NEWTONIAN COSMOLOGY

The equations that describe the time evolution of an expanding universe which is homogeneous and isotropic can be deduced from Newtonian dynamics and gravitation. Although the derivation is not strictly self-consistent in that it requires the use of a result from general relativity (GR)—Birkhoff's theorem—it nevertheless provides some intuitive insights and is a valuable first step.

First we show that the Hubble expansion is a natural property of an expanding universe that obeys the cosmological principle. Referring to Figure 2.1:

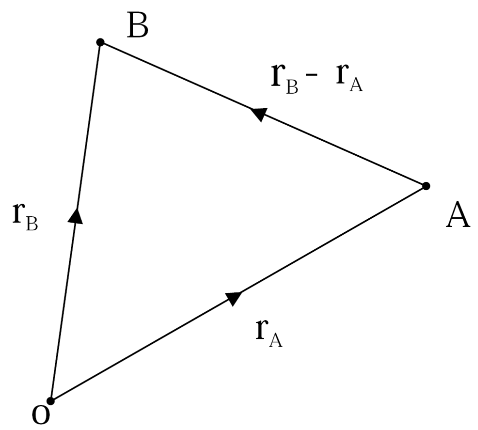

Figure 2.1: All observers see galaxies expanding with the same Hubble law.

$$\mathbf{v}_A = H_0 \cdot \mathbf{r}_A \tag{2.1}$$

and

$$\mathbf{v}_B = H_0 \cdot \mathbf{r}_B \tag{2.2}$$

where $\mathbf{v}$ and $\mathbf{r}$ are respectively the velocity and position vectors and the subscript 0 is used to indicate the present time.

By the rule of vector addition, the recession velocity of galaxy B as seen by an observer on galaxy A is

$$\mathbf{v}_{BA} = \mathbf{v}_B - \mathbf{v}_A = H_0\mathbf{r}_B - H_0\mathbf{r}_A = H_0(\mathbf{r}_B - \mathbf{r}_A) \tag{2.3}$$

So the observer on galaxy A sees all other galaxies in the universe receding with velocities described by the *same* Hubble law as on Earth.

In a homogeneous universe every particle moving with the substratum has a purely radial velocity proportional to its distance from the observer. As eq. 2.3 can be written for *any* two particles, we can change to a more convenient coordinate system, known as **comoving** coordinates. These are coordinates that are carried along with the expansion, so that we can express the distance $\mathbf{r}$ as a product of the comoving distance $\mathbf{x}$ and a term $a(t)$ which is a function of time only:

$$\mathbf{r}_{BA} = a(t) \cdot \mathbf{x}_{BA} \tag{2.4}$$

The original $\mathbf{r}$ coordinate system, which does not expand, is usually known as physical coordinates.

The term $a(t)$ is the scale factor of the universe, and it tells us how physical separations grow with time, since the coordinate distances $\mathbf{x}$ are by definition fixed. Deriving an equation for the universal expansion thus reduces to determining a function which describes $a(t)$.

In Newtonian cosmology this is done by considering the forces acting on masses A, B, C, D on a sphere of radius r centred at O, as in Figure 2.2.

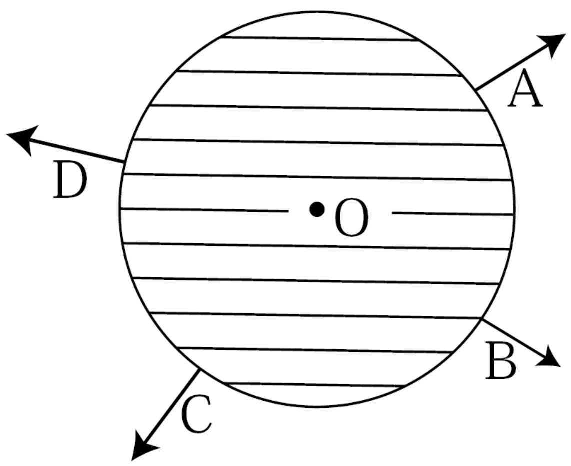

Figure 2.2: Birkhoff's theorem: the force acting on A, B, C, D—which are particles located on the surface of a sphere of radius r—is the gravitational attraction from the matter internal to r only, acting as a point mass at O.

Birkhoff's theorem states that the net gravitational effect of a uniform external medium on a spherical cavity is zero—in other words, the force acting on A, B, C, D is the gravitational attraction from the matter M internal to r only, which acts as a point mass at O. We can then write the total energy of a particle of mass m at A, B, C, D as the usual sum of kinetic and gravitational potential energy

$$U = T + V = \frac{1}{2}m\dot{r}^2 - \frac{GMm}{r} = \frac{1}{2}m\dot{r}^2 - \frac{4\pi}{3}G\rho r^2 m \tag{2.5}$$

where the dot denotes differentiation with respect to time, ρ is the density of matter within the sphere of radius r, and G is Newton's gravitational constant.

Substituting (2.4) into (2.5) we have:

$$U = \frac{1}{2}m\dot{a}^2x^2 - \frac{4\pi}{3}G\rho a^2x^2m \tag{2.6}$$

which can be re-arranged into the familiar form of the **Friedmann equation**

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \tag{2.7}$$

where

$$kc^2 = -\frac{2U}{mx^2} \tag{2.8}$$

The parameter k is interesting. Note that k must be independent of x, since the other terms in the equation are. Thus $U \propto x^2$; homogeneity requires that U, while constant for a given particle, does change if we look at different comoving separations x. From eq. 2.8 we can also see that $k \neq f(t)$, since for a given particle the total energy U is conserved and $\dot{x} = 0$ by definition. Thus k is just a constant, unchanging with either space or time.

An expanding universe has a unique value of k which it maintains throughout its evolution. The value of k determines the form of this evolution. It can be appreciated immediately from eq. 2.8 that:

- A positive k implies negative U, so that $V > T$ in eq. 2.5—the expansion will at some time t halt and reverse itself
- A negative k implies positive U, so that $V < T$ in eq. 2.5—the expansion will continue forever

- If $k = 0$, the total energy is also $U = 0$ and the expansion of the universe will slow down, but only halt at $t = \infty$

In the definition of eq. 2.8 k has the units of an inverse area.

Combining eqs. 2.1 and 2.4, we can write:

$$\mathbf{v} = \frac{|\dot{\mathbf{r}}|}{|\mathbf{r}|}\mathbf{r} = \frac{\dot{a}}{a}\mathbf{r} \tag{2.9}$$

Comparison with (2.1) identifies the Hubble parameter as:

$$H = \frac{\dot{a}}{a} \tag{2.10}$$

and Friedmann equation can be re-written in a form that explicitly expresses the time evolution of the Hubble parameter $H(t)$:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \tag{2.11}$$

which defines a *critical density* today:

$$\boxed{\rho_c = \frac{3H_0^2}{8\pi G}} \tag{2.12}$$

Returning to eq. 2.7, it is clear that we cannot use this equation to describe the time evolution of the scale factor of the universe, $a(t)$, without an additional equation describing the time evolution of the density ρ of material in the universe.

From thermodynamics we know that:

$$dE + pdV = TdS \tag{2.13}$$

where p is the pressure. Applying it to an expanding volume V of unit comoving radius[1], and using $E = mc^2$, the energy within the volume is:

$$E = \frac{4\pi}{3}a^3\rho c^2$$

[1]Beware of two more symbol ambiguities: earlier we used p to indicate momentum and V potential energy.

The change of energy in a time dt is:

$$\frac{dE}{dt} = 4\pi a^2 \rho c^2 \frac{da}{dt} + \frac{4\pi}{3} a^3 \frac{d\rho}{dt} c^2$$

while the change in volume with time is:

$$\frac{dV}{dt} = 4\pi a^2 \frac{da}{dt}$$

Assuming a reversible expansion, i.e. $dS = 0$ in eq. 2.13, we obtain:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0 \tag{2.14}$$

Equation 2.14 is known as the **fluid equation**. It tells us that there are two terms contributing to the change in density as the universe expands. The first term in the brackets corresponds to the dilution in the density because the volume has increased—that's straightforward. The second term corresponds to the loss in energy because the *pressure* of the material has done work as the universe's volume increased.

What about conservation of energy? Of course, energy is always conserved—in this case the energy lost from the fluid via the work done has gone into gravitational potential energy.

The term 'pressure' here, does not mean a pressure *gradient* which supplies the force driving the expansion—there are no such pressure forces in a homogeneous universe because density and pressure are the same everywhere. In cosmology the assumption is usually made that there is a unique pressure associated with each density, so that $p \equiv p(\rho)$. Such a relationship is known as the **equation of state**. The form of the equation of state depends on the nature of the constituent of the universe we are considering:

- Non-relativistic matter has negligible pressure, $p = 0$. Examples are galaxies (their only interaction is through gravity) and atoms in general once the universe has expanded and cooled. Cosmologists sometimes refer to this component as '**dust**', precisely to indicate matter with negligible pressure.

- Light, or more generally any highly relativistic particle, has an associated pressure (**radiation** pressure) $p = \rho c^2/3$.

Armed with the fluid equation, we can now derive an expression for the acceleration of the universe, $\ddot{a}$, as follows:

Differentiating w.r.t. time the Friedmann equation (2.7), we obtain:

$$2\frac{\dot{a}}{a}\frac{a\ddot{a}-\dot{a}^2}{a^2} = \frac{8\pi G}{3}\dot{\rho} + 2\frac{kc^2\dot{a}}{a^3} \tag{2.15}$$

Substituting in $\dot{\rho}$ from 2.14, leads to:

$$\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = -4\pi G\left(\rho + \frac{p}{c^2}\right) + \frac{kc^2}{a^2} \tag{2.16}$$

Using again 2.7, we arrive at the **acceleration equation**:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) \tag{2.17}$$

Note that pressure here acts to *increase* the gravitational force, and so further *decelerates* the expansion. Note also that the constant k, which is so important in Friedmann equation, does not appear anywhere in 2.17—it cancelled out in the derivation.

It is worthwhile considering in passing the behaviour of the Friedmann equation in the simplest case where $k = 0$, for the two types of constituents of the universe we mentioned above: (a) pressureless dust and (b) radiation.

Dust. With $p = 0$, the fluid equation (2.14) becomes:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho = 0$$

which can be re-written as:

$$\frac{1}{a^3}\frac{d}{dt}(\rho a^3) = 0$$

which implies

$$\frac{d}{dt}(\rho a^3) = 0$$

That is, ρa^3 is a constant and therefore:

$$\rho \propto \frac{1}{a^3} \tag{2.18}$$

This rather obvious result tells us that the density of matter falls off in proportion to the volume of the expanding universe. If we choose as the unit scale factor $a(t)$ the scale factor today, that is $a_0 = 1$, so that physical and comoving coordinates coincide today (recall our definition of both at 2.4), we have:

$$\rho = \rho_0/a^3 \tag{2.19}$$

and subsituting 2.19 into 2.7 with $k = 0$ we obtain:

$$\dot{a}^2 = \frac{8\pi G\rho_0}{3}\frac{1}{a} \tag{2.20}$$

which has the solution $a \propto t^{2/3}$. As we have fixed $a_0 = 1$, the full solution is therefore

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3}; \qquad \rho(t) = \frac{\rho_0}{a^3} = \frac{\rho_0 t_0^2}{t^2} \tag{2.21}$$

In this solution, the universe expands forever, but an ever decreasing rate:

$$H(t) \equiv \frac{\dot{a}}{a} = \frac{2}{3t} \tag{2.22}$$

This is one of the classic cosmological solutions and is referred to as an 'Einstein-de Sitter' cosmology.

Notice from 2.22 that:

$$t_0 = \frac{2}{3}\frac{1}{H_0}$$

where t_0 is the current age of the universe and $1/H_0$ is the **Hubble time**. For $H_0 = 70$ km s^{-1} Mpc^{-1}, $H_0^{-1} = 14\,\mathrm{Gyr}$ and $t_0 = 9.3\,\mathrm{Gyr}$.

Radiation. Substituting the radiation pressure $p = \rho c^2/3$ into the fluid equation (2.14) we have:

$$\dot{\rho} + 4\frac{\dot{a}}{a}\rho = 0$$

which, following the same steps as above for the dust-dominated universe, leads to the conclusion that:

$$\rho \propto \frac{1}{a^4} \tag{2.23}$$

and

$$a(t) = \left(\frac{t}{t_0}\right)^{1/2}; \qquad \rho(t) = \frac{\rho_0}{a^4} = \frac{\rho_0 t_0^2}{t^2} \tag{2.24}$$

This is the second classic cosmological solution.

Note that the universe expands more slowly if it is radiation-, rather than dust-dominated, because of the additional *deceleration* which the pressure supplies (see 2.17). However, in each case the density falls off as t^2. From 2.23, we see that the radiation density drops as the fourth power of the scale factor. Three of these powers we have already identified as the increase in volume, while the fourth power is due to the redshift of the light [by a factor $(1+z) = 1/a$], which decreases its associated energy $E = h\nu$.

The above relations also show that when the universe is radiation dominated (i.e. earliest epochs, when matter is relativistic),

$$a(t) \propto t^{1/2}; \qquad \rho_{\mathrm{rad}} \propto \frac{1}{t^2}; \qquad \rho_{\mathrm{dust}} \propto \frac{1}{a^3} \propto \frac{1}{t^{3/2}} \tag{2.25}$$

which is an unstable situation since the density of radiation is falling off faster than the density of dust.

On the other hand, once the universe becomes dust-dominated,

$$a(t) \propto t^{2/3}; \qquad \rho_{\mathrm{dust}} \propto \frac{1}{t^2}; \qquad \rho_{\mathrm{rad}} \propto \frac{1}{a^4} \propto \frac{1}{t^{8/3}} \tag{2.26}$$

which is a stable situation—the dust becomes increasingly dominant over the radiation as time progresses.