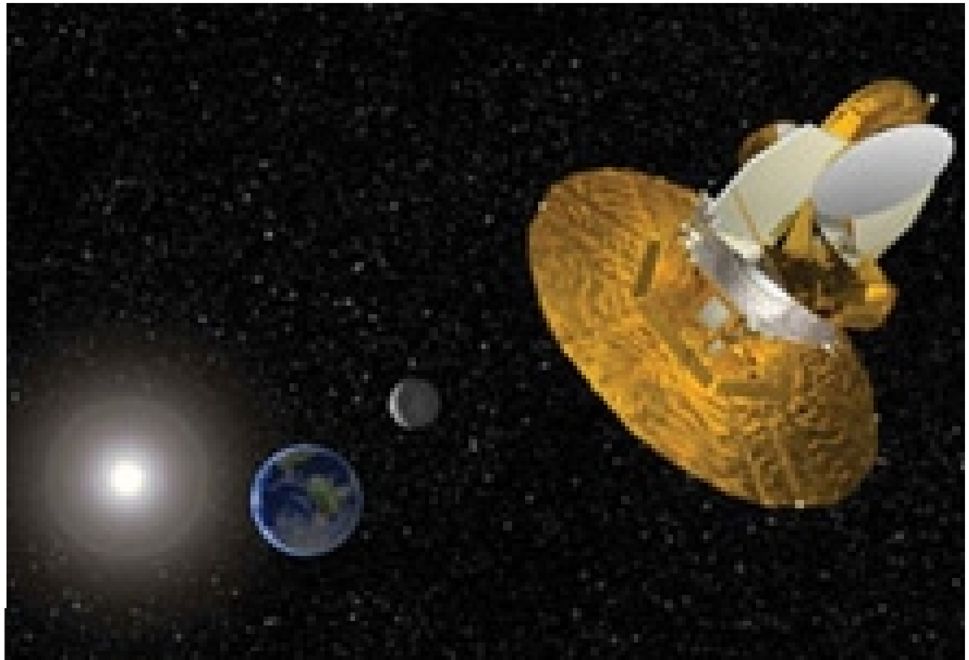
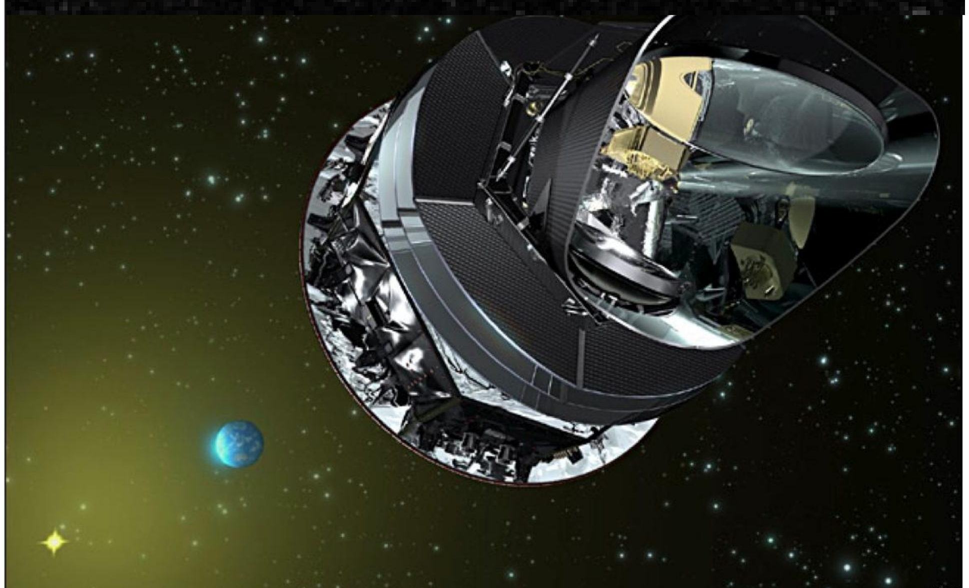
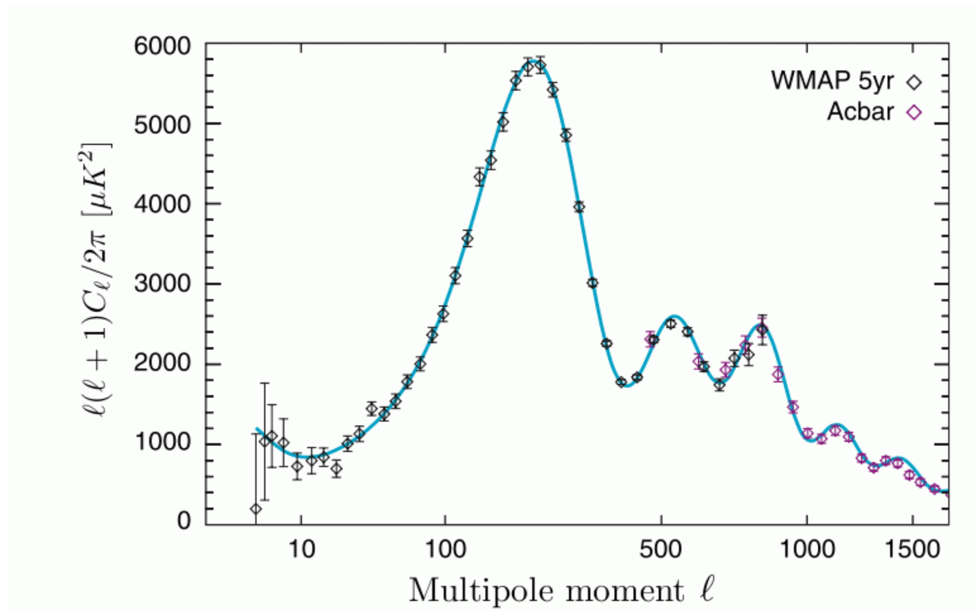
PARTICLE ASTROPHYSICS LECTURE 9

Tests of Inflation







Origin of Fluctuations I: The quantum harmonic oscillator

The action:

$$S = \int \frac{1}{2} dt (\dot{x}^2 - \omega^2 x^2),$$

gives us the classical equation of motion

$$\ddot{x} + \omega^2 = 0. \quad (1)$$

To describe the quantum harmonic oscillator, we make $\widehat{x}$ (and its conjugate momentum) into an operator by expanding in annihilation and creation operators:

$$\widehat{x} = v(t)\widehat{a} + v^*(t)\widehat{a}^\dagger.$$

The coefficients $v(t)$ satistfy the *classical* equation of motion (1), with solution

$$v(t) = \sqrt{\frac{\hbar}{2\omega}} \mathrm{e}^{-i\omega t}.$$

The normalization is fixed by requiring the vacuum state to be the ground state of the Hamiltonian

$$\widehat{H} = \hbar\omega\left(\widehat{a}^\dagger\widehat{a} + \frac{1}{2}\right).$$

In the ground state, there are zero point fluctuations – for the position operator

$$\langle|\widehat{x}|^2\rangle = \langle 0|\widehat{x}^\dagger\widehat{x}|0\rangle = |v(t)|^2\rangle = \frac{\hbar}{2\omega}.$$

Origin of Fluctuations II: Quantum fluctuations in de Sitter space,

The description of quantum fluctuations in de Sitter space is algebraically more complicated, but no fundamentally new ideas are involved. We begin with the action

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\,[R - (\nabla\phi)^2 - 2V(\phi)],$$

R is the Ricci scalar (not to be confused with the scale factor $R(t)$. Now, we need to handle perturbations, so write the metric as

$$g_{ij} = R^2(t)[(1-2\mathcal{R})\delta_{ij} + h_{ij}].$$

scalar perturbations

tensor perturbations

and then after a lot of algebra, we can rewrite the action to second order in $\mathcal{R}$

$$S = \frac{1}{2}\int d\tau d^3x\left[(v')^2 + (\partial_i v)^2 + \frac{z''}{z}v^2\right],$$

where

$$v = z\mathcal{R}, \qquad z^2 = \frac{R^2(t)}{H^2}\dot{\phi}^2,$$

and primes denote differentiation with respect to conformal time, τ, $d\tau = \int dt/R(t)$. [Note that for de Sitter space

$$\tau = -\frac{1}{R(t)H} \propto -e^{-Ht}$$

and runs from $\tau = -\infty \rightarrow 0$.] As with the harmonic oscillator, express v and the conjugate momentum v' as quantum operators,

$$\hat{v} = \frac{1}{(2\pi)^3}\int d^3\mathbf{k}[v_{\mathbf{k}}(\tau)\hat{a}_{\mathbf{k}}\mathrm{e}^{i\mathbf{k}.\mathbf{x}} + v^*_{\mathbf{k}}(\tau)\hat{a}^\dagger_{\mathbf{k}}\mathrm{e}^{-i\mathbf{k}.\mathbf{x}}].$$

We then find that $v(t)$ satisfies the equation

$$v''_{\mathbf{k}} + \left(k^2 - \frac{2}{\tau^2}\right)v_{\mathbf{k}} = 0,$$

(*cf* harmonic oscillator) with solution

$$v_{\mathbf{k}} = \frac{e^{-ik\tau}}{\sqrt{2}k}\left(1 - \frac{i}{k\tau}\right).$$

(The normalization is chosen so that in the limit $k\tau \gg 1$, the vacuum state is the Minkowski vacuum). We can then compute the power spectrum of the scalar metric perturbation $\mathcal{R}$ at *horizon crossing* $R(t)H = k$ ($k|\tau| = 1$):

$$\langle(\mathcal{R}_{\mathbf{k}})^2\rangle = \frac{\langle|v_{\mathbf{k}}|^2}{z^2} = \frac{H^2}{R^2\dot{\phi}^2}\frac{1}{2k}\left(1 + \frac{1}{k^2\tau^2}\right) = \frac{H^4}{\dot{\phi}^2 k^3},$$

where $\dot{\phi}$ is evaluated at $RH = k$. In GR, $\mathcal{R}$ does not evolve on superhorizon scales, so once fixed, it remains constant. This then gives us the amplitude $\Delta^2_{\mathcal{R}}$of the power-spectrum of *scalar perturbations* generated during inflation:

$$\frac{1}{(2\pi)^3}\int\langle(\mathcal{R}_{\mathbf{k}})^2\rangle d^3\mathbf{k} = \frac{1}{(2\pi)^2}\int\langle(\mathcal{R}_{\mathbf{k}})^2\rangle k^2 dk = \int \Delta^2_S d\ln k,$$

$$\boxed{\Delta_S^2 = \frac{H^4}{2\pi^2\dot{\phi}^2}.} \tag{2}$$

In an analogous fashion, the fluctuations h_{ij} in the metric give rise to *tensor* (gravitational wave) perturbations with power-spectrum:

$$\boxed{\Delta_T^2 = \frac{2}{\pi}\frac{H^2}{M_{\mathrm{Pl}}^2}.} \tag{3}$$

We know observationally that $\Delta_S^2 \approx 10^{-9}$, so a detection of tensor perturbations would fix the energy scale of inflation. Since $H^2 \approx V$,

$$V^{1/4} \approx 3.3 \times 10^{16} r^{\frac{1}{4}}\ \mathrm{GeV}, \qquad r = \frac{\Delta_T^2}{\Delta_T^2}.$$

If the amplitudes (2) and (3) are constants (*i.e.* independent of wavenumber k), the fluctuations are *scale-invariant*.

Slow roll parameters

If H varies slightly during inflation, the scalar and tensor fluctations will not be exactly scale-invariant.

More generally we can write

$$\Delta_S^2 \propto k^{n_s - 1}, \qquad \Delta_T^2 \propto k^{n_T},$$

where the *spectral indices* are:

$$n_S - 1 \approx 2\eta - 6\epsilon, \quad n_T = -2\epsilon,$$

and the *tensor-scalar ratio* is

$$r = 16\epsilon,$$

where the quantities ϵ and η are *slow-roll* parameters,

$$\epsilon = \frac{M_{\mathrm{Pl}}^2}{2}\left(\frac{1}{V}\frac{dV}{d\phi}\right)^2, \qquad \eta = \frac{M_{\mathrm{Pl}}^2}{V}\frac{d^2V}{d\phi^2},$$

which must necessarliy be small during inflation.

Observations of the CMB anisotropies offer the best prospects for measuring these quantities and learning about the dynamics of inflation.

Fluctuations in the CMB

We can get a good idea of the evolution of fluctuations prior to recombination by making the following assumptions:

• Radiation and baryons are tightly coupled by Thomson scattering. We will negect weakly interacting dark matter.

• Neglect gravity. This is accurate for fluctuations with scales much smaller than the Hubble radius $\lambda < ct$, since the dynamics is dominated by pressure not gravity.

• Neglect the expansion of the Universe. This is *not* a particularly good approximation because recombination is quite an extended process taking $\Delta z \sim 200$ at $z \sim 1000$.

Let us write the photon distribution function as

$$f(\mathbf{x}, \mathbf{q}, t) = f_0 + f_1,$$

where f_1 is a perturbation on the black-body function f_0, and q is the comoving photon momentum. We can define the perturbation to the radiation brightness as

$$\Delta(\mathbf{x}, \mathbf{q}, t) = f_1 \left(\frac{T_0}{4} \frac{\partial f_0}{\partial T_0} \right)^{-1}.$$

Then since Thomson scattering is independent of photon energy, the Boltzmann equation for the perturbation Δ is

$$\frac{\partial \Delta}{\partial t} + \frac{\gamma_i}{R} \frac{\partial \Delta}{\partial x^i} = \sigma_T n_e [\Delta_0 + 4\gamma_i v_b^i - \Delta],$$

where the γ^i are the direction cosines of $\hat{\mathbf{q}}$, σ_T is the Thomson cross-section, n_e is the free electron density, v_b is the matter velocity, and Δ_0 is the isotropic part of Δ:

$$\Delta_0 = \frac{1}{4\pi} \int \Delta d\Omega.$$

Fourier transforming the Boltzmann equation:

$$\frac{d\Delta}{dt} + \frac{ik\mu}{R}\Delta = \sigma_T n_e [\Delta_0 + 4\mu v_b - \Delta], \quad \mu = \hat{\mathbf{k}}.\hat{\mathbf{q}}. \tag{1}$$

The equation of motion for the matter (neglecting expansion) is:

$$\frac{dv_b}{dt} = \sigma_T n_e \frac{\overline{\rho}_\gamma}{\overline{\rho}_b} \left[\Delta_1 - \frac{4}{3} v_b \right], \tag{2}$$

where

$$\Delta_1 = \frac{1}{2} \int_{-1}^{1} \Delta \mu d\mu,$$

is the photon energy flux. (This equation tells that it is difficult for baryons to move relative to the radiation if they are strongly coupled to the photons). Finally, mass conservation gives the familiar equation of continuity

$$\frac{d\delta_b}{dt} = -\frac{ikv_b}{R}, \tag{3}$$

where δ_b is the baryon overdensity ($\delta_b = (\rho_b - \overline{\rho}_b)/\overline{\rho}_b$). Equations (1) - (3) allow us to calculate the evolution of the fluctuations:

Prior to recombination, the baryon component is highly ionised and the mean-free time for Thomson scattering

$$t_c = \frac{1}{\sigma_T n_e},$$

is much smaller than the expansion rate, $t_c \ll t$. This is what we mean when we say that the matter and radiation are *tightly coupled*. The equations (1) and (2) are therefore very 'stiff', and so to first order, the terms in square brackets must be close to zero:

$$\Delta = \Delta_0 + 4\mu v.$$

Now define the quantity $X = \Delta_0 + 4\mu v$ and insert back into (1) and (2). Then to second order in t_c

$$\Delta = X - t_c\left[\dot{X} + i\frac{\mu k X}{R}\right] + t_c^2\left[\ddot{X} + 2\frac{i\mu k \dot{X}}{R} - \frac{\mu^2 k^2 X}{R^2}\right] + \mathcal{O}(t_c^3),$$

i.e. for imperfect coupling (non-zero t_c) the radiation develops quadrupolar, octopolar and higher order perturbations. This allows us to get a closed set of equations for Δ_0 and v, valid to first order in t_c:

$$\begin{aligned}
\dot{\Delta}_0 &= \frac{-ik}{R}\left[\frac{4}{3}v - t_c\left(\frac{4}{3}\dot{v} + \frac{ik\Delta_0}{3R}\right)\right], \\
\dot{v} &= \frac{\overline{\rho}_\gamma}{\overline{\rho}_m}\left[-\frac{4}{3}\dot{v} - \frac{ik\Delta_0}{3R} + t_c\left(\frac{4}{3}\ddot{v} + \frac{2ik\dot{\Delta}_0}{3R} - \frac{4k^2 v}{5R^2}\right)\right].
\end{aligned}$$

These equations have solutions of the form:

$$\left.\begin{array}{c} v \\ \Delta \end{array}\right\} \propto \exp(-\Gamma t),$$

with

$$\Gamma = \pm\frac{ik}{R\sqrt{3B}} - \frac{k^2 t_c}{6R^2}\left(1 - \frac{6}{5B} + \frac{1}{B^2}\right), \qquad (*)$$

where

$$B = 1 + \frac{3\overline{\rho}_b}{4\overline{\rho}_\gamma}.$$

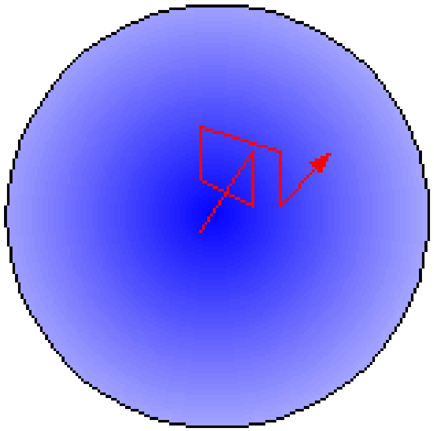
The first term in (*) describes *acoustic oscillations* with adiabatic sound speed

$$c_s = \frac{c}{\sqrt{3}}\left(1 + \frac{3\overline{\rho}_b}{4\overline{\rho}_\gamma}\right)^{-1/2}.$$

In the tightly coupled regime, baryons and photons act as a single fluid, so the inertia of the baryons reduces the sound speed below the relativistic value $c/\sqrt{3}$.

The second term in (*) describes *damping* of small-scale fluctuations by *photon diffusion*.

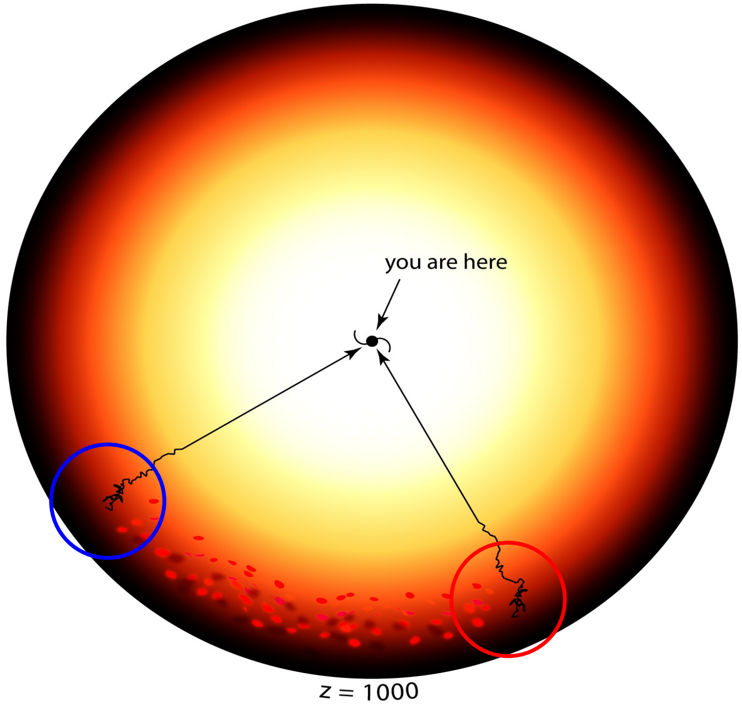
A photon within a perturbation will random walk. The mean time between collisions is t_c, so the number of collisions in time t is $N = t/t_c$. Therefore photons will diffuse over a length $\sqrt{N} c t_c = c\sqrt{t t_c}$ carrying the matter with them, in agreement with (*).



Note at recombination $z_{\rm rec} \approx 1000$, $B \approx 1.65$, and the characteristic damping scale is $k_d/R_0 \sim (15\ \mathrm{Mpc})^{-1}$.

CMB fluctations on large angular scales

On large scales, we need to perform a full general relativistic analysis. However, the result is easy to understand. The temperature anisotropies measure the potential fluctuations (via gravitational redshift) on the last scattering surface *the Sachs-Wolfe effect*:



$$\frac{\Delta T}{T} \propto \frac{(\delta\phi)_\lambda}{c^2} \sim \frac{G\delta M}{\lambda} \propto \delta\rho\lambda^2 \propto \lambda^{(1-n_s)/2}.$$

Where the last expression follows for fluctuations with power spectrum $P(k) \propto k^{n_s}$.

In practice, we compute the *CMB power spectrum*, C_ℓ, on the sky by expanding in spherical harmonics:

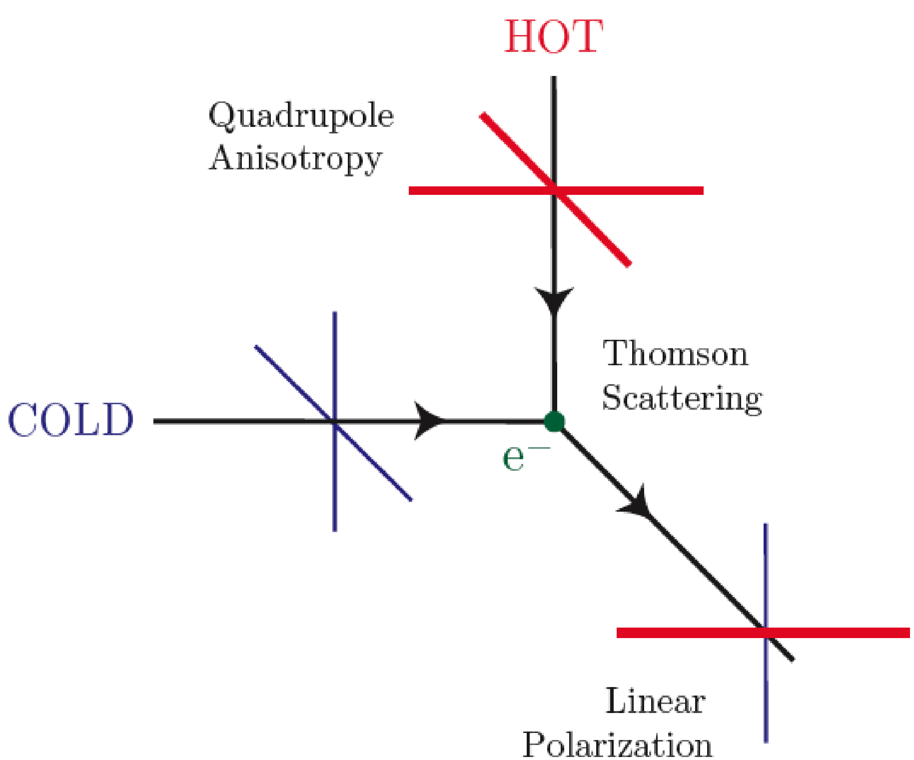
$$\begin{aligned} \frac{\Delta T}{T} &= \sum_{\ell m} a_{\ell m} Y_{\ell m}(\theta, \phi) \\ C_\ell &= \langle |a_{\ell m}|^2 \rangle. \end{aligned} \tag{1}$$

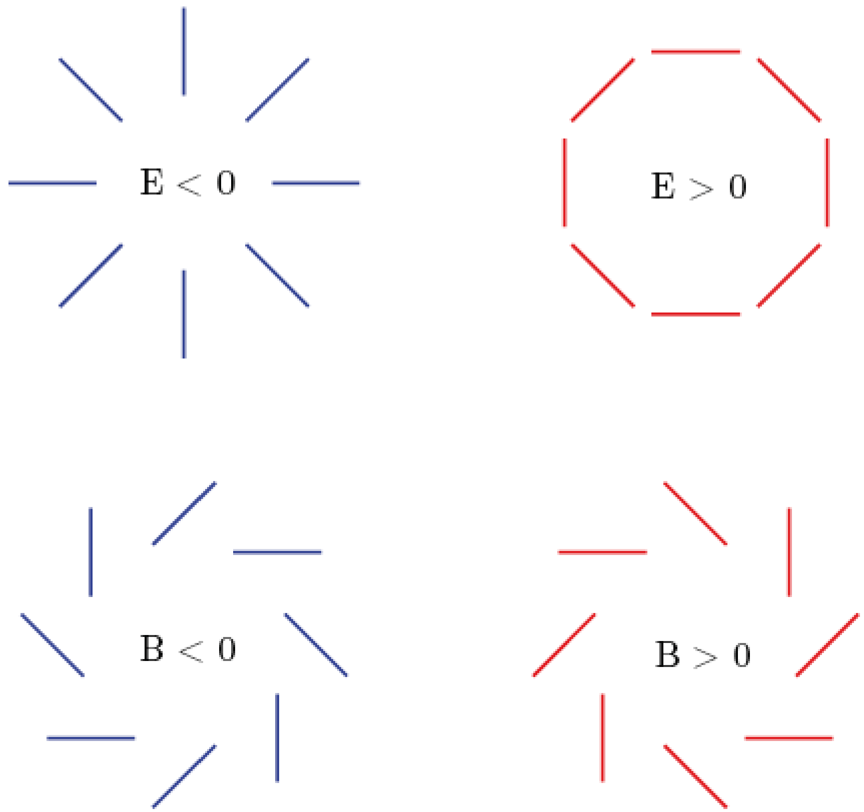
If $\Omega_K = \Omega_\Lambda = 0$, then the potential fluctuations are independent of time from the last scattering surface to the present day and we then find

$$C_\ell \propto \frac{\Gamma\left(\ell + \left(\frac{n_s - 1}{2}\right)\right)}{\Gamma\left(\ell + \left(\frac{5 - n_s}{2}\right)\right)} \propto \ell^{(n_s - 3)} \text{ if } \ell >> 1.$$

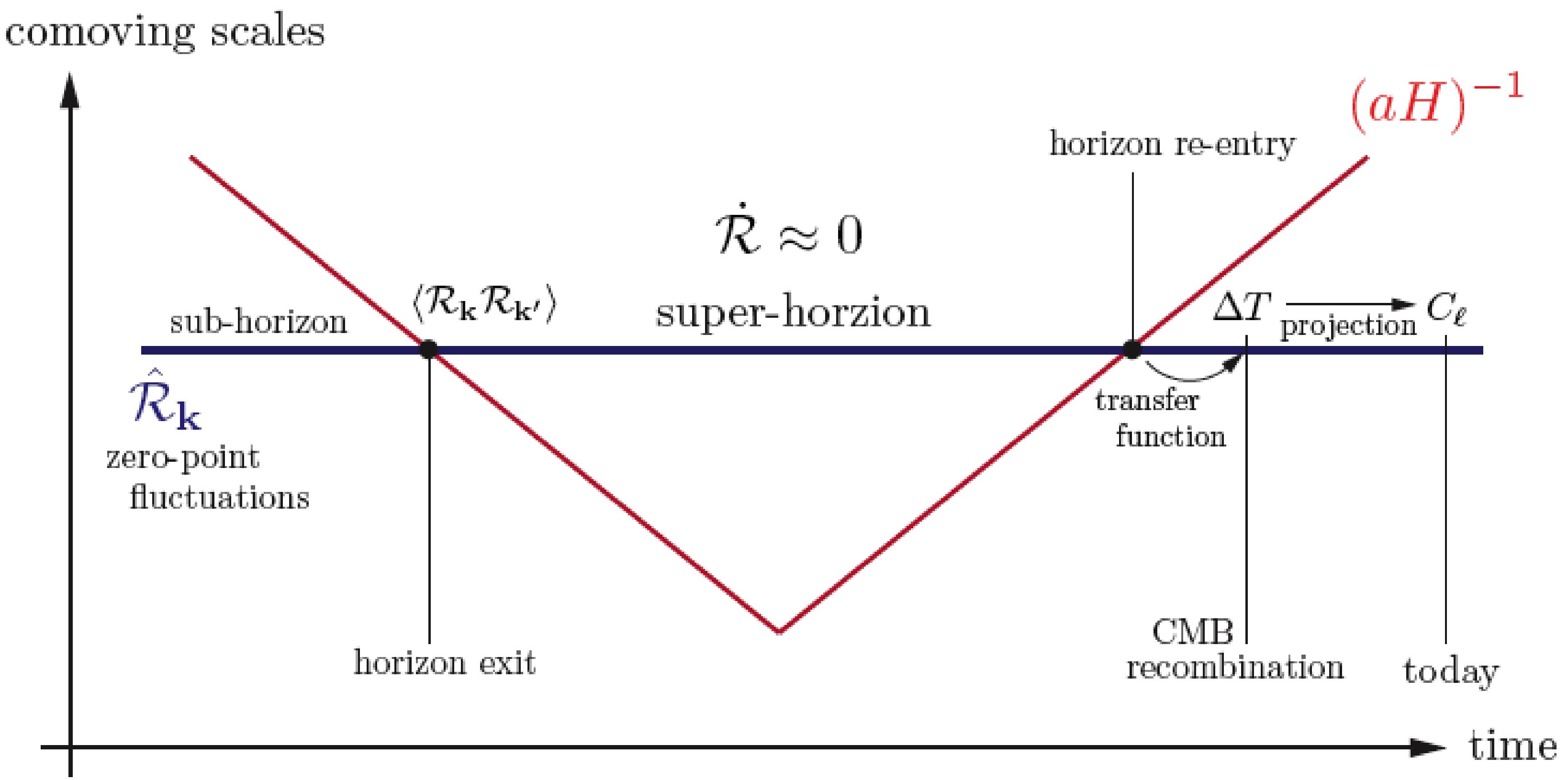
Scale-invariant fluctuations therefore lead to $\ell^2 C_\ell \approx$ constant at multipoles less than ~ 100.

Polarization: So far, we have assumed that Thomson scattering is isotropic. In fact, it is anisotropic. A quadrupolar radiation field will produce a *linearly polarized* radiation field after Thomson scattering. The CMB should be linearly polarized at the few percent level. The pattern of polarization can be decomposed into an 'electric' type pattern (*E-mode*) and a 'magnetic' type (*B-mode*). Scalar perturbations generate only E-modes.

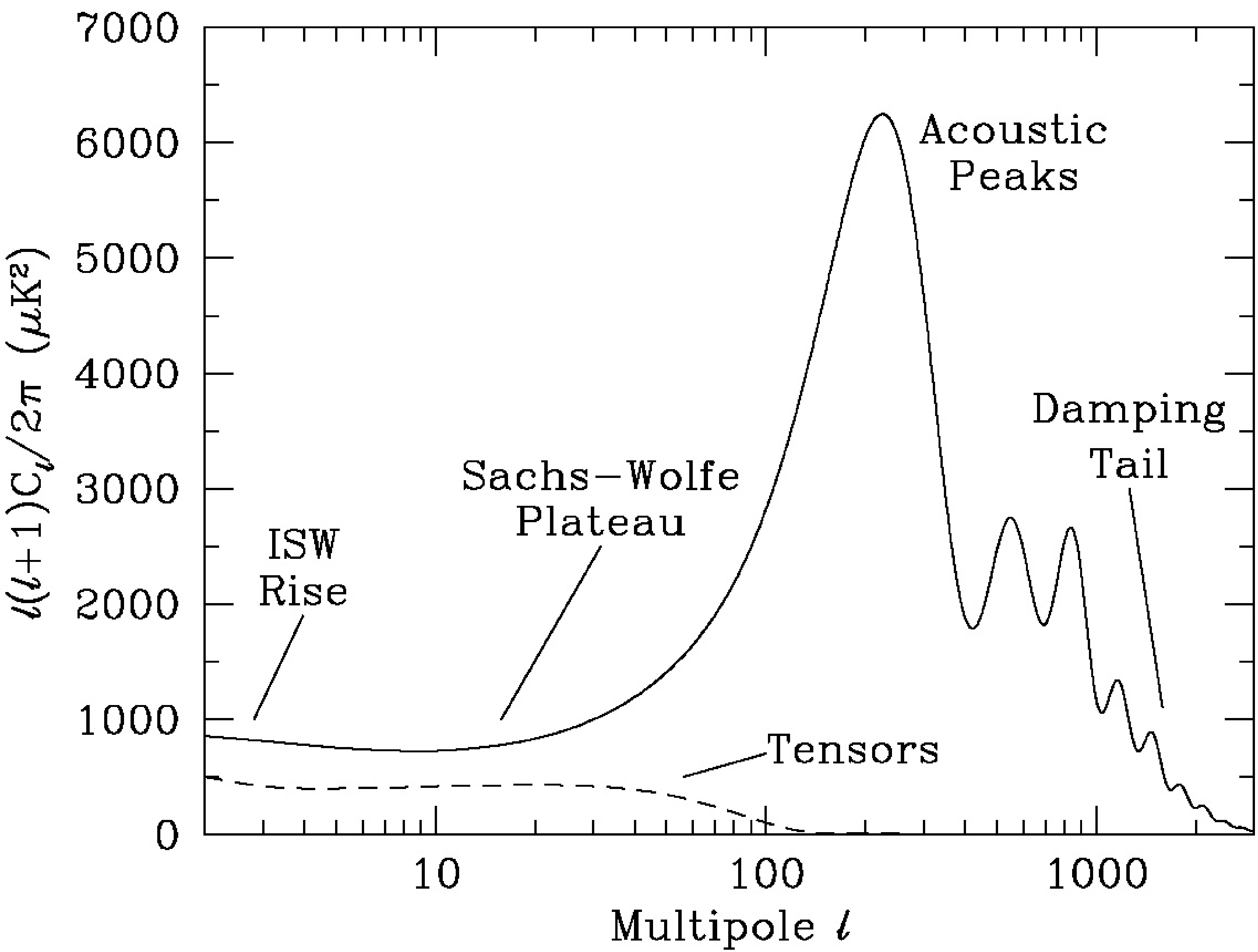




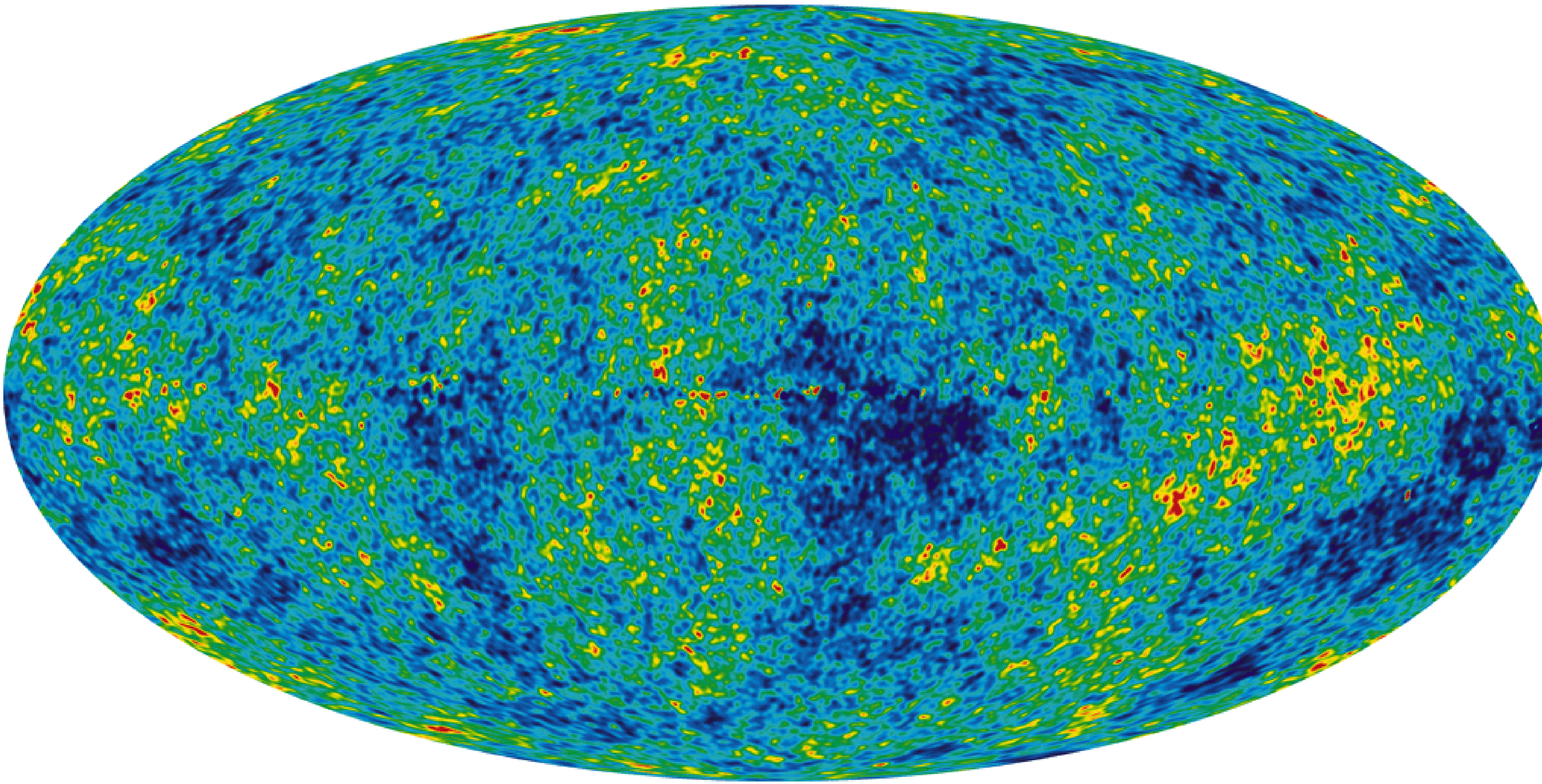
Summary:

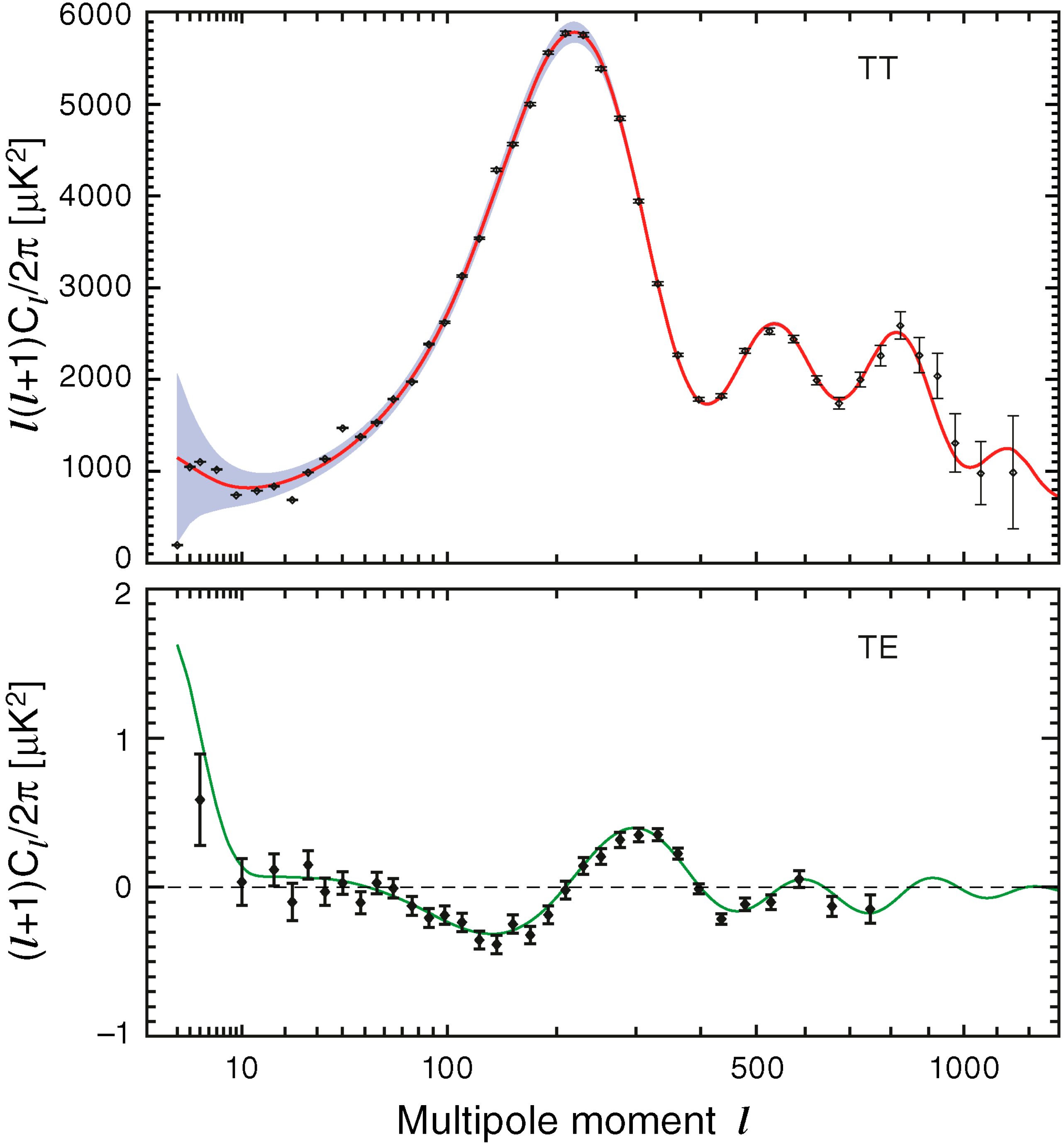


comoving scales
$(aH)^{-1}$
horizon re-entry
$\dot{\mathcal{R}} \approx 0$
sub-horizon
$\langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{\mathbf{k}'} \rangle$
super-horzion
ΔT
projection
C_ℓ
$\hat{\mathcal{R}}_{\mathbf{k}}$
zero-point
fluctuations
transfer
function
horizon exit
CMB
recombination
today
time

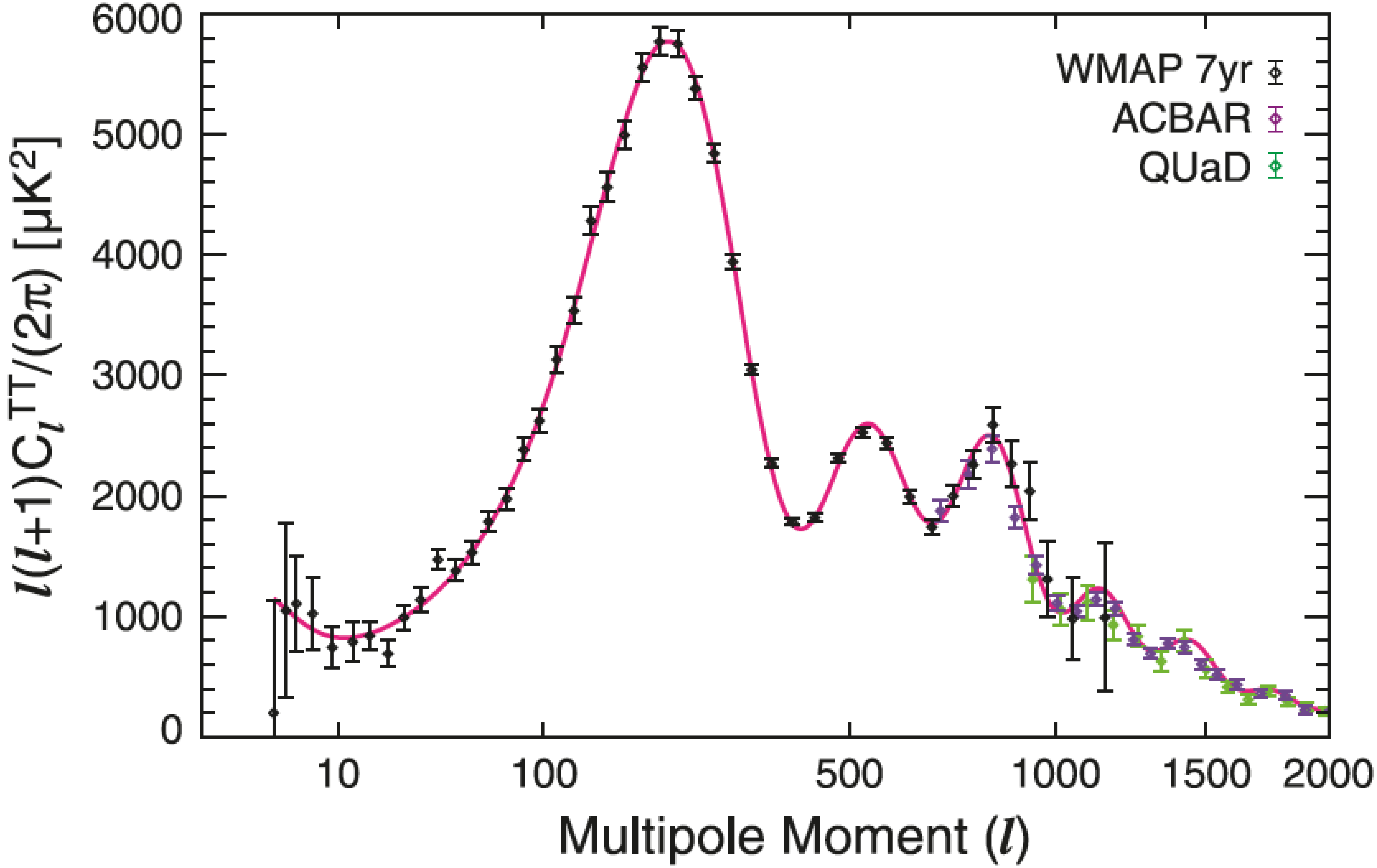


7000
6000
5000
4000
3000
2000
1000
0
10
100
1000
Multipole l
$l(l+1)C_l/2\pi$ (μK^2)
Acoustic Peaks
Damping Tail
Sachs−Wolfe Plateau
ISW Rise
Tensors

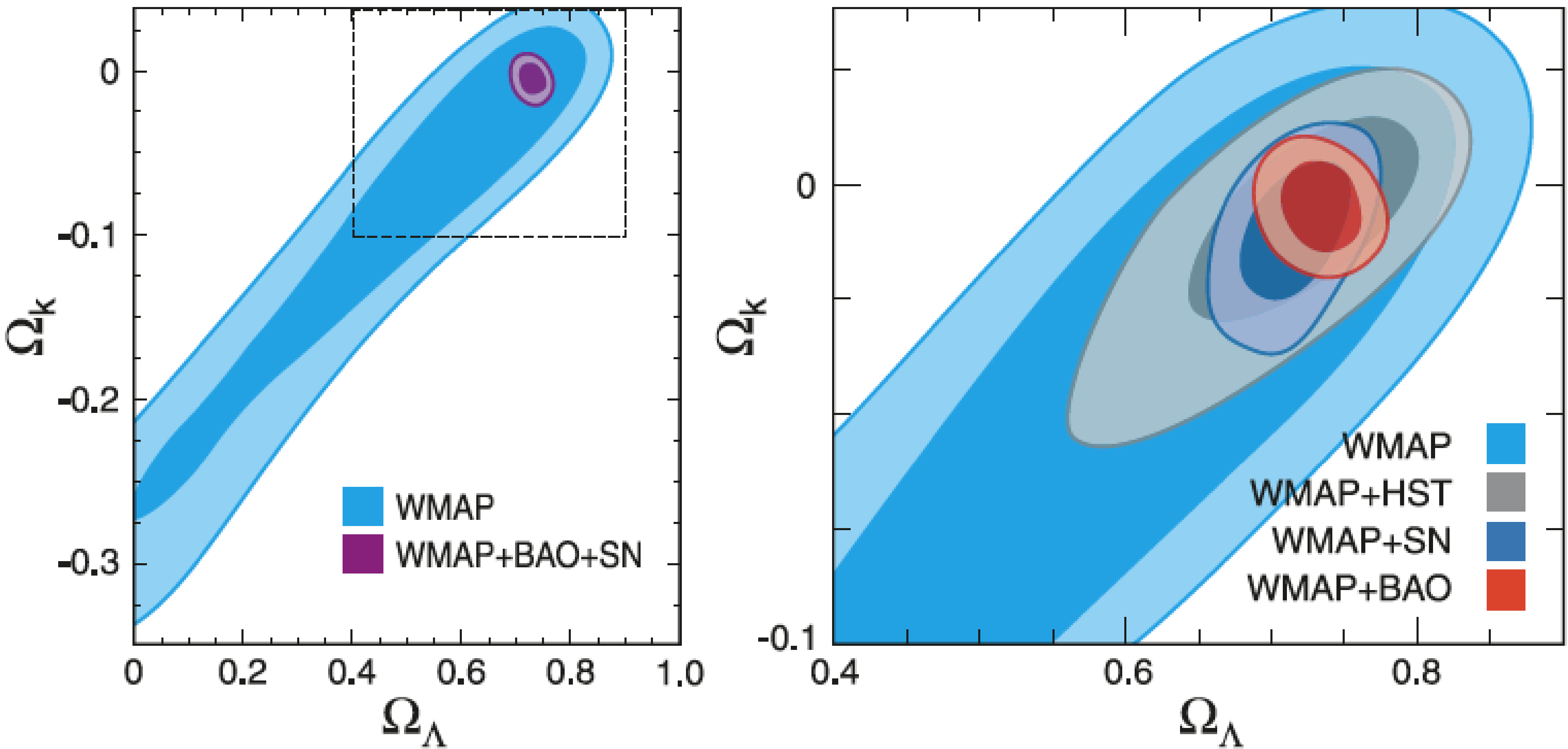




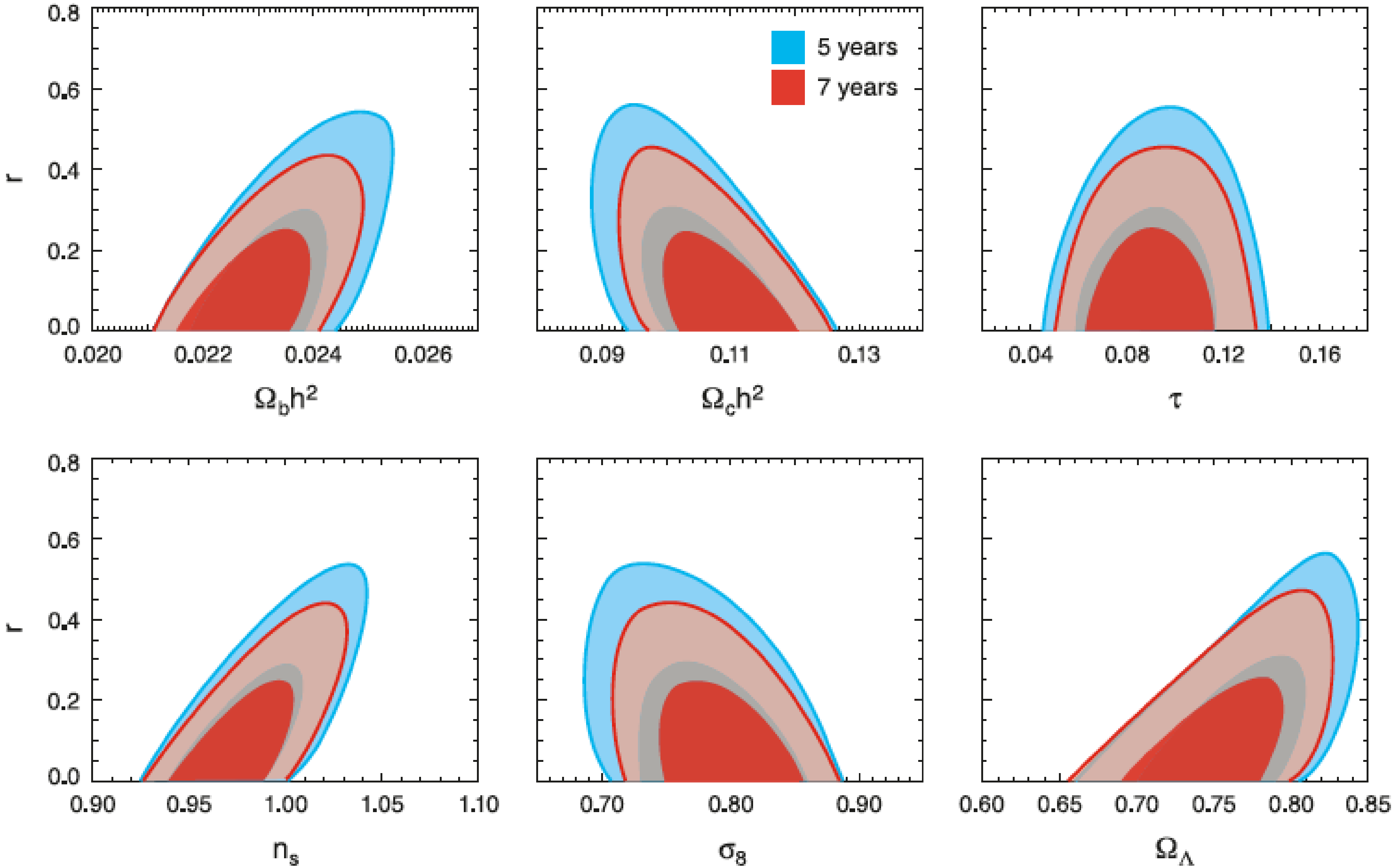
TT
TE
6000
5000
4000
3000
2000
1000
0
2
1
0
−1
10
100
500
1000
$l(l+1)C_l/2\pi$ [μK^2]
$(l+1)C_l/2\pi$ [μK^2]
Multipole moment l



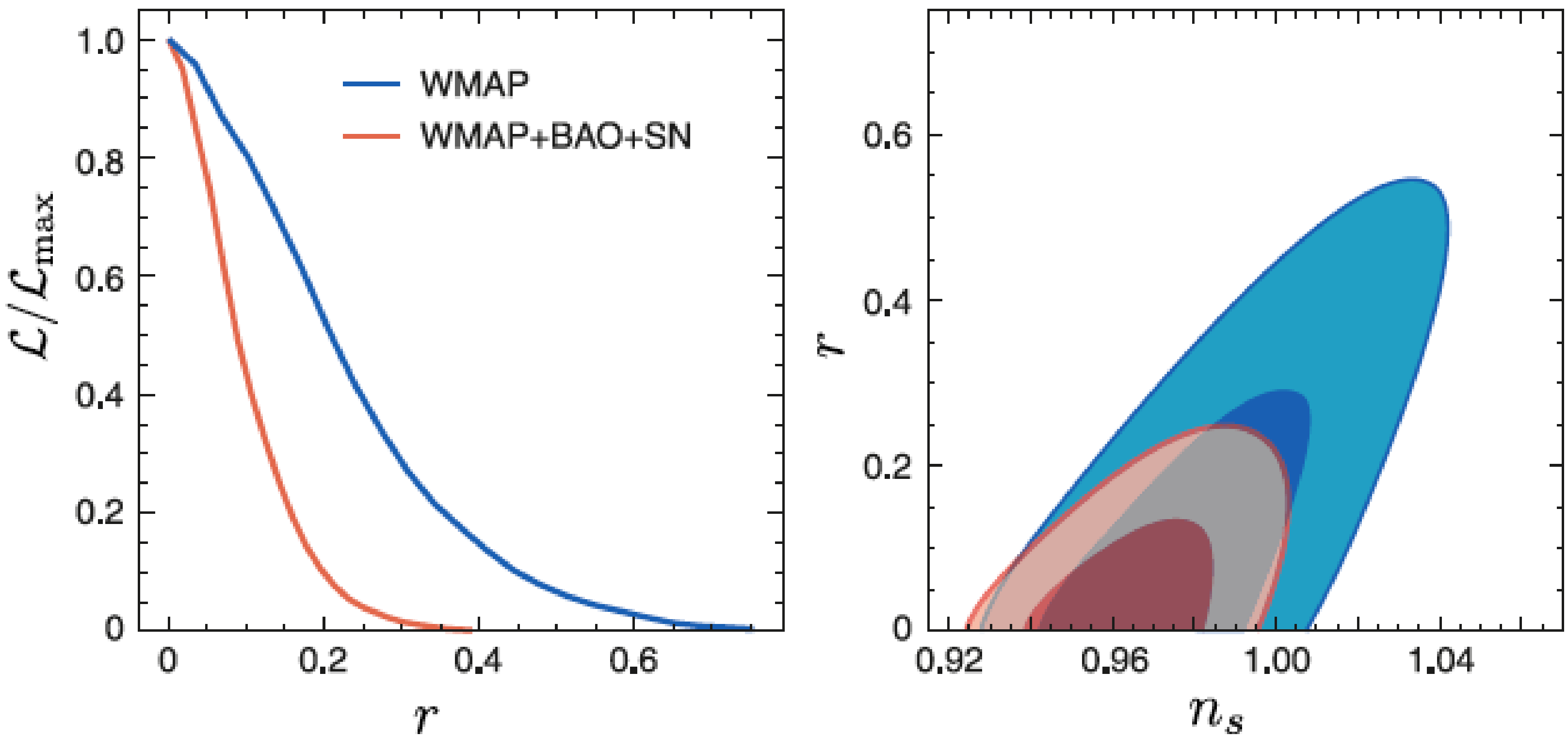
$l(l+1)C_l^{TT}/(2\pi)$ [μK^2]
6000
5000
4000
3000
2000
1000
0
10
100
500
1000
1500
2000
Multipole Moment (l)
WMAP 7yr
ACBAR
QUaD



0
-0.1
-0.2
-0.3
Ω_K
0
0.2
0.4
0.6
0.8
1.0
Ω_Λ
WMAP
WMAP+BAO+SN
0
-0.1
Ω_K
0.4
0.6
0.8
Ω_Λ
WMAP
WMAP+HST
WMAP+SN
WMAP+BAO



5 years
7 years
r
0.0
0.2
0.4
0.6
0.8
0.020
0.022
0.024
0.026
$\Omega_b h^2$
0.09
0.11
0.13
$\Omega_c h^2$
0.04
0.08
0.12
0.16
τ
0.90
0.95
1.00
1.05
1.10
n_s
0.70
0.80
0.90
σ_8
0.60
0.65
0.70
0.75
0.80
0.85
Ω_Λ



WMAP
WMAP+BAO+SN
$\mathcal{L}/\mathcal{L}_{\rm max}$
r
n_s
1.0
0.8
0.6
0.4
0.2
0
0.92
0.96
1.00
1.04