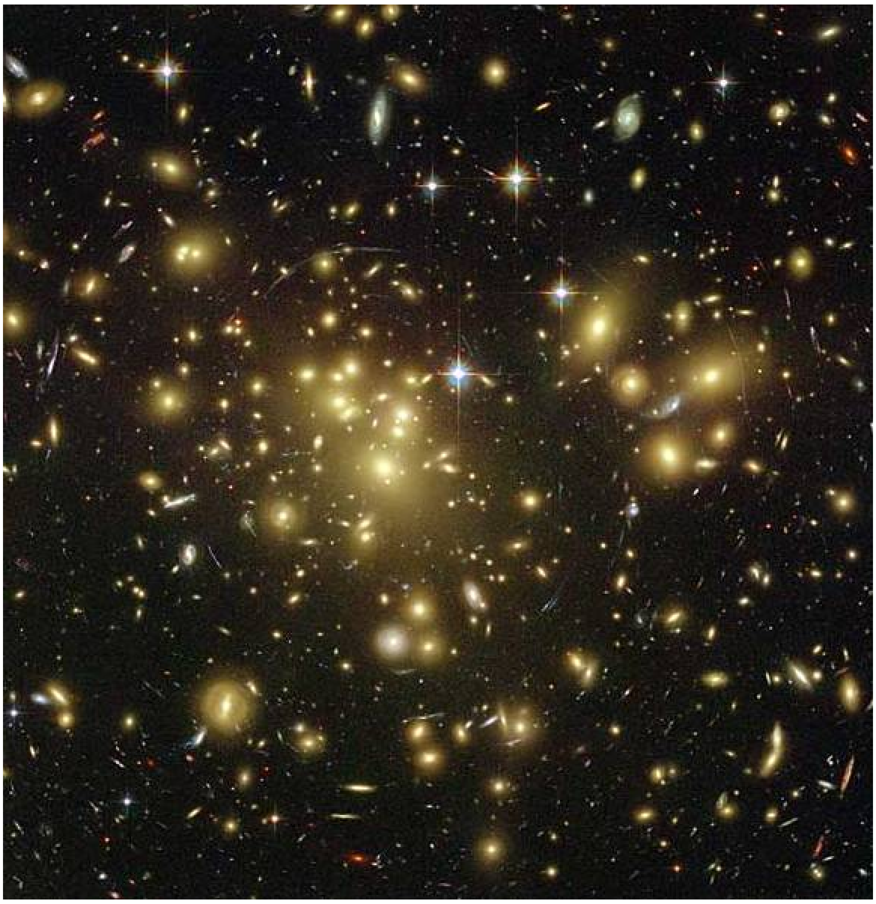

Figure 5.6: The galaxy cluster Abell 1689 at redshift $z = 0.183$, as pictured by the Advanced Camera for Surveys on board the *Hubble Space Telescope*. With a mass $M_{\rm cluster} \simeq 1 \times 10^{15} M_\odot$, Abell 1689 is one of the most massive galaxy clusters known. This very deep (13 hours) *HST*-ACS exposure provides a stunning demonstration of gravitational lensing of distant galaxies by the cluster potential.

The dependence of the angular diameter distance on $\Omega_{\mathrm{k},0}$ prompted a number of tests of the geometry of the Universe based on measuring the angular size of different sources (galaxies, clusters of galaxies, radio sources) as a function of redshift. However any conclusions from such tests are made very uncertain by the possibility (likelihood?) of intrinsic evolution in the proper size D of such sources with look-back time. In other words, we have yet to find an astronomical object that can used reliably as a 'standard ruler'.

An excellent probe of the geometry is the angular power spectrum of the temperature fluctuations of the Cosmic Microwave Background (CMB). We will consider the CMB in some detail later on in the course. For the present purpose, it is sufficient to say that the CMB radiation is a snapshot of the oldest light in our Universe, imprinted on the sky when the Universe was just 372 000 years old (at $z_{\mathrm{em}} = 1090$). Following the epoch of recombination, when the temperature of the expanding Universe had fallen sufficiently for electron and protons to form neutral hydrogen, photons were finally able to stream free through the Universe.

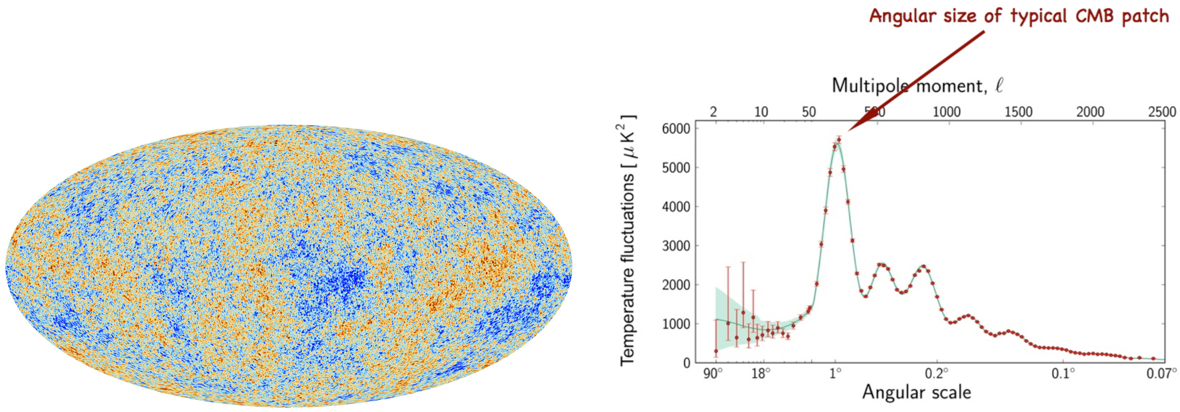

Figure 5.7: Temperature fluctuations of the Cosmic Microwave Background recorded by the Planck satellite (Images: ESA).

The spectrum of the CMB is a near-perfect blackbody with a temperature of $2.7255 \pm 0.0006\,\mathrm{K}$. As the precision of the CMB maps improved, however, astronomers found *tiny* variations in the CMB temperature at different locations on the sky, amounting on average to only about one part in 100 000. These temperature fluctuations reflect tiny fluctuations in the matter density, already present at this early epoch. It is these tiny fluctuations that, under the influence of gravity, grew into the large-scale structure in the distribution of galaxies that we see around us today.

One can calculate the typical size of an overdense (or underdense) region at the time the microwave photons started to stream free. As we also know the distance to this last scattering surface, we can compare their ratio to the observed angular size and hence obtain a very accurate measurement of the curvature of the Universe. The favoured solution is that we live in a flat Universe, with $k = 0$.

The power spectrum of temperature and polarization anisotropies of the cosmic microwave background radiation is a true treasure trove of information on the most important cosmological parameters that describe our Universe. For this reason, the CMB radiation has been studied extensively, and at increasingly higher spatial resolution, since its discovery in 1965. We shall return to this key topic in cosmology later on in the course.

5.3.4 Luminosity Distance

The luminosity distance $d_{\rm L}$, which is commonly used in measurements of sources at cosmological distances, is defined to be the distance satisfying the relation:

$$F_{\rm obs} = \frac{L}{4\pi d_{\rm L}^2}\,, \tag{5.39}$$

where $F_{\rm obs}$ is the observed flux from an astronomical source and L is its absolute luminosity. Definitions are important here. We define flux as the *energy* that passes per unit time through a unit area (so, for example, the energy per unit time, or the power, collected by a telescope of area T is $F \times T$), and luminosity as the total power (energy per unit time) emitted by the source at all wavelengths (sometimes referred to as the *bolometric* luminosity). Eq. 5.39 is a straightforward consequence of the fact that, at a distance r_1 from a source, its photons are spread out over the surface of a sphere of area

$$A = r_1^2 \int\!\!\int \sin\theta \, d\theta \, d\phi \; = \; 4\pi r_1^2\,.$$

In a cosmological context, we have to consider that the total received power, integrated over the 4π solid angle of the surface of the sphere, is *not* the same as the emitted power. This is because photons emitted with wavelength λ_1 at time intervals δt_1 are received (by an observer on the surface of the sphere) at time intervals δt_0 and with wavelength λ_0.

As we have already seen, both wavelengths and time intervals are related by:

$$\frac{\lambda_1}{\lambda_0} = \frac{\delta t_1}{\delta t_0} = \frac{a_1}{a_0}$$

So, considering a single photon of energy $h\nu = hc/\lambda$,

$$\text{Emitted power :} \qquad P_{\text{em}} = \frac{h\nu_1}{\delta t_1}$$

$$\text{Received power :} \qquad P_{\text{obs}} = \frac{h\nu_0}{\delta t_0} = \frac{h\nu_1}{\delta t_1} \cdot \frac{a_1^2}{a_0^2}$$

Thus, the flux (power per unit area) measured on the surface of the sphere at distance r_1 from the source will be:

$$F_{\text{obs}} = L \cdot \frac{1}{4\pi a_0^2 r_1^2} \cdot \frac{a_1^2}{a_0^2} \tag{5.40}$$

Comparing (5.40) with (5.39), and with our usual convention that $a_0 = 1$, we can see that:

$$d_{\text{L}} = \frac{r_1}{a} = (1+z)\, r_1\,.$$

Repeating the steps at (5.33) and following equations for the radial coordinate as a function of redshift, we finally obtain:

$$d_{\text{L}}(z) = \frac{c(1+z)}{\sqrt{|\Omega_{\text{k},0}|}H_0}\, S_k\left(H_0\sqrt{|\Omega_{\text{k},0}|}\int_0^z \frac{dz}{H(z)}\right) \tag{5.41}$$

and

$$d_{\text{L}} = (1+z)^2 \cdot d_{\text{A}} \tag{5.42}$$

5.4 The deceleration parameter

As we have seen, the concept of distance is a difficult one in an expanding Universe. Of course, the distinctions between proper, angular, and luminosity distance fade away when we are dealing with nearby objects. For $z \ll 1$ and small r_1 we have:

$$d_{\text{P}} \simeq d_{\text{A}} \simeq d_{\text{L}} \simeq r_1\,.$$

Recall eq. 5.33:

$$\int_{t_1}^{t_0} \frac{cdt}{a(t)} = c\int_0^z \frac{dz}{H(z)} = \frac{1}{|k|^{1/2}} S_k^{-1}(|k|^{1/2} r_1) \tag{5.43}$$

and eq. 5.12 for $H(z)$:

$$H(z) = H_0 \left[\Omega_{\mathrm{m},0} \cdot (1+z)^3 + \Omega_{\mathrm{k},0} \cdot (1+z)^2 + \Omega_{\Lambda,0}\right]^{1/2} = H_0 \cdot E(z)^{1/2} \,. \tag{5.44}$$

For small values of z, we can write:

$$E(z) = \Omega_{\mathrm{m},0}(1+3z) + (1 - \Omega_{\mathrm{m},0} - \Omega_{\Lambda,0})(1+2z) + \Omega_{\Lambda,0}$$

and

$$E(z) = 1 + 2z\left(\frac{1}{2}\Omega_{\mathrm{m},0} - \Omega_{\Lambda,0} + 1\right) = 1 + 2z(q_0 + 1) \tag{5.45}$$

by defining:

$$q_0 = \frac{\Omega_{\mathrm{m},0}}{2} - \Omega_{\Lambda,0} \,. \tag{5.46}$$

The parameter

$$q(t) = -\frac{1}{H^2}\frac{\ddot{a}}{a} = -a\,\frac{\ddot{a}}{\dot{a}^2} \tag{5.47}$$

is called the *deceleration parameter* and, as the name implies, it describes whether the expansion of the Universe is slowing down ($q > 0$) or accelerating ($q < 0$). With the currently favoured values for $\Omega_{\mathrm{m},0}$ and $\Omega_{\Lambda,0}$ (Table 1.1), we see from eq. 5.46 that at present we live in an accelerating Universe.

Returning to (5.43), we can now write:

$$\int_0^z \frac{dz}{H(z)} = \frac{1}{H_0}\int_0^z \frac{dz}{E(z)^{1/2}} \approx \frac{1}{H_0}\int_0^z [1 - z(q_0+1)]\, dz \approx \frac{1}{H_0}\left[z - (q_0+1)\frac{z^2}{2}\right]$$

and the luminosity distance becomes:

$$d_{\mathrm{L}} = (1+z)r_1 \approx \frac{c}{H_0}\left[z + \frac{1}{2}(1-q_0)z^2 + \cdots\right] \tag{5.48}$$