WORLD MODELS

In this lecture we are going to solve the Friedmann equations to deduce the evolution of the Universe, $a(t)$, in different cosmological models. We are also going to simplify our notation by adopting the so-called 'natural units', where $c = 1$. (In this notation, mass density and energy density, related by the equation $E = mc^2$, become the same).

Note that eq. 3.21 (the first Friedmann equation) then becomes:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi}{3} G\rho \tag{4.1}$$

and the constant k now has units of $[\text{time}]^{-2}$ — setting $c = 1$ makes time and length units interchangeable.

We recall that:

$$H(t) \equiv \frac{\dot{a}}{a} \tag{4.2}$$

and that we can formally associate an energy density with the cosmological constant:

$$\rho_\Lambda \equiv \frac{\Lambda}{8\pi G} \tag{4.3}$$

(see discussion after eqs. 3.28 and 3.29).

With this notation we can re-write Friedmann equation:

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\left(\sum_i \rho_i + \rho_\Lambda\right) \tag{4.4}$$

where the index i is a label for the type of particle fluid under study, such as matter or radiation. Note that in general we have to sum over all the 'particle' species or energy components in the Universe in order to obtain the total energy-momentum tensor.

If the geometry of the universe is flat, $k = 0$, and therefore:

$$\rho_{\rm tot} \equiv \sum_i \rho_i + \rho_\Lambda = \frac{3H^2}{8\pi G} \equiv \rho_{\rm crit} \tag{4.5}$$

which we can use to define the fraction of the critical density contributed by each component of the Universe:

$$\Omega_i \equiv \frac{\rho_i}{\rho_{\rm crit}} \tag{4.6}$$

Thus we have $\Omega_{\rm m}$, $\Omega_{\rm r}$, and Ω_Λ for matter, radiation, and dark energy respectively. These quantities are of course all time dependent [because $H = f(t)$ in eq. 4.5]; we shall indicate their values today with the additional subscript 0, i.e. $\Omega_{i,0}$.[1]

With this notation we can rewrite eq. 4.4 in the more compact form:

$$\frac{k}{a^2H^2} = \sum_i \Omega_i + \Omega_\Lambda - 1$$

and we can do even better by defining:

$$\Omega_k \equiv -k/(aH)^2 \tag{4.7}$$

which then leads to the elegant result:

$$\sum_i \Omega_i + \Omega_\Lambda + \Omega_k = 1 \tag{4.8}$$

Note that the sign of the definition of Ω_k varies in the literature.

Out of the many world models described by the Friedmann equations, we shall consider the evolution of the scale factor, $a(t)$, in two general cases. In both cases we shall assume pressureless matter ($p = 0$). The two general cases are those for $\Omega_k = 0$ (flat cosmologies—often referred to as Friedmann-Robertson-Walker, or FRW, models), and those for $\Omega_\Lambda = 0$ (no cosmological constant).

4.1 Flat FRW Cosmologies

We have already seen (eqs. 2.18 and subsequent ones) that in this case the density of matter evolves as:

$$\rho_{\rm m} = \rho_{\rm m,0}\left(\frac{a}{a_0}\right)^{-3}$$

[1]Note that this notation is different from the more common, but less precise, convention whereby Ω_i is taken to indicate $\Omega_{i,0}$.

and that by setting $a_0 \equiv 1$, the first Friedmann equation becomes:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_{\rm m,0}a^{-3} + \frac{\Lambda}{3} \tag{4.9}$$

which can be re-written as:

$$\dot{a}^2 = H_0^2\Omega_{\rm m,0}a^{-1} + H_0^2\Omega_{\Lambda,0}a^2 \tag{4.10}$$

Note that in a flat universe $\Omega_{\rm m,0} + \Omega_{\Lambda,0} = 1$ (eq. 4.8).

We consider three cases, corresponding to $\Lambda = +$ve, $-$ve, and 0 in turn.

$\boldsymbol{\Lambda > 0}$. Using the substitution:

$$u = \frac{2\Omega_{\Lambda,0}}{\Omega_{\rm m,0}}\,a^3$$

we have:

$$\dot{u}^2 = 9H_0^2\Omega_{\Lambda,0}\left[2u + u^2\right] = 3\Lambda\left[2u + u^2\right]$$

If we take the positive root of this equation, we obtain:

$$\int_0^u \frac{du}{(2u+u^2)^{1/2}} = \int_0^t (3\Lambda)^{1/2}\,dt = (3\Lambda)^{1/2}t$$

(for a Big Bang model with $a = 0$ at $t = 0$). This can be integrated by completing the square in the u-integral and with substitutions $v = u + 1$ and $\cosh w = v$:

$$\int_0^u \frac{du}{\left[(u+1)^2 - 1\right]^{1/2}} = \int_1^v \frac{dv}{(v^2-1)^{1/2}} = \int_0^w \frac{\sinh w\,dw}{\left(\cosh^2 w - 1\right)^{1/2}} = \int_0^w dw = w$$

and we finally obtain the time evolution of the scale factor:

$$a^3 = \frac{\Omega_{\rm m,0}}{2\Omega_{\Lambda,0}}\left[\cosh(3\Lambda)^{1/2}t - 1\right] \tag{4.11}$$

$\boldsymbol{\Lambda < 0}$. In this case we introduce

$$u = -\frac{2\Omega_{\Lambda,0}}{\Omega_{\rm m,0}}\,a^3$$

and proceed as above to obtain:

$$a^3 = \frac{\Omega_{\rm m,0}}{2(-\Omega_{\Lambda,0})}\left\{1 - \cos\left[3(-\Lambda)\right]^{1/2} t\right\} \tag{4.12}$$

$\mathbf{\Lambda = 0}$. This is now the Einstein-deSitter case which we have already encountered:

$$a = \left(\frac{9}{4}H_0^2 t^2\right)^{1/3} \tag{4.13}$$

The three solutions are shown graphically in Figure 4.1.

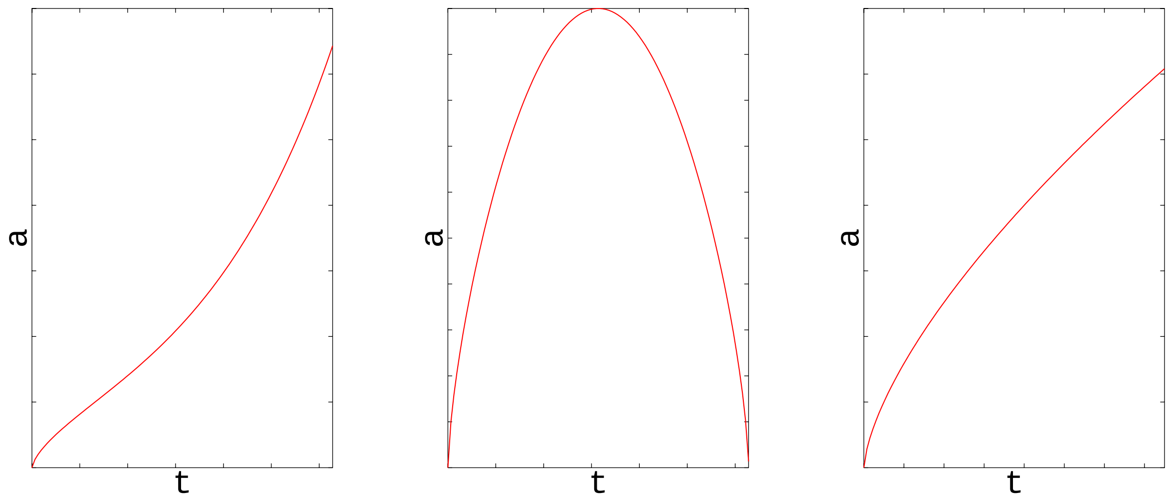

Figure 4.1: The three flat, pressureless cosmological models. On the left $\Lambda > 0$, in the middle $\Lambda < 0$ and on the right the Einstein de-Sitter model with $\Lambda = 0$.

A flat, pressureless universe with a small, but non-zero, cosmological constant initially evolves as if it were Einstein-deSitter. This can be appreciated immediately from eq. 4.10 where the first term on the right-hand side dominates when a is small (provided $\Omega_{\rm m,0} \sim \Omega_{\Lambda,0}$).

More generally, we can assess qualitatively the behaviour of these flat, pressureless solutions by considering the right-hand sides of eqs. 4.9 and 4.10. If $\Lambda \geq 0$, a grows without bound. For $\Lambda > 0$, the second term on the right-hand side of these equations (the Λ term) dominates at large values of t and the universe grows exponentially:

$$a \propto \exp\left[(\Lambda/3)^{1/2}\ t\right]$$

On the other hand, for negative values of Λ, $\dot{a} = 0$ when:

$$a = a_{\max} = \left(-\frac{\Omega_{\mathrm{m},0}}{\Omega_{\Lambda,0}}\right)^{1/3}$$

which is a local maximum.

4.2 Cosmologies with $\mathbf{k \neq 0}$ and $\boldsymbol{\Lambda = 0}$

In this case, we have to solve:

$$\dot{a}^2 = \Omega_{\mathrm{m},0} H_0^2 a^{-1} - k = \Omega_{\mathrm{m},0} H_0^2 a^{-1} + \Omega_{\mathrm{k},0} H_0^2 \,. \tag{4.14}$$

Note that now we have $\Omega_{\mathrm{k},0} = 1 - \Omega_{\mathrm{m},0}$ (eq. 4.8).

Before solving eq. 4.14 we can again look qualitatively at its behaviour.

If $\Omega_{\mathrm{k},0} > 0$ (which, recalling the definition of Ω_{k} in eq. 4.7, corresponds to *negative* curvature), the second term on the right-hand side of 4.14 dominates at late times, when the scale factor a is large. In this limit,

$$\dot{a}^2 \simeq \Omega_{\mathrm{k},0} H_0^2 = -k$$

and therefore $a \propto t$ — the scale factor grows linearly with time.

On the other hand, if $\Omega_{\mathrm{k},0} < 0$ (positive curvature), then there will be a time when the two terms on the right-hand side of 4.14 are equal and $\dot{a} = 0$. This is again a local maximum which occurs at:

$$a_{\max} = \frac{\Omega_{\mathrm{m},0}}{|\Omega_{\mathrm{k},0}|} \tag{4.15}$$

To solve eq. 4.14 for $k > 0$ ($\Omega_{\mathrm{k},0} < 0$), we use the substitution $u^2 = -a/a_{\max} = a\,k/(\Omega_{\mathrm{m},0} H_0^2)$ and obtain:

$$\dot{u}^2 = \frac{u^{-2} H_0^2 |\Omega_{\mathrm{k},0}|^3}{4\Omega_{\mathrm{m},0}^2} \left[u^{-2} - 1\right] .$$

When we substitute $u = \sin\theta$, we can integrate this differential equation and obtain:

$$t = c_1 \left\{ \sin^{-1}\left[\frac{a}{a_{\max}}\right]^{1/2} - \left[\frac{a}{a_{\max}}\right]^{1/2} \left[1 - \frac{a}{a_{\max}}\right]^{1/2} \right\} \tag{4.16}$$

with $c_1 = \Omega_{\mathrm{m},0}/(|\Omega_{\mathrm{k},0}|^{3/2} H_0)$.

Similarly, for $k < 0$ ($\Omega_{\mathrm{k},0} > 0$) we obtain:

$$t = c_1 \left\{ -\sinh^{-1} \left[\frac{a}{a_{\mathrm{max}}} \right]^{1/2} + \left[\frac{a}{a_{\mathrm{max}}} \right]^{1/2} \left[1 + \frac{a}{a_{\mathrm{max}}} \right]^{1/2} \right\} . \tag{4.17}$$

The behaviour of these two functions is shown graphically in Figure 4.2 .

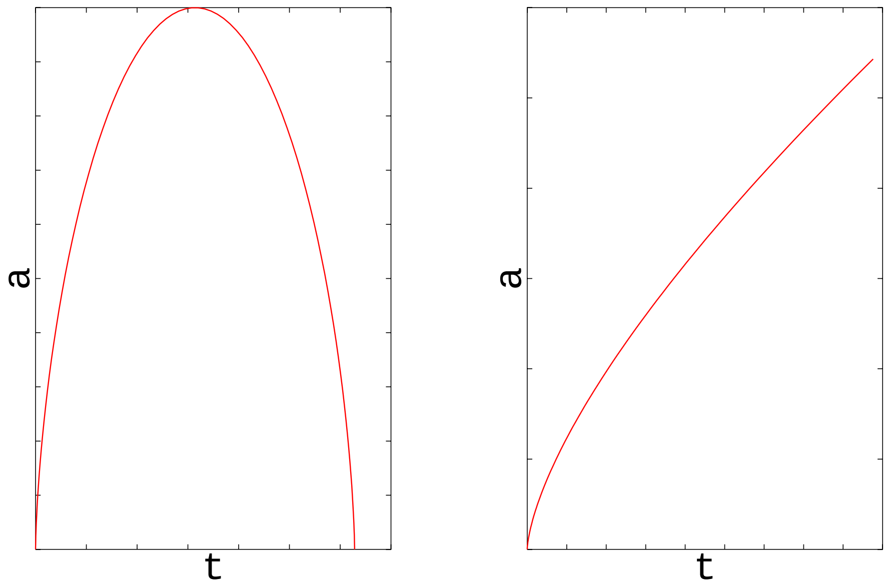

Figure 4.2: The two non-flat FRW models with vanishing cosmological constant $\Lambda = 0$. On the left with $k = +1$ and on the right with $k = -1$.

Finally, we mention the cosmology proposed by de Sitter: a flat Universe, devoid of matter ($\rho = 0$), whose expansion is propelled by a positive cosmological constant. In this case we have:

$$a = \exp \left[\left(\frac{1}{3} \Lambda \right)^{1/2} t \right] \tag{4.18}$$

where we chose $a = 1$ at $t = 0$.

In Figure 4.3 we show the expansion history of the Universe for different values of the parameters $\Omega_{\mathrm{m},0}$, $\Omega_{\Lambda,0}$, and $\Omega_{\mathrm{k},0}$; it is clear that the acceleration caused by a non-zero $\Omega_{\Lambda,0}$ leads to an older Universe.

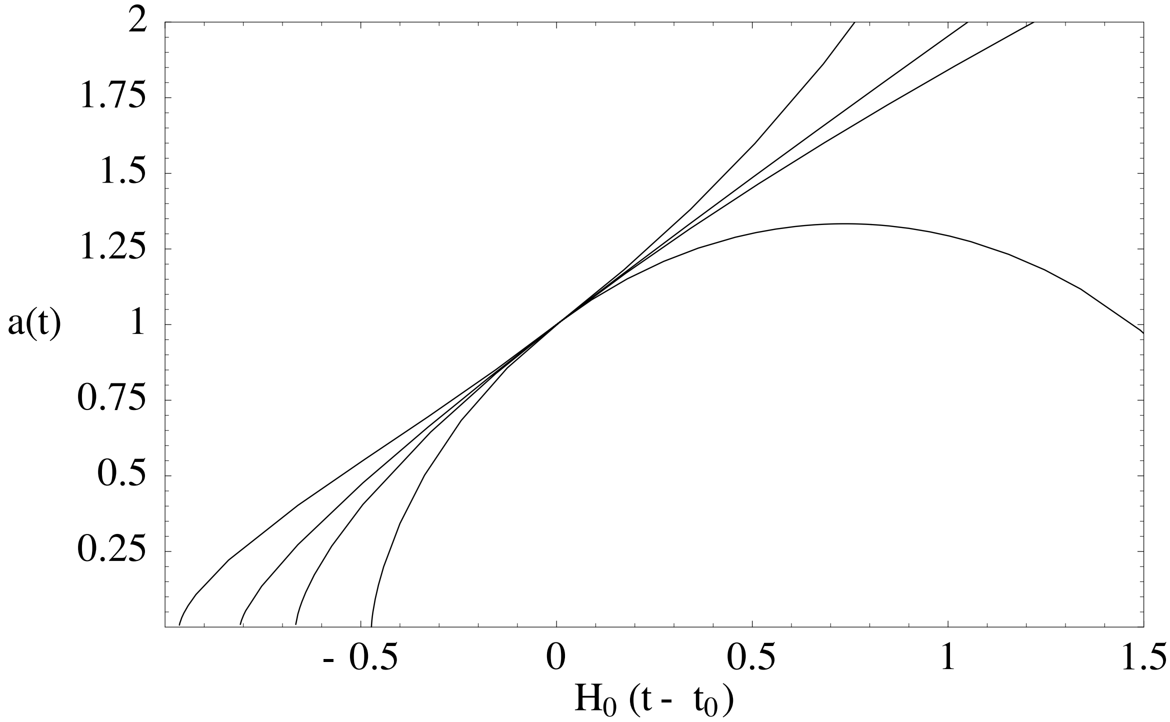

Figure 4.3: Expansion histories for different values of $\Omega_{\rm m,0}$, $\Omega_{\Lambda,0}$, and $\Omega_{\rm k,0}$. From top to bottom, the curves describe $\Omega_{\rm m,0}$, $\Omega_{\Lambda,0}$, $\Omega_{\rm k,0} = (0.3, 0.7, 0.0)$, $(0.3, 0.0, 0.7)$, $(1.0, 0.0, 0.0)$, and $(4.0, 0.0, -3.0)$.