3.1

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - V(\phi)$$

$$S = \int \sqrt{-g}\, \mathcal{L}\, d^4x$$

$$\delta S = \int \sqrt{-g}\, \delta\mathcal{L}\, d^4x$$

$$= \int \sqrt{-g}\,.\left[g^{\mu\nu} \phi_{,\mu} \delta\phi_{,\nu} - \frac{dV(\phi)}{d\phi}\delta\phi \right] d^4x$$

integrating this by parts

$$\sqrt{-g}\, g^{\mu\nu}\phi_{,\mu}\, \delta\phi\, d^3x - \int (\sqrt{-g}\, g^{\mu\nu}\phi_{,\mu})_{,\nu}\, d^4x\, \delta\phi$$

$$= 0 .$$

so,

$$\delta S = -\int d^4x\, d\phi \left[(\sqrt{-g}\, g^{\mu\nu}\phi_{,\mu})_{,\nu} + \sqrt{-g} V' \right] = 0 .$$

so, the field equation is,

$$(\sqrt{-g}\, g^{\mu\nu}\phi_{,\mu})_{,\nu} + \sqrt{-g}\, V' = 0 .$$

Now, recall from the definition of $\Gamma^{\alpha}_{\beta\gamma}$ –

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\kappa}\left[\frac{\partial g_{\kappa\gamma}}{\partial x^{\beta}} + \frac{\partial g_{\beta\kappa}}{\partial x^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\kappa}} \right]$$

so,

$$\Gamma^{\alpha}_{\alpha\gamma} = \frac{1}{2} g^{\alpha\kappa}\left[\frac{\partial g_{\kappa\gamma}}{\partial x^{\alpha}} + \frac{\partial g_{\alpha\kappa}}{\partial x^{\gamma}} - \frac{\partial g_{\alpha\gamma}}{\partial x^{\kappa}} \right]$$

these terms cancel, just swap indices α & κ.

So,

$$\Gamma^{\alpha}_{\alpha\gamma} = \frac{1}{2} g^{\alpha\kappa} \frac{\partial g_{\alpha\kappa}}{\partial x^{\gamma}}$$

Note also, the identity –

$$\frac{\partial \sqrt{-g}}{\partial x^{\gamma}} = \sqrt{-g}\,\frac{1}{2} g^{\alpha\kappa} \frac{\partial g_{\alpha\kappa}}{\partial x^{\gamma}} = \sqrt{-g}\;\Gamma^{\alpha}_{\alpha\gamma}.$$

The covariant derivative of $g^{\mu\nu}\phi_{,\mu}$ is

$$(g^{\mu\nu}\phi_{,\mu})_{;\nu} = (g^{\mu\nu}\phi_{,\mu})_{,\nu} + \Gamma^{\nu}_{\alpha\nu}\,(g^{\mu\alpha}\phi_{,\mu})$$

$$= (g^{\mu\nu}\phi_{,\mu})_{,\nu} + \frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial x^{\alpha}}\,(g^{\mu\alpha}\phi_{,\mu})$$

$$= \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,\phi_{,\mu}\right)_{,\nu}.$$

hence, the field equation is

$$(g^{\mu\nu}\phi_{,\mu})_{;\nu} + V'(\phi) = 0.$$

3.2.

In the slow roll limit,

$$3H\dot{\phi} = -V'$$
$$H^2 = 1/3\, V$$

so,

$$3H^2 \frac{\dot{\phi}}{H} = -V'$$

$$\therefore \quad \frac{\dot{\phi}}{H} = -\frac{V'}{V}$$

so,

$$R \frac{d\phi}{dt} \cdot \frac{dt}{dR} = -\frac{V'}{V}.$$

i.e.

$$-d\phi \frac{V}{V'} = \frac{dR}{R}$$

since,

$$V(\phi) = \frac{\lambda}{n} \phi^n.$$

$$\int -d\phi \cdot \frac{1}{n} \phi = \int \frac{dR}{R}$$

i.e.

$$\ln R = -1/n \; \phi^2/2$$

and so,

$$R(\phi) \simeq R_i \exp\left(\frac{1}{2n} \left(\phi_i^2 - \phi^2(t) \right) \right).$$

3.3.

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0$$

Now, neglect expansion and multiply by ϕ

$$\phi\ddot{\phi} + \phi V' = 0$$

which we can write as

$$(\phi\dot{\phi})' - \dot{\phi}^2 + \phi V' = 0 .$$

Averaging over a period of rapid oscillation, the first term vanishes so,

$$\langle \dot{\phi}^2 \rangle = \langle \phi V' \rangle .$$

Since

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$

$$\omega = \frac{p}{\rho} = \frac{\langle \dot{\phi}^2 \rangle - 2V}{\langle \dot{\phi}^2 \rangle + 2V}$$

$$= \frac{\langle \phi V' \rangle - 2V}{\langle \phi V' \rangle + 2V}$$

if $V(\phi) \propto \phi^n$

$$\omega = \frac{n-2}{n+2}$$

If $n = 4$, $\omega = 1/3$ so effective equation of state is like a radiation dominated universe

if $n = 2$, $\omega = 0$, i.e. matter dominated.

The "equation of state" prior to reheating depends on the shape of the potential.

3.4

The scalar field obeys

$$\ddot{\phi} + 3H\dot{\phi} = -V' \qquad (1)$$

$$R \propto t^{1/2}$$

so, $$H = \frac{\dot{R}}{R} = \frac{1}{2t}$$

$$V = A/\phi^{\alpha}$$

so, (1) reads

$$\ddot{\phi} + \frac{3}{2t}\dot{\phi} = \alpha A/\phi^{\alpha+1}$$

put $$\phi = \phi_0 t^{p}$$

$$p(p-1)t^{p-2} + \frac{3}{2}p\,t^{p-2} = \frac{\alpha A}{\phi_0^{\alpha+2}} t^{-p(\alpha+1)}$$

For a power-law solution

$$p - 2 = -p(\alpha+1)$$

ie. $$p = \frac{2}{(2+\alpha)}$$

$$\rho_\phi \propto t^{2p-2}$$

$$\rho \propto R^{-4} \propto t^{-2}$$

so, $$\rho_\phi/\rho \propto t^{4/(2+\alpha)}$$

4.1 This is example 6.4 from Perkins (particle astrophysics).

Let E_{th} be the threshold photon energy
E_0 the energy of the target photon.

Threshold for pair production

$$\gamma\gamma \to e^+e^-$$

is fixed by

$$s = (E_{th} + E_0)^2 - (p_{th} + p_0)^2 > 4m_e^2$$

For a head-on collision

$$E_{th} = \frac{m_e^2}{E_0}.$$

For microwave photons, $E_0 = y(kT_0)$ $T_0 = 2.73$

and so

$$E_{th} \sim \frac{10^3}{y}\,\text{TeV}$$

The absorption length for γ-rays is

$$\lambda \sim 1/\rho\sigma$$

where ρ is the density of target photons
σ is the interaction cross section.

The cross section is given by the Klein-Nishina formula (see perkins). At threshold, $\sigma = 0$, but it rises to a max. of $\sim 0.25\,\sigma_{Thomson}$ at $s = 8m^2$. For Thomson cross-section, and $\rho \simeq 400\ \text{cm}^{-3}$,

$$\lambda \sim 10^{22}\,\text{cm} \sim 4\,\text{kpc}.$$

So, for very high energy photons ($> 10^3$ TeV) the Universe will be opaque.

(4.2)

The quadrupole formula is

$$L_{GW} \sim \frac{G}{5c^5} \langle \dddot{I}^2 \rangle .$$

If we have an object of mass M, size L, then

$$I \sim \rho L^5 = \frac{M L^5}{L^3}$$

$$\dddot{I} \sim M L^2 \omega^3$$

So,

$$L_{GW} \sim \frac{G}{5c^5} M^2 L^4 \omega^6$$

Now,

$$\omega^2 \sim \frac{GM}{L^3}$$

So,

$$L_{GW} \sim \frac{G}{5c^5} M^2 L^4 \frac{G^3 M^3}{L^9}$$

$$\sim \frac{c^5}{G} \left(\frac{GM}{c^2 L} \right)^5$$

If we set $L = \frac{GM}{c^2}$ (Schwarzschild radius)

then max. L_{GW} is

$$L_{GW} \sim c^5/G \, , \qquad T_{01} \sim \frac{c^5}{G \cdot 4\pi r^2}$$

giving

$h \sim 10^{-14}$ at $r = 1\,\mathrm{kpc}$

$h \sim 10^{-17}$ at $r = 1\,\mathrm{Mpc}$

for a frequency of $1\,\mathrm{kHz}$