Problem Set II: G. Efstathiou, Particle Astrophysics

3. Inflation and Dark Energy

(3.1) The Lagrangian for a scalar field ϕ with potential $V(\phi)$ is

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - V(\phi).$$

By varying the action

$$S = \int \sqrt{-g} \mathcal{L} d^4 x,$$

with respect to a change in the field $\delta\phi$, show that the equation of motion is

$$(g^{\mu\nu} \phi_{,\mu})_{;\nu} = -\frac{dV}{d\phi}.$$

[Note you may find this calculation a bit tricky. If so, consult for example, Peebles, Physical Cosmology, pages 256-257.]

(3.2) For slow roll inflation ($\ddot{\phi} \ll 3H\dot{\phi}$, $\dot{\phi}^2 \ll V(\phi)$), with a power-law potential

$$V(\phi) = \frac{1}{n} \lambda \phi^n,$$

(in Planck units $8\pi G = M_{pl}^{-2} = 1$), show that the scale factor varies as

$$R(\phi) \approx R_i \mathrm{exp} \left(\frac{1}{2n} (\phi_i^2 - \phi^2(t) \right).$$

(3.3) At the end of inflation, the field oscillates around the minimum of the potential. Assuming that the period of oscillation is much smaller than the expansion timescale, show that the effective equation of state for the power-law potential of problem (3.2) is

$$w = \frac{P}{\rho} \approx \frac{(n-2)}{(n+2)},$$

and comment on this result.

(3.4) Suppose that at early times the Universe is radiation dominated and the scale-factor evolves as $R(t) \propto t^{1/2}$. Suppose also that the dark energy is a scalar field, ϕ, evolving in a potential

$$V(\phi) = \frac{A}{\phi^\alpha}, \qquad \alpha > 0.$$

Show that there is a power-law solution of the scalar field equation, $\phi \propto t^p$, with

$$p = \frac{2}{(2+\alpha)}.$$

Show that for this solution, the ratio of the energy density of the scalar field to the radiation density increases with time as

$$\frac{\rho_\phi}{\rho} \propto t^{\frac{4}{(2+\alpha)}}.$$

4. High Energy Astrophysics and Gravitational Waves

(4.1) High energy γ-rays from very distant sources can collide with photons of the cosmic microwave background to produce electron-positron pairs. Estimate the threshold energy involved and the typical absorption length.

(4.2) The gravitational strain of a gravitational wave of frequency ω and energy flux T_{01} is

$$h = \left(\frac{32\pi G T_{01}}{c^3\omega^2}\right)^{1/2}.$$

Use the quadrupole formula

$$L_{\rm GW} \sim \frac{G}{5c^5}\langle \dddot{I}^2 \rangle,$$

to show that the maximum gravitational energy flux from a source is independent of mass and of order

$$T_{01} \sim \frac{c^5}{G}\frac{1}{4\pi r^2}.$$

Hence calculate the maximum gravitational strain expected at 1 kHz for (i) a source at a distance of 1 kpc; (ii) a source at a distance of 10 Mpc.