Solutions.

(1.1) If you can't do this, you had better quit now.

(1.2) In calculating distances in the FRW model it is convenient to transform radial coordinate –

$$ds^2 = dt^2 - R^2\left[\frac{dr^2}{(1-Kr^2)} + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2\right]$$

put $r = \frac{1}{\sqrt{|K|}}\,\mathrm{sin}_K\,\chi$ where $\left\{\begin{array}{lll}\mathrm{sin}_K = \sin & & K>0\\ \mathrm{sin}_K = \sinh & & K<0\end{array}\right.$

then the metric is

$$ds^2 = dt^2 - \frac{R^2}{|K|}\left[d\chi^2 - \mathrm{sin}_K^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

The coordinate distance χ to an object at redshift z is

$$\chi = \sqrt{|K|}\int_{t_e(z)}^{t_0}\frac{dt}{R(t)} = \sqrt{|K|}\int\frac{dR}{\dot{R}R} = \sqrt{|K|}\int\frac{dR}{R^2H}$$

and

$$H^2 = \frac{8\pi G\rho}{3} - K/R^2 + \Lambda/3$$

So,

$$H^2 = H_0^2\left[\Omega_m(0)\left(R_0/R\right)^3 + \Omega_K(0)\left(R_0/R\right)^2 + \Omega_\Lambda\right]$$

$$\Omega_K(0) = -K/R_0^2H_0^2$$

Hence,

$$\chi = \frac{\sqrt{|K|}}{H_0R_0}\int\frac{dR/R_0}{(R/R_0)^2\left[\Omega_m(R_0/R)^3 + \Omega_K(R_0/R)^2 + \Omega_\Lambda\right]^{1/2}}$$

$$= \sqrt{|\Omega_K|}\int\frac{dz}{\left[\Omega_m(1+z)^3 + \Omega_K(1+z)^2 + \Omega_\Lambda\right]^{1/2}}$$

The angular diameter and luminosity distances are

$$d_A = \frac{\mathrm{sin}_K\,\chi}{H_0|\Omega_K|^{1/2}(1+z)}$$

$$d_L = \frac{(1+z)\,\mathrm{sin}_K\,\chi}{\left[H_0|\Omega_K|^{1/2}\right]}$$

If $\Omega_K = 0$, $\Omega_\Lambda = 0$, $\Omega_m = 1$, then

$$d_A = \frac{c}{H_0(1+z)} \int_0^z \frac{dz}{(1+z)^{3/2}}$$

$$= \frac{2c}{H_0(1+z)} \cdot \left[1 - \frac{1}{(1+z)^{1/2}} \right]$$

and differentiating, $d_A(z)$ has a maximum at

$$\frac{1}{(1+z)^2} = \frac{3}{2(1+z)^{5/2}}$$

i.e. $z_{max} = 5/4$.

This is because at high redshifts, the Universe was physically smaller, so a high redshift object appears physically larger to us.

(1.3) Start with the Friedmann equations

$$\frac{\dot{R}^2}{R^2} + \frac{K}{R^2} = \frac{8\pi G\rho}{3}$$

$$3\frac{\ddot{R}}{R} = -4\pi G(\rho+3P) = -4\pi G\rho(1+3w)$$

$$\frac{dR}{dt} = \frac{dR}{d\tau}\cdot\frac{d\tau}{dt} = \frac{1}{R}\frac{dR}{d\tau} = \mathcal{H}, \qquad d\tau = dt/R$$

Hence,

$$\frac{K}{R^2} = \frac{8\pi G\rho}{3} - \frac{\mathcal{H}^2}{R^2}.$$

i.e.

$$\frac{K}{\mathcal{H}^2} = (\Omega - 1) \qquad (1)$$

$$\ddot{R} = \frac{1}{R}\frac{d\mathcal{H}}{d\tau}$$

so,

$$\frac{3}{R^2}\frac{d\mathcal{H}}{d\tau} = -4\pi G\rho(1+3w)$$

i.e.

$$2\frac{d\mathcal{H}}{d\tau} = -\mathcal{H}^2\Omega(1+3w) = -(\mathcal{H}^2+K)(1+3w) \qquad (2)$$

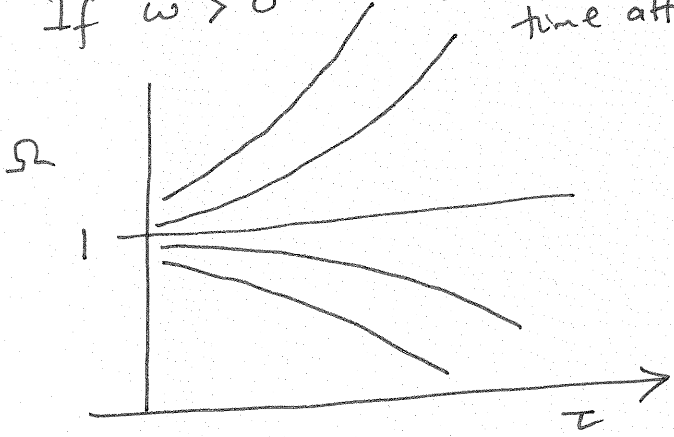
From (1)

$$\frac{d\Omega}{d\tau} = -\frac{2K}{\mathcal{H}^3}\frac{d\mathcal{H}}{d\tau} = \frac{(\mathcal{H}^2+K)(1+3w)}{\mathcal{H}^3} \quad \text{using (2)}$$

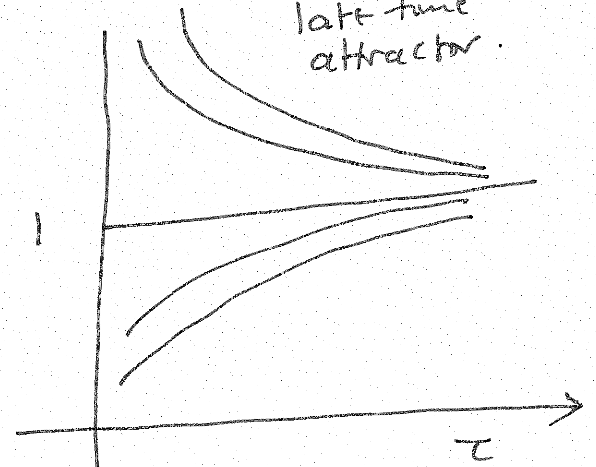
and replacing K

$$\frac{d\Omega}{d\tau} = (1+3w)(\Omega-1)\Omega\mathcal{H}.$$

If $w > 0$. $\Omega=1$ is an early time attractor



If $w < 0$, Ω is a late time attractor.



Note that for de-Sitter space

$$H = \dot{R}/R = \text{constant} \qquad R \propto e^{Ht}$$

so, conformal time is

$$\int d\tau = \int dt/R = -\frac{e^{-Ht}}{H}$$

so, for de-Sitter space,

$$\tau = -\frac{1}{RH}$$

and runs from $\tau = -\infty \to 0$. This is because de-Sitter space is eternal, + running from $-\infty \to \infty$.

(1.4)

$$\mathcal{L} = [g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta]^{1/2} = \frac{ds}{dp} \qquad (1)$$

$$\frac{d}{dp}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}\right) - \frac{\partial \mathcal{L}}{\partial x^\alpha} = 0$$

$$\frac{\partial \mathcal{L}^2}{\partial \dot{x}^\alpha} = 2\mathcal{L}\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}$$

so,

$$\frac{d}{dp}\left(\frac{\partial \mathcal{L}^2}{\partial \dot{x}^\alpha}\right) = 2\frac{d\mathcal{L}}{dp}\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} + 2\mathcal{L}\frac{d}{dp}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}\right)$$

$$= 2\frac{d\mathcal{L}}{dp}\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} + 2\mathcal{L}\frac{\partial \mathcal{L}}{\partial x^\alpha}$$

Hence,

$$\frac{d}{dp}\left(\frac{\partial \mathcal{L}^2}{\partial \dot{x}^\alpha}\right) - \frac{\partial \mathcal{L}^2}{\partial x^\alpha} = 2\frac{d\mathcal{L}}{dp}\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha}$$

but if p is an affine parameter $p = \alpha s + \beta$
hence from (1) $\frac{d\mathcal{L}}{dp} = 0$, and so $\mathcal{L}^2$ obeys the Euler-Lagrange equ.

$$\frac{d}{dp}\left(\frac{\partial \mathcal{L}^2}{\partial \dot{x}^\alpha}\right) - \frac{\partial \mathcal{L}^2}{\partial x^\alpha} = 0$$

So,

$$\frac{d}{dp}\left(g_{\mu\nu}\dot{x}^\nu \delta_{\alpha\mu} + g_{\mu\nu}\dot{x}^\mu \delta_{\alpha\nu}\right) - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\dot{x}^\mu \dot{x}^\nu = 0$$

$$dg_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x^\beta}dx^\beta$$

hence,

$$g_{\mu\nu}\ddot{x}^\nu \delta_{\alpha\mu} + g_{\mu\nu}\ddot{x}^\mu \delta_{\alpha\mu} + \frac{\partial g_{\mu\nu}}{\partial x^\beta}\dot{x}^\nu \dot{x}^\beta \delta_{\alpha\mu}$$
$$+ \frac{\partial g_{\mu\nu}}{\partial x^\beta}\dot{x}^\mu \dot{x}^\beta \delta_{\alpha\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\dot{x}^\mu \dot{x}^\nu = 0$$

so,

$$2g_{\alpha\beta}\ddot{x}^\beta + \frac{\partial g_{\alpha\nu}}{\partial x^\beta}\dot{x}^\nu \dot{x}^\beta + \frac{\partial g_{\mu\alpha}}{\partial x^\beta}\dot{x}^\mu \dot{x}^\beta - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\dot{x}^\mu \dot{x}^\nu = 0$$

hence,

$$\ddot{x}^\gamma + \frac{1}{2}g^{\gamma\alpha}\left[\frac{\partial g_{\alpha\nu}}{\partial x^\mu} + \frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right]\dot{x}^\mu \dot{x}^\nu$$

giving the familiar form.

$$\ddot{x}^\gamma + \Gamma^\gamma_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0$$

(1.5) Lensing does not involve emission or absorption of photons – it just moves light rays around mapping the source plane to the image plane. It therefore preserves surface brightness so we can compute magnification of images by computing the Jacobian of the transformation for the source to image planes

$$\mu = \left| \frac{\partial \theta_s}{\partial \underset{\sim}{\theta}} \right|^{-1}$$

Now in our dimensionless variables, $\underset{\sim}{y} = \underset{\sim}{\theta}_s/\theta_E$, $\underset{\sim}{x} = \underset{\sim}{\theta}/\theta_E$.

and
$$\underset{\sim}{x} = \frac{1}{2}\left(\underset{\sim}{y} + \frac{z}{y}\underset{\sim}{y}\right) \qquad \text{where } z = \sqrt{4+y^2}$$

hence,
$$\frac{\partial x_1}{\partial y_1} = \frac{1}{2}\left[1 \pm z/y \pm \left(\frac{1}{yz} - z/y^3\right) y_1^2\right]$$

$$\frac{\partial x_2}{\partial y_1} = \frac{\partial x_1}{\partial y_2} = \pm\frac{1}{2}\left(\frac{1}{yz} - \frac{z}{y^3}\right) y_1 y_2$$

$$\frac{\partial x_2}{\partial y_2} = \frac{1}{2}\left[1 \pm z/y \pm \left(1/yz - z/y^3\right) y_2^2\right]$$

Now compute the Jacobian. $J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$

$$J = \frac{1}{4}\left[(1 \pm z/y) \pm \left(\frac{1}{yz} - \frac{z}{y^3}\right) y_1^2\right]\left[(1 \pm z/y) \pm \left(\frac{1}{yz} - \frac{z}{y^3}\right) y_2^2\right] - \frac{1}{4}\left(\frac{1}{yz} - \frac{z}{y^3}\right)^2 y_1^2 y_2^2$$

$$= \frac{1}{4}\left[(1 \pm z/y)^2 \pm (1 \pm z/y)\left(1/yz - z/y^3\right)(y_1^2 + y_2^2)\right]$$

$$= \frac{1}{4}\left[1 \pm 2z/y + z^2/y^2 \pm \left(y/z - z/y\right) + 1 - z^2/y^2\right]$$

$$= \frac{1}{4}\left[2 \pm y/z \pm z/y\right]$$

So, the magnifications are $\mu_\pm = |J| = \frac{1}{4}\left[y/z + z/y \pm 2\right]$

The ~~lens~~ source must be "small" for local Jacobian approx to be valid. This requires that the physical size of the ~~lens~~ source is $R_S < \sqrt{D_L R_{Sw}}$ where R_{Sw} is the Schwarzschild radius of the lens.

(1.6) If we compare the energy momentum tensor to that of a perfect fluid, we see that this describes an object with negative pressure (ie. tension) along the string.

From the metric

$$ds^2 = dt^2 - dr^2 - f^2(r)\, d\theta^2 - dz^2$$

we see that there are only two non-zero affine connections

$$\Gamma^r{}_{\theta\theta} = -ff' \quad , \quad \Gamma^\theta{}_{\theta r} = f'/f$$

We therefore expect only two non-zero components for the Ricci tensor,

$$R_{rr} = \frac{\partial \Gamma^\theta{}_{\theta r}}{\partial r} + (\Gamma^r{}_{r\theta})^2$$

$$= -f'^2/f^2 + f''/f + f'^2/f^2 = f''/f$$

and

$$R_{\theta\theta} = -\frac{\partial \Gamma^r{}_{\theta\theta}}{\partial r} + \Gamma^r{}_{\theta\theta}\Gamma^\theta{}_{r\theta} + \cancel{\Gamma^\theta{}_{\theta r}\Gamma^r{}_{\theta\theta} - \Gamma^r{}_{\theta\theta}\Gamma^\theta{}_{r\theta}}$$

$$= f'^2 + ff'' - ff'f'/f = ff''$$

We write the Einstein equation in the form.

$$R_{\mu\nu} = -8\pi G\left(T_{\mu\nu} - 1/2\, g_{\mu\nu} T^\kappa{}_\kappa\right)$$

$$T_{00} = \rho, \quad T_{zz} = -\rho \quad , \quad T^0{}_0 = \rho \, , \quad T^z{}_z = \rho$$

So,

$$S_{rr} = \rho \, , \quad S_{\theta\theta} = \rho f^2(r)$$

So the field equations give <u>one</u> independent equation.

$$f''/f = -8\pi G\rho(r)$$

So, if $r<r_0$

$$f''/f = -8\pi G\rho_0$$

with solution

$$f = 1/k \sin kr \qquad k=\sqrt{8\pi G\rho_0}$$

(note $f\to 0$ as $r\to 0$)

For $r>r_0$

$$f''=0$$

with solution $f = a+br$

Requiring f, f' to be continuous at $r=r_0$,

$$f = \frac{\sin kr_0}{k} - r_0\cos kr_0 + r\cos kr_0$$

and so,

$$f^2 = \left[\frac{\sin kr_0}{k} + (r-r_0)\cos kr_0\right]^2$$

if $kr_0 \ll 1$, $r \gg r_0$

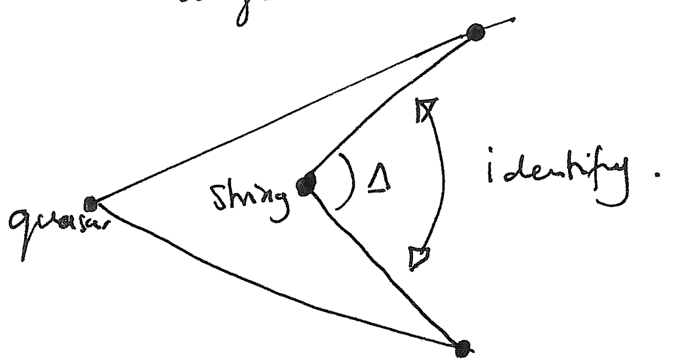
$$f^2 \simeq r^2\left(1-\frac{(kr_0)^2}{2}\right)^2$$

$$\simeq r^2\left(1-8\pi G r_0^2\rho\right)$$

$$= r^2\left(1-8\mu G\right)$$

where $\mu = \pi\rho_0 r_0^2$ is the mass/unit length of the string.

This describes a conical geometry with a "deficit angle" in the θ-direction. This leads to lensing



(2.1) Entropy in cosmology, and its relation to energy conservation, sometime causes confusion.

The density and pressure for a uniform distribution of particles in thermal equilibrium at temperature T is

$$\rho = \frac{4\pi}{h^3} g_i \int E p^2 f \, dp , \quad P = \frac{4\pi}{3h^3} g_i \int \frac{p^4}{E} f \, dp \quad (1)$$

where

$$f = \frac{1}{\left[\exp\left(\frac{E-\mu}{kT}\right) \pm 1\right]}$$

Energy conservation requires

$$\frac{d \rho R^3}{dR} = -3PR^2$$

which we can rewrite as.

$$\frac{d(\rho + P)R^3}{dR} = R^3 \frac{dP}{dR} .$$

Hence to prove that

$$\frac{d}{dR}\left(R^3 \frac{(\rho+P)}{T}\right) = 0 ,$$

$$\frac{1}{T}\frac{d}{dR}(R^3(\rho+P)) - \frac{1}{T^2} R^3(\rho+P)\frac{dT}{dR} = \frac{R^3}{T}\frac{dP}{dR} - \frac{R^3(\rho+P)}{T^2}\frac{dT}{dR}$$

we need to prove that

$$\frac{dP}{dT} = \frac{(\rho + P)}{T} .$$

We can do this from the integrals (1).

$$\frac{dP}{dT} = A \int \frac{p^4}{3E} \frac{df}{dT} \, dp$$

but

$$\frac{df}{dT} = -\frac{E}{T}\frac{df}{dE} = -\frac{E}{T}\frac{df}{dp}\frac{E}{p}$$

so

$$T \frac{dP}{dT} = -\frac{A}{3}\int \frac{df}{dp} p^3 E \, dp$$

and integrating by parts

$$= A\int f \, p^2 E \, dp + \frac{A}{3}\int \frac{f p^4 \, dp}{E}$$

$$= (\rho + P)$$

Hence, the constancy of $R^3(\rho + P)/T$ for particles in thermal equilibrium is simply a restatement of conservation of energy/momentum.

We can re-express this as

$$T \, d\left[R^3(\rho + P)/T\right] = d(\rho R^3) + P \, dR^3$$

and in comparing with the second law of thermodynamics

$$T dS = dU + P dV$$

we can identify $S = R^3(\rho + P)/T$ as the entropy of the particles in a box of volume $V = R^3$.

(2.2) The number density is

$$n = \frac{4\pi}{h^3} \int_0^\infty \frac{p^2\,dp}{\left[e^{(pc-\mu)/kT}+1\right]}$$

If $\mu/kT \gg$, then we can approximate f as a step function at $pc = \mu$, so

$$n = \frac{4}{3}\frac{\pi}{h^3 c^3}\mu^3$$

and we must have $\mu \propto T$ in thermal equilibrium. Note that the number density of the corresponding antineutrino is $n_{\bar\nu} \approx 0$, since $\mu_{\bar\nu} = -\mu_\nu$ and so the number density is exponentially suppressed if $\mu/kT \gg 1$.

The contribution to the energy density is

$$\rho c^2 = \frac{4\pi}{h^3} c \int p^3\,dp\, f$$

$$= \frac{\pi c}{h^3 c^4}\mu^4.$$

So,

$$\rho = \frac{\pi}{h^3}\frac{\mu^4}{c^5}$$

Hence,

$$\Omega_\nu = \rho_\nu/\rho_c = \frac{\pi (kT_\nu)^4}{h^3 c^5 \rho_c}\left(\frac{\mu}{kT_\nu}\right)^4$$

$$= (4/11)^{4/3}\frac{15}{(8\pi)}\rho_\gamma/\rho_c\,(\mu/kT_\nu)^4 \approx 0.15 \times 2.4\times10^{-5} h^{-2} (\mu/kT_\nu)^4$$

So, $\Omega_\nu < 1$ requires $(\mu/kT_\nu) < 23\, h^{1/2}$.

Neutrino degeneracy <u>increases</u> the expansion rate at time of nucleosynthesis.

(2.3) Recall the statistical mechanical derivation of the equilibrium distribution functions. The chemical potential is introduced as a Lagrange multiplier to enforce conservation of the number of particles, $\delta N = 0$. Now, if we have mixtures of particles that can react with each other, conserved quantities (eg. charge) will impose constraints on the numbers of particles of different types and hence on the chemical potentials of different particle species. For example, a photon can pair produce and so photon numbers are not preserved. The photon distribution function in thermal equilibrium must be a function only of p, T. Hence for photons,

$$\mu_\gamma = 0.$$

Similarly, particles and antiparticles can annihilate into photons, so the chemical potentials of a particle and antiparticle must be equal and opposite., $\mu_x = -\mu_{\bar{x}}$.

In the non-relativistic and non-degenerate limit,

$$E \approx mc^2 + p^2/2m$$

and

$$f = \frac{1}{e^{(E-\mu)/kT} \pm 1} \rightarrow e^{-(\mu - mc^2)/kT} e^{-p^2/2mkT}$$

So,

$$n = \frac{4\pi g_s}{h^3} e^{(\mu - mc^2)/kT} \int_0^\infty p^2 e^{-p^2/2mkT} \, dp$$

$$= \frac{4\pi g_s}{h^3}(2m kT)^{3/2}\, e^{(\mu - mc^2)/kT} \int_0^\infty x^2 e^{-x^2}\, dx$$

$$\int_0^\infty x^2 e^{-x^2}\, dx = \sqrt{\pi}/4$$

So,

$$n = g_s \left(\frac{2\pi m kT}{h^2} \right)^{3/2} e^{(\mu - mc^2)/kT}.$$

Note that if $\mu = 0$, the number density of non-relativistic particles in thermal equilibrium is exponentially suppressed as $T \to 0$.

(2.4)

$$p + n \to D + \gamma$$

and the chemical potentials satisfy

$$\mu_p + \mu_n = \mu_D$$

Now from problem (2.3) in thermal equilibrium,

$$\frac{n_D}{n_n n_p} \simeq \frac{g_D}{g_n g_p}\left(\frac{m_D}{m_n m_p}\frac{h^2}{2\pi k T}\right)^{3/2}$$
$$\times\; e^{\frac{\mu_D - m_D c^2}{kT}}\, e^{-\frac{\mu_p + m_p c^2}{kT}}\, e^{-\frac{\mu_n + m_n c^2}{kT}}$$

$$\frac{n_D}{n_n n_p} = \left(\frac{\pi k T m_p}{h^2}\right)^{-3/2} e^{B_D/kT}$$

where $m_D \approx 2m_p$, $B_D = (m_p + m_n - m_D)c^2$.

Now, $X_D = n_D/n_B$, $X_n = n_n/n_B$, $X_p = n_p/n_B$

So,
$$\frac{X_D}{X_n X_p} = n_B\left(\frac{\pi k T m_p}{h^2}\right)^{-3/2} e^{B_D/kT}$$

and
$$\frac{n_\gamma}{n_B} \simeq \frac{a T_\gamma^3}{k n_b} = \frac{8\pi^5 k^3 T_\gamma^3}{15 h^3 c^3 n_B} = \frac{1}{\eta}$$

and we showed in Lecture (1) that η is a very tiny number $\eta \sim 10^{-10}$. Ignoring factors of dimensionless number order unity —

$$n_B\left(\frac{k T m_p}{h^2}\right)^{-3/2} \sim \frac{k^3 T^3}{h^3 c^3}\,\eta \cdot \frac{h^3}{(kTm)^{3/2}} \sim \eta\left(\frac{kT}{m_p c^2}\right)^{3/2}$$

So,
$$\frac{X_D}{X_n X_p} \sim \left(\frac{kT}{m_p c^2}\right)^{3/2} \eta\, e^{B_D/kT}$$

and it is because η is such a tiny number that Deuterium ~~can only~~ forms at a lower temperature than $B_D \sim kT$.

(2.5) The Boltzmann distribution for non-relativistic particles is

$$n_i = g_i \left(\frac{2\pi k T m c^2}{h^2} \right)^{3/2} \exp\left(\frac{\mu_i - mc^2}{kT} \right)$$

So,
$$\frac{n_e n_p}{n_H} = \frac{g_e g_p}{g_H} \left(\frac{2\pi k T m_e c^2}{h^2} \right)^{3/2} \exp\left(- \frac{(m_e + m_p - m_H) c^2}{kT} \right)$$

where we have assumed $m_p \simeq m_H$, $\mu_e + \mu_p = \mu_H$ since this is required by charge conservation.

$g_e = g_p = 2$

$g_H = 4$ since H has two hyperfine ground states of spin 0 and 1.

Hence,
$$\frac{n_e n_p}{n_H} = \left(\frac{2\pi k T m_e c^2}{h^2} \right)^{3/2} \exp\left(- B_H / kT \right)$$

where $B_H = (m_e + m_p - m_H) c^2$ is the binding energy of hydrogen.

$$X_e = \frac{n_e}{n_e + n_H}$$

So,
$$n_e = n_p = \frac{n_H X_e}{(1 - X_e)}$$

and so,
$$\frac{n_e n_p}{n_H} = \frac{n_H X_e^2}{(1 - X_e)^2}$$

but
$$n_H + n_p = (1 - Y_e) \rho_b / m_p$$

Binding energy of helium is much greater than hydrogen, so it is already neutral at the time of H recombination.

So,
$$n_H + \frac{n_H X_e}{(1 - X_e)} = (1 - Y_e) \rho_b / m_p.$$

$$\therefore \quad \frac{n_H}{(1 - X_e)} = \left[(1 - Y_e) \rho_b / m_p \right]$$

and so,

$$\frac{X_e^2}{1-X_e} \approx \left[(1-Y_e)\rho_b/m_p\right]^{-1}\left(\frac{2\pi k T m_e c^2}{h^2}\right)^{3/2} \exp\left(-\frac{B_H}{kT}\right)$$

Evaluating we get

$$\frac{X_e^2}{(1-X_e)} = A \exp\left(-B_H/kT\right)$$

$$A \sim 10^{22}\, T^{-3/2}$$ with T in Kelvin.

Since $A \gg 1$, recombination occurs at a much lower temperature than B_H/k.

This derivation assumes thermal equilibrium. However, it is a fairly poor approximation to the recombination history because thermal equilibrium is not maintained for low ionization levels. Photons emitted when free electrons are captured by a photon can ionize another hydrogen atom. The full physics of recombination is complicated and is still being worked on today.