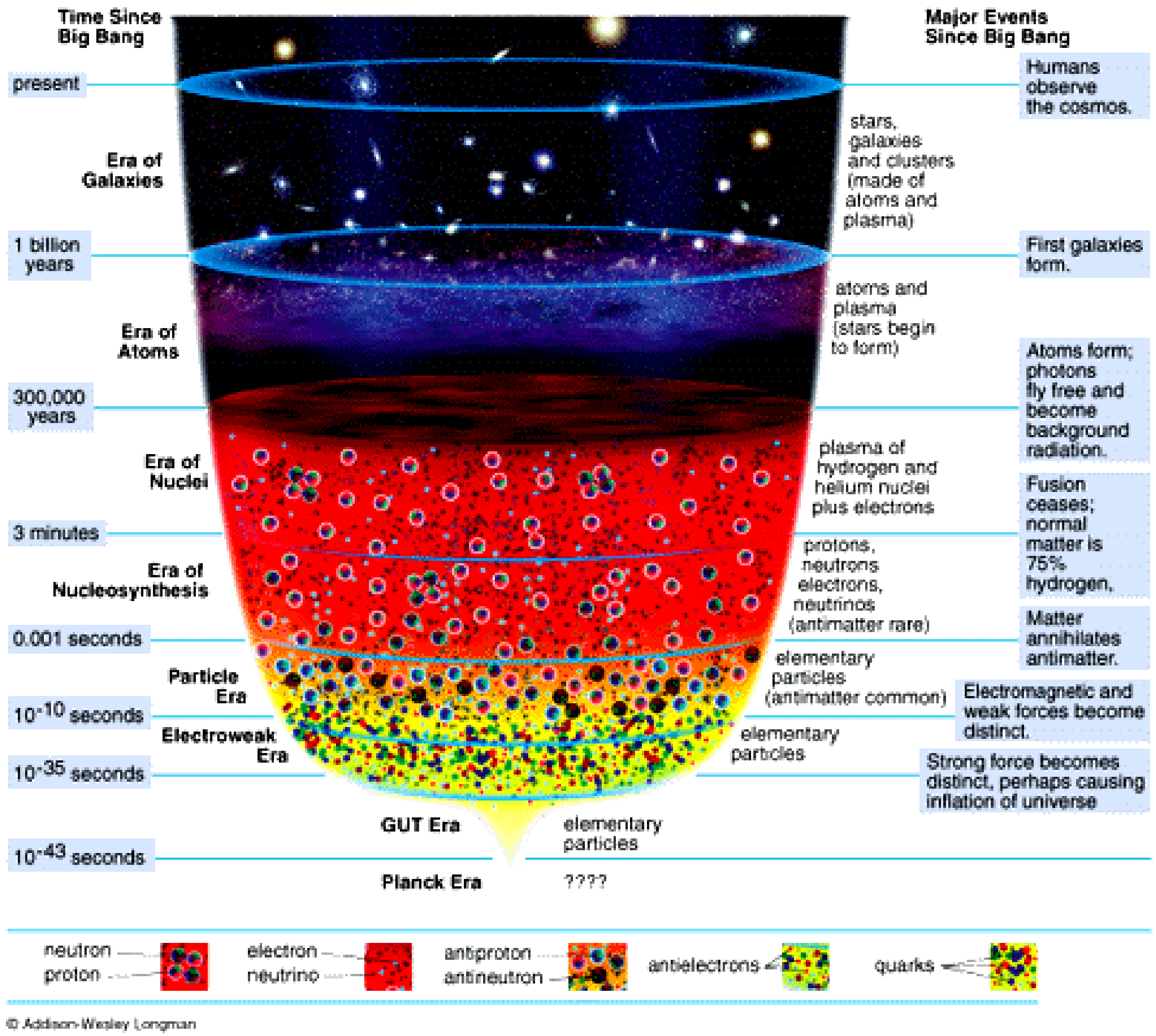
PARTICLE ASTROPHYSICS LECTURE 3

Thermal history and nucleosynthesis



Summary of Thermal History

Event	T	kT	g_{eff}	z	t
Now	2.7 K	0.0002 eV	3.3	0	13 Gyr
First Galaxies	16 K	0.001 eV	3.3	5	1 Gyr
Recombination	3000 K	0.3 eV	3.3	1100	300,000 yr
$\rho_M = \rho_R$	9500 K	0.8 eV	3.3	3500	50,000 yr
$e^+ e^-$ pairs	$10^{9.7}$ K	0.5 MeV	11	$10^{9.5}$	3 s
Nucleosynthesis	10^{10} K	1 MeV	11	10^{10}	1 s
Nucleon pairs	10^{13} K	1 GeV	70	10^{13}	10^{-7} s
E-W unification	$10^{15.5}$ K	250 GeV	100	10^{15}	10^{-12} s
Grand unification	10^{28} K	10^{15} GeV	100(?)	10^{28}	10^{-36} s
Quantum gravity	10^{32} K	10^{19} GeV	100(?)	10^{32}	10^{-43} s

Motion of a free particle

We can derive the equations of motion of a free particle from the Lagrangian:

$$\mathcal{L} = m(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{1/2}, \quad \delta\int m\,ds = \delta\int m\left(g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right)^{1/2} dt.$$

The equations of motion of a free particle are

$$\frac{dp_\alpha}{dt} = \frac{1}{2}g_{\mu\nu,\alpha}p^\mu\dot{x}^\nu$$

(valid for any gravitational field). The conjugate momenta are

$$p_\alpha = \frac{m g_{\alpha\nu}\dot{x}^\nu}{(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{1/2}} = m\frac{dx_\alpha}{d\tau}.$$

Note that $\dot{x}^\nu = (1, \dot{x}^i)$, and so we can write the coordinate velocity as

$$\dot{x}^{\alpha} = \frac{p^{\alpha}}{p_0}.$$

In addition

$$g^{\mu\nu} p_{\mu} p_{\nu} = m^2.$$

For a freely falling partice in FRW metric

$$\begin{aligned} \frac{dp_i}{dt} &= 0 \\ \frac{dp_0}{dt} &= \frac{1}{2} g_{jk,0} p^j \dot{x}^k = -R\dot{R} p^j \dot{x}^j \\ &= -R\dot{R} \frac{p^j p^j}{p_0}. \end{aligned}$$

Hence we can write

$$\frac{dp_0}{dt} = -\frac{\dot{R}}{R} \frac{p^2}{p_0}$$

where

$$p^2 = \frac{p_i p_i}{R^2}. \tag{1}$$

From the normalization condition $g^{\mu\nu} p_\mu p_\nu = m^2$,

$$p_0^2 - \frac{p_i p_i}{R^2} = p_0^2 - p^2 = m^2.$$

Hence:

$$p_0 \frac{dp_0}{dt} = p \frac{dp}{dt} = -\frac{\dot{R}}{R} p^2.$$

So $p \propto R^{-1}$, consistent with (1).

So, the equations of motion simply tell us that the (proper) momentum of a particle decays *adiabatically* as the Universe expands, *independent of the mass of the particle.*

The distribution function

$$\delta N = f(x^i, p_i, t)\delta x^1 \delta x^2 \delta x^3 \delta p_1 \delta p_2 \delta p_3$$

The phase space volume element

$$\delta x^1 \delta x^2 \delta x^3 \delta p_1 \delta p_2 \delta p_3$$

is an *invariant* under coordinate transformations. In the absence of collisions, the distribution function f is therefore constant along the path of a particle (Lioville's theorem). The stress-energy tensor of a gas of particles is

$$\begin{aligned}
T^{\mu\nu} &= \int \frac{d^4p}{(-g)^{1/2}} 2\delta(g^{\mu\nu}p_\mu p_\nu - m^2) p^\mu p^\nu f, \\
&= \int dp_0^2 \delta(p_0^2 - p^2 - m^2) \frac{1}{R^3} \frac{p^\mu p^\nu}{p_0} f dp_1 dp_2 dp_3 \\
&= \int p^\mu v^\nu f \frac{dp_1 dp_2 dp_3}{R^3}, \qquad v^\nu = \dot{x}^\nu = \frac{p^\nu}{p_0}.
\end{aligned}$$

Hence if the distribution function is isotropic

$$\rho = T^{00} = 4\pi \int E f p^2 dp, \qquad (2a)$$

$$P = \frac{1}{3} T^{ii} = \frac{4\pi}{3} \int \frac{f}{E} p^4 dp. \qquad (2b)$$

where $E = p_0$.

Note that for ultra-relativistic matter, $E = p$, hence necessarily

$$\boxed{P = \tfrac{1}{3}\rho c^2}$$

(A stiffer equation of state requires strong collective interactions.)

For particles of species i in *thermal equilibrium* at temperature T, the distribution function is

$$n_i(p)dp = \frac{4\pi}{h^3} \frac{g_i p^2 dp}{\left[\exp\left(\frac{E-\mu_i}{kT}\right) \pm 1\right]}$$

where $E^2 = p^2 + m^2$, g_i is the number of spin states, and

$$\begin{array}{lll} +1 & \text{for Fermions} & (\text{Fermi – Dirac distribution}) \\ -1 & \text{for Bosons} & (\text{Bose – Einstein distribution}) \end{array}$$

The quantities μ_i are the *chemical potentials* , and are fixed by conserved quantum numbers (see later).

Ignoring chemical potentials, we can evaluate the integrals

$$\rho = \frac{4\pi}{h^3} g_i \int \frac{E p^2 dp}{\left[\exp\left(\frac{E}{kT}\right) \pm 1\right]},$$

$$P = \frac{4\pi}{3h^3} g_i \int \frac{p^4 dp}{E\left[\exp\left(\frac{E}{kT}\right) \pm 1\right]}.$$

for ultra-relativistic particles ($E = p$):

$$\rho = aT^4 \left(\frac{g_i}{2}\right) \qquad \text{(Bosons)}$$

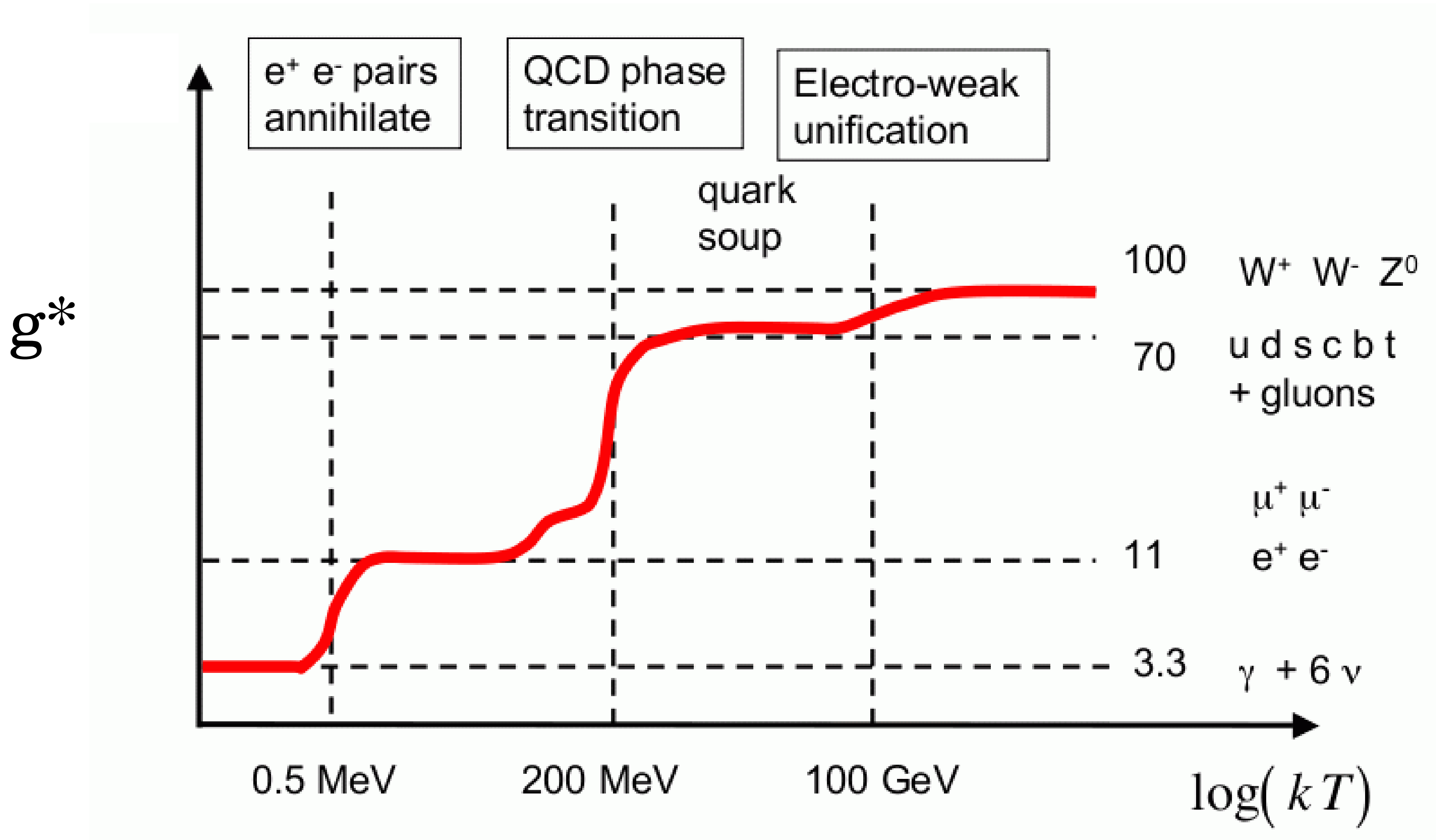
$$\rho = \frac{7}{16} aT^4 g_i \qquad \text{(Fermions)}$$

where

$$a = \frac{\pi^2 k^4}{15 \hbar^3 c^3}.$$

Hence we can define an *effective statistical weight* for relativistic particles:

$$g^*(T) = \Sigma_{\text{bosons}}\, g_i \left(\frac{T_i}{T}\right)^4 + \Sigma_{\text{fermions}}\, \frac{7}{8} g_i \left(\frac{T_i}{T}\right)^4$$



Recall that energy conservation requires

$$\frac{d(\rho R^3)}{dR} = -3PR^2.$$

We can rewrite this as:

$$d((\rho + P)R^3) = R^3 dP.$$

Hence

$$\begin{aligned} TdS &= d(\rho R^3) + PdR^3 \\ &= d[(\rho + P)R^3] - R^3 dP \\ &= 0. \end{aligned}$$

So conservation of energy is equivalent to conservation of entropy within a comoving volume element. If $\rho(T)$, $P(T)$, then it is straightforward to show that

$$S = \frac{R^3}{T}(\rho + P) + \text{constant}.$$

Example: Computing the neutrino temperature

At $T \gtrsim 10^{10} K$ the relativistic species are

$$\gamma,\ e^{\pm},\ \nu_e, \overline{\nu}_e, \nu_\mu, \overline{\nu}_\mu, \nu_\tau, \overline{\nu}_\tau,$$

and the neutrinos are in thermal equilibrium via the weak reactions

$$e^{+} + e^{-} \Longleftrightarrow \nu_i + \overline{\nu}_i$$

The collision timescale is

$$t_{\text{coll}}^{-1} \sim n_e \langle \sigma_w v \rangle \propto T^5$$

and the expansion timescale is

$$t_{\text{exp}}^{-1} \sim \frac{\dot{R}}{R} \propto T^2 \qquad \left(H^2 = \frac{8\pi G}{3} \rho \right).$$

So, electrons and neutrinos decouple when

$$t_{\rm coll} \gtrsim t_{\rm exp},$$

which occurs at a temperature of $kT \sim 3$MeV. The e^+e^- annihilate at $kT \sim 1$MeV boosting the radiation density.

Before e^+e^- annihilation

$$\rho = aT^4 + \frac{7}{4}aT^4 + \rho_\nu(T).$$

After e^+e^- annihilation

$$\rho = aT_\gamma^4 + \rho_\nu(T).$$

For relativistic particles in thermal equilibrium

$$S = \frac{R^3}{T}(\rho + P) = \frac{4R^3}{3T}\rho,$$

Hence

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma = 1.93K.$$

Collisionless Boltzmann equation

For homogeneous components, the distribution function is a function of E ($E \equiv p_0$) and t, $f(E,t)$. The collisionless Boltzmann equation is then

$$\frac{\partial f}{\partial t} + \frac{dp_0}{dt}\frac{\partial f}{\partial p_0} = 0.$$

Recall that the equations of motion give

$$\frac{dp_0}{dt} = -\frac{\dot{R}}{R}\frac{p^2}{p_0}.$$

So,

$$\frac{\partial f}{\partial t} - \frac{\dot{R}p^2}{Rp_0}\frac{\partial f}{\partial p_0} = 0.$$

Now integrate over $p^2 dp$

$$4\pi \int \frac{\partial f}{\partial t} p^2 dp - 4\pi \frac{\dot{R}}{R} \int \frac{p^4}{p_0} \frac{\partial f}{\partial p_0} dp = 0,$$

i.e.

$$\frac{\partial n}{\partial t} - 4\pi \frac{\dot{R}}{R} \int p^3 dp \frac{\partial f}{\partial p} = 0, \qquad p_0 dp_0 = p dp$$

integrating the second term by parts

$$\int p^3 \frac{df}{dp} dp = \left[p^3 f \right]_{-\infty}^{\infty} - 3 \int p^2 f dp,$$

we finally get,

$$\frac{dn}{dt} + 3 \frac{\dot{R}}{R} n = 0.$$

This simply tells us that in the absence of collisions, the number density is diluted by the expansion of the Universe ($n \propto R^{-3}$).

Collisional Boltzmann equation

Consider, for example, particle-antiparticle annihilations

$$X + \overline{X} \Longleftrightarrow Y + \overline{Y}.$$

Assume that Y and $\overline{Y}$ are in thermal equilibrium and that CP invariance holds. Then $n_x = \overline{n}_x$ and including collisions, the rate equation can be written as

$$\frac{dn}{dt} + 3\frac{\dot{R}}{R}n = \langle \sigma v \rangle (n_{\text{equ}}^2 - n^2) \qquad (1)$$

where

$$\langle \sigma v \rangle n\overline{n} \approx \frac{1}{h^6} \int \sigma(p, \overline{p}) v f \overline{f} d^3p d^3\overline{p}.$$

and $\langle \sigma v \rangle n_{\text{equ}}^2$ is the production rate and n_{equ} is the LTE (local thermodynamic equilibrium) abundance of $X, \overline{X}$ at temperature T.

It is easy to see what the solutions of (1) look like. The relevant timescales are the collision timescale, $t_{\rm coll}$, and the expansion timescale, $t_{\rm exp}$:

$$t_{\rm coll}^{-1} \sim n\langle\sigma v\rangle, \qquad t_{\rm exp}^{-1} \sim \frac{\dot{R}}{R}.$$

• If $t_{\rm coll} << t_{\rm exp}$, then the rate equation is very stiff and the solution is

$$n = n_{\rm equ}(T).$$

• If $t_{\rm coll} << t_{\rm exp}$, then the particles are collisionless and the solution is

$$n \propto R^{-3}.$$

So the *relic abundance* is

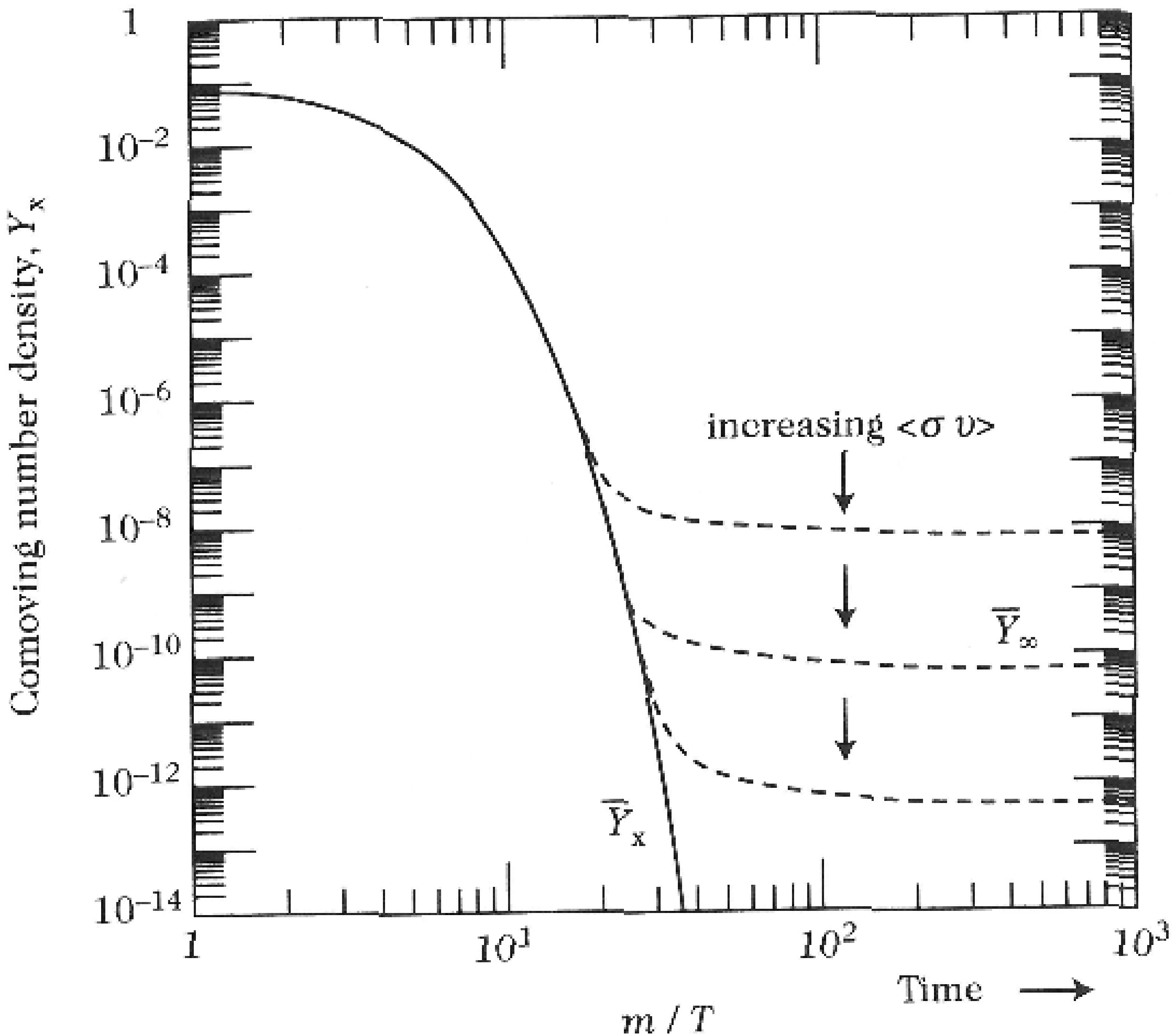
$$n(t) \approx n_{\text{equ}}(t_f)\frac{R^3(t_f)}{R^3(t)},$$

where t_f is the *'freeze-out' time* defined by

$$t_{\text{coll}} = t_{\text{exp}}.$$

Note that if the particles X are massive at t_f,

$$n_{\text{equ}}(T_f) \propto \left(\frac{m_X}{T_f}\right)^{3/2} \text{exp}\left(-\frac{m}{kT_f}\right).$$



Comoving number density, Y_x
1
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
10^{-14}
increasing $<\sigma v>$
$\overline{Y}_\infty$
$\overline{Y}_x$
1
10^1
10^2
10^3
Time →
m / T

Primordial Nucleosynthesis

- Stage 1: $T >> 1$MeV:

Weak interactions

$$\begin{aligned} n + \nu_e &\Longleftrightarrow p + e \\ e^+ + n &\Longleftrightarrow p + \overline{\nu}_e \\ n &\Longleftrightarrow p + e + \overline{\nu}_e \end{aligned}$$

and EM interactions

$$e + e^+ \Longleftrightarrow \gamma + \gamma$$

maintain all components in thermal equilibrium ($g* = 10.75$). The neutron-proton ratio in thermodynamic equilibrium is

$$\frac{n}{p} = \left(\frac{m_n}{m_p}\right)^{3/2} \exp\left(-\frac{Q}{kT}\right)$$

where

$$\begin{aligned} Q &= (m_n - m_p)c^2 = 1.29\text{MeV} \\ m_n &= 936.6\text{MeV}, \quad m_p = 938.3\text{MeV} \end{aligned}$$

- Stage 2: $T \approx 0.8$MeV, freeze-out:

$$\begin{aligned} \sigma_w &= 10^{-47}(kT/1\text{MeV})^2 m^2 \\ n\sigma_w c &\approx H \quad \text{at} \quad kT \approx 0.8\text{MeV}, \quad (t \approx 1s) \end{aligned}$$

so the neutron-proton ratio freezes out at

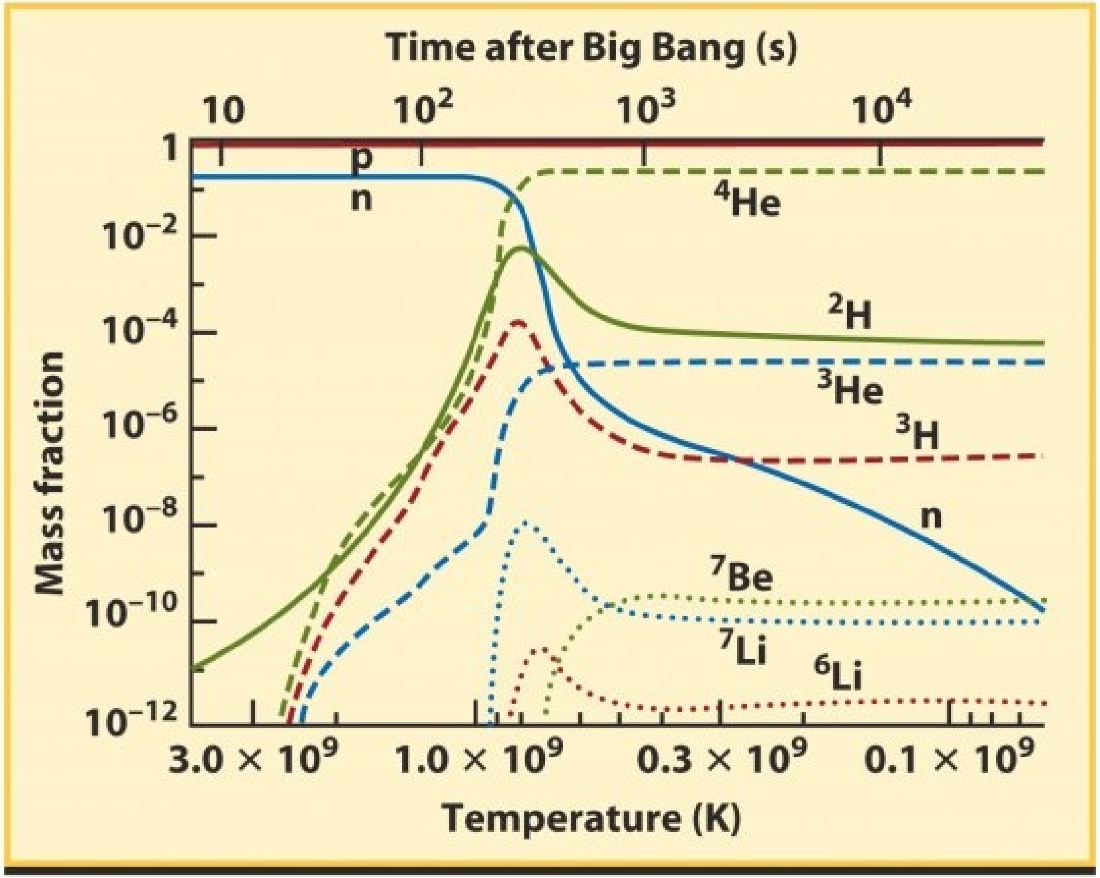
$$\frac{n}{p} = \exp\left(-\frac{1.29}{0.8}\right) \approx \frac{1}{5}.$$

If all of the neutrons end up bound into Helium (almost correct), then the Helium abundance by mass is

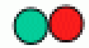
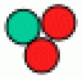
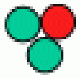
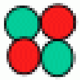
$$Y_{\text{He}} = \frac{4m_p n_{\text{He}}}{m_p n_H + 4m_p n_{\text{He}}} = \frac{2(n/p)}{1 + n/p} \approx 0.25.$$

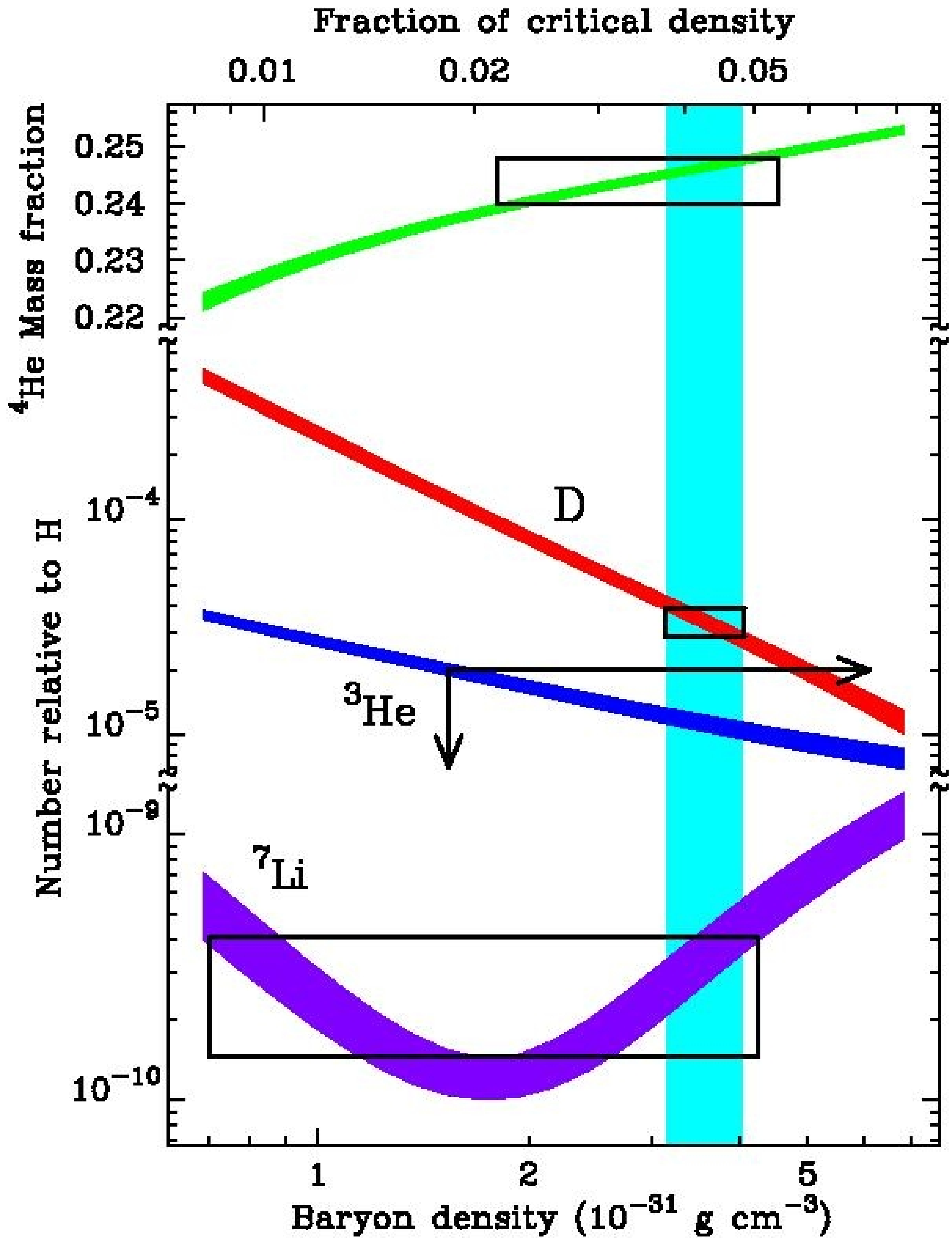
- Stage 3: $T \approx 0.7 - 0.05$MeV:

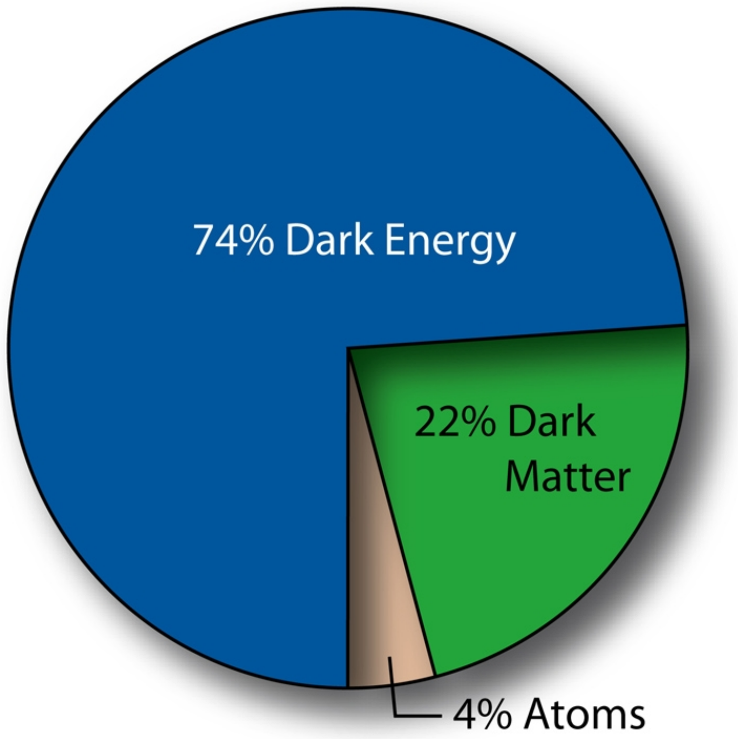
Fusion begins. But, there is a 'deuterium bottleneck', because deuterium has a low binding energy and is easily dissociated by the blackbody radiation. Deuterium production is therefore delayed until blackbody photons are redshifted.



Key Fusion Reactions

Reaction	product:	binding energy:
$n + p \rightarrow D + \gamma$	Deuterium (pn) 	2.2 MeV
$D + D \rightarrow {}^3He^{++} + n$ $p + D \rightarrow {}^3He^{++} + \gamma$	${}^3He$ (ppn) 	7.72 MeV
$n + D \rightarrow T + \gamma$ $D + D \rightarrow T + p$ $n + {}^3He^{++} \rightarrow T + p$	Tritium (pnn) 	8.48 MeV
$n + {}^3He^{++} \rightarrow {}^4He^{++} + \gamma$ $D + {}^3He^{++} \rightarrow {}^4He^{++} + p$ $p + T \rightarrow {}^4He^{++} + \gamma$ $D + T \rightarrow {}^4He^{++} + n$ ${}^3He^{++} + {}^3He^{++} \rightarrow {}^4He^{++} + 2p$	${}^4He$ (ppnn) 	28.3 MeV


Fraction of critical density
0.01
0.02
0.05
^{4}He Mass fraction
0.25
0.24
0.23
0.22
D
^{3}He
^{7}Li
Number relative to H
10^{-4}
10^{-5}
10^{-9}
10^{-10}
1
2
5
Baryon density (10^{-31} g cm^{-3})


74% Dark Energy
22% Dark Matter
4% Atoms

Sakharov criteria for generating baryon (lepton) number asymmetry

- Need a physical process that violates baryon (lepton) number.

- Needs a physical process that violates invariance under **C** and **CP**. (Necessary to ensure a different reaction (or decay rate) for particles and antiparticles.)

- Needs a departure from thermal equilibrium (**CPT** requires particle and antiparticles have *exactly* equal mass). In absence of conserved numbers, chemical potentials are zero. Hence in thermal equilibrium, the distribution functions of particles and antiparticles are identical.

Appendix I: What is the origin of the source term in the Boltzmann equation?

Assume a simple reaction:

$$\psi\overline{\psi} \to X\overline{X},$$

and assume Maxwell-Boltzmann statistics. The net production rate of particles ψ and $\overline{\psi}$ is

$$-\int d\Pi_\psi d\Pi_{\overline{\psi}} d\Pi_X d\Pi_{\overline{X}} (|\mathcal{M}|^2_{\psi\overline{\psi}\to X\overline{X}} f_\psi f_{\overline{\psi}} - |\mathcal{M}|^2_{X\overline{X}\to\psi\overline{\psi}} f_X f_{\overline{X}}) \delta^4(p_\psi + p_{\overline{\psi}} - p_X - p_{\overline{X}}),$$

where $d\Pi = g d^3p/(h^3 E)$.

If **CP** is conserved (**T** invariance) then

$$|\mathcal{M}|^2_{\psi\overline{\psi}\to X\overline{X}} = |\mathcal{M}|^2_{X\overline{X}\to\psi\overline{\psi}} = |\mathcal{M}|^2.$$

And if $X\overline{X}$ are in *thermal equilibrium*

$$f_X f_{\overline{X}} = \exp\left(-\frac{E_X}{kT}\right)\exp\left(-\frac{E_{\overline{X}}}{kT}\right) = \exp\left(-\frac{E_\psi}{kT}\right)\exp\left(-\frac{E_{\overline{\psi}}}{kT}\right),$$

since energy conservation requires $E_X + E_{\overline{X}} = E_\psi + E_{\overline{\psi}}$.

So, the net production rate is

$$-\int d\Pi_\psi d\Pi_{\overline{\psi}} |\mathcal{M}|^2 (f_\psi f_{\overline{\psi}} - f_\psi^{\text{equ}} f_{\overline{\psi}}^{\text{equ}}),$$

and if the cross-section varies slowly with momentum, we can do the integrals giving the equation in the notes:

$$\boxed{\langle \sigma v \rangle (n_{\text{equ}}^2 - n^2).}$$

Appendix II: Why do we need C *and* CP violation for baryogenesis?

If $\mathbf{C}$ is a symmetry, then every B violating reaction $X \to Y + Z$ has the same rate as the **C** conjugate reaction:

$$\Gamma(X \to Y + Z) = \Gamma(\overline{X} \to \overline{Y} + \overline{Z}).$$

If this is true, then baryon number is conserved.

However, violation of **C** is not enough. Consider a baryon violating process that creates *left-handed* baryons $X \to q_L q_L$. If **CP** is a symmetry then the rates for this reaction is the same as the CP-conjugate reaction $\overline{X} \to \overline{q}_R \overline{q}_R$. If CP is conserved

$$\Gamma(X \to q_L q_L) + \Gamma(X \to q_R q_R) = \Gamma(\overline{X} \to \overline{q}_L \overline{q}_L) + \Gamma(\overline{X} \to \overline{q}_R \overline{q}_R),$$

and so again, no net baryon number.

So, we need violation of both **C** and **CP**.