Solutions to HEP problems from the Astroparticle Minor Option

These notes are intended to guide you through the questions, and do not always have complete working. There are also some comments to help you understand the context of the questions.

1. a) Expand the exponential:

$$V = -(1 - \alpha^2\phi^2 + \frac{1}{2}\alpha^4\phi^4 - \frac{1}{6}\alpha^6\phi^6 + \ldots)$$

Comparing to the Klein-Gordon equation, we find the mass term in ϕ^2 giving

$$\alpha^2 = \frac{1}{2}\left(\frac{mc}{\hbar}\right)^2 \Rightarrow m = \frac{\sqrt{2}\alpha\hbar}{c}$$

The remaining terms are couplings (4 point, six point etc).

b) Under the gauge transformation we have

$$\begin{aligned} F^{\mu\nu} &\rightarrow \partial^\mu\left[A^\nu + \partial^\nu\lambda\right] - \partial^\nu\left[A^\mu + \partial^\mu\lambda\right] \\ &= \partial^\mu A^\nu - \partial^\nu A^\mu + \partial^\mu\partial^\nu\lambda - \partial^\nu\partial^\mu\lambda \\ &= \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu} \\ A^\nu A_\nu &\rightarrow \left[A^\nu + \partial^\nu\lambda\right]\left[A_\nu + \partial_\nu\lambda\right] \neq A^\nu A_\nu \end{aligned}$$

Hence the gauge bosons must be massless.

2. a) there are standard SU(2) transitions (LH W, Z) up and down the columns of the multiplet, and SU(3) transistions (gluons) along the rows between the quarks. There are new leptoquarks which allow transitions along the rows between the coloured quark states and the colourless leptons. In addition, the right-handed multiplet has RH W and Z states.

b) the X bosons conserve weak isospin. They carry colour and 2/3 electric charge. They change baryon number by 1/3 and lepton number by 1 unit.

c) the neutrino has zero electric charge, no colour and no LH weak interactions, and so is sterile in the standard model. It will have RH weak interactions, but these are suppressed by the high mass of the RH W and Z states.

d) Since the leptoquarks change baryon number by 1/3, at least 3 X interactions would be needed to create proton decay. Each interaction is highly suppressed by the X mass, and so in practical terms the proton is stable in this model. However, supersymmetric versions of this model allow proton lifetimes close to the current experimental limits.

3. Write the amplitude for the kaon beam with equal numbers of long and short states as

$$|K_L + K_S\rangle = \exp(i(\omega_s t + \phi_s))\exp\left(-\frac{t}{2\tau_s}\right) + \exp(i(\omega_L t + \phi_L))\exp\left(-\frac{t}{2\tau_L}\right)$$

Squaring this amplitude leads to

$$N(t) = N(0)\left\{e^{-t/\tau_S} + e^{-t/\tau_L} + 2\exp\left(-t\left(\frac{1}{2\tau_S} + \frac{1}{2\tau_L}\right)\right)\cos\left((\omega_S - \omega_L)t + (\phi_S - \phi_L)\right)\right\}$$

From the data, half a wavelength of the oscillation term is about 6 K_S lifetimes, leading to a mass difference of 3.4×10^{-6} eV. The phase at the origin gives an initial phase difference of about 60 degrees (true value 43.5 degrees). By comparing the rate at t=0, where the decays are completely dominated by the K_S term, with the rate at large t, dominated by the K_L term, we obtain a value for eta of around 10^{-3}. (NB, eta is defined as the ratio of amplitudes, not rates, as incorrectly stated in the question).

4 a) Consider $m_1 \to m_2 + m_3$

$$m_1^2 = (E_2 + E_3, \vec{0})^2 \qquad \text{in frame of 1}$$
$$= E_2^2 + E_3^2 + 2E_2E_3$$
$$= m_2^2 + p^2 + m_3^2 + p^2 + 2E_2E_3$$
$$(m_{1w}^2 - m_2^2 - m_3^2 - 2p^2)^2 = 4(m_2^2 + p^2)(m_3^2 + p^2)$$
$$4m_1^2p^2 = m_1^4 + m_2^4 + m_3^4 - 2m_1^2m_2^2 - 2m_1^2m_3^2 - 2m_2^2m_3^2$$
$$= (m_1^2 - m_2^2 + m_3^2)^2 - 4m_1^2m_3^2$$
$$4m_1^2E_3^2 = (m_1^2 - m_2^2 + m_3^2)^2$$
$$E_3 = \frac{m_1^2 - m_2^2 + m_3^2}{2m_1}$$

If m_3 is small, $E_3=p_3=p$.

b) Consider $\tilde{\chi}_2^0 \to \tilde{l}l_2$ and work in slepton rest frame. The four vectors are:

$$\tilde{\chi}_2^0 \equiv (E_2, p_2)$$
$$\tilde{l} \equiv (m_{\tilde{l}}, 0)$$
$$l_2 \equiv (p_2, p_2)$$

Conserve E, p in this frame:

$$m_{\tilde{\chi}_2^0}^2 = (m_{\tilde{l}} + p_2, p_2)^2$$
$$= m_{\tilde{l}}^2 + 2m_{\tilde{l}}p_2$$
$$p_2 = \frac{m_{\tilde{\chi}_2^0}^2 - m_{\tilde{l}}^2}{2m_{\tilde{l}}}$$

p_2 is the momentum of the outgoing lepton (labelled 2 because it is produced with the Chi2).

Now consider the outgoing particles from the decay $\tilde{l} \to \tilde{\chi}_1^0 l_1$. The four vectors are:

$$\tilde{l} \equiv (m_{\tilde{l}}, 0)$$
$$\tilde{\chi}_1^0 \equiv (E_1, p_1)$$
$$l_1 \equiv (p_1, -p_1)$$

Conserve E, p again:

$$\begin{aligned} m_{\tilde{l}}^2 &= (E_1 + p_1, 0)^2 \\ &= E_1^2 + p_1^2 + 2E_1 p_1 \\ (m_{\tilde{l}}^2 - m_{\tilde{\chi}_1^0}^2 - 2p_1^2)^2 &= 4p_1^2(m_{\tilde{\chi}_1^0}^2 + p_1^2) \\ m_{\tilde{l}}^4 + m_{\tilde{\chi}_1^0}^4 - 2m_{\tilde{l}}^2 m_{\tilde{\chi}_1^0}^2 &= 4p_1^2 m_{\tilde{l}}^2 \\ p_1 &= \frac{m_{\tilde{l}}^2 - m_{\tilde{\chi}_1^0}^2}{2m_{\tilde{l}}} \end{aligned}$$

This gives us the momentum of lepton 1 (again in the slepton rest frame).
Now we can make the invariant mass of the lepton pair:

$$\begin{aligned} m_{ll}^2 &= (E_1 + E_2, \vec{p}_1 + \vec{p}_2) \\ &= 2p_1 p_2 (1 - \cos\theta) \end{aligned}$$

where we have neglected lepton masses. Theta is the angle between the two leptons. Clearly the maximum is when $\cos\theta = -1$, giving

$$\begin{aligned} m_{ll}^2(\max) &= 4p_1 p_2 \\ &= 4\left(\frac{m_{\tilde{l}}^2 - m_{\tilde{\chi}_1^0}^2}{2m_{\tilde{l}}}\right)\left(\frac{m_{\tilde{\chi}_2^0}^2 - m_{\tilde{l}}^2}{2m_{\tilde{l}}}\right) \\ &= m_{\tilde{\chi}_2^0}^2\left(1 - \frac{m_{\tilde{l}}^2}{m_{\tilde{\chi}_2^0}^2}\right)\left(1 - \frac{m_{\tilde{\chi}_1^0}^2}{m_{\tilde{l}}^2}\right) \end{aligned}$$

This configuration gives the leptons antiparallel, with lepton 2 along the chi1 direction.

c) Stay in this frame. The invariant mass of the llγ system is given by

$$\begin{aligned} m_{ll\gamma}^2 &= (E_1 + E_2 + E_\gamma, \vec{p}_1 + \vec{p}_2 + \vec{p}_3)^2 \\ &= 2p_1 p_2 (1 - \cos\theta_{12}) + 2p_1 p_\gamma (1 - \cos\theta_{1\gamma}) + 2p_2 p_\gamma (1 - \cos\theta_{2\gamma}) \end{aligned}$$

Whatever the angles between the particles, this has a maximum for the highest possible photon momentum in this frame. This occurs when the photon is emitted along the line of the chi1, thereby being boosted forwards. Since the chi1 is antiparallel to lepton 1, $\theta_{1\gamma} = \pi$ and $\theta_{2\gamma} = \theta_{12} - \pi$.

We therefore have two possible cases for the maximum:

$$\theta_{12} = 0: \qquad m_{ll\gamma}^2 = 4p_1p_\gamma + 4p_2p_\gamma$$
$$\theta_{12} = \pi: \qquad m_{ll\gamma}^2 = 4p_1p_\gamma + 4p_1p_2$$

The first is the maximum if $p_\gamma > p_1$, in which case the two leptons are parallel and antiparallel to the photon.

We just need the momentum of the photon. It is emitted by the chi1, which is travelling at velocity $\beta = p_{\tilde{\chi}_1^0} / E_{\tilde{\chi}_1^0}$ in the slepton rest frame. The momentum of the photon in the chi1 rest frame is $p^* = m_{\tilde{\chi}_1^0}/2$. Transform this to the slepton rest frame, remembering that the frame is travelling in the opposite direction to the chi1:

$$\begin{aligned}
p_\gamma &= \gamma p^*(1+\beta) \\
&= \frac{E_{\tilde{\chi}_1^0}}{m_{\tilde{\chi}_1^0}} \frac{m_{\tilde{\chi}_1^0}}{2} \left(1 + \frac{p_{\tilde{\chi}_1^0}}{E_{\tilde{\chi}_1^0}}\right) \\
&= \frac{1}{2}\left(E_{\tilde{\chi}_1^0} + p_{\tilde{\chi}_1^0}\right) \\
p_{\tilde{\chi}_1^0} &= p_1 \\
E_{\tilde{\chi}_1^0}^2 &= \left(\left(\frac{m_{\tilde{l}}^2 - m_{\tilde{\chi}_1^0}^2}{2m_{\tilde{l}}}\right)^2 + m_{\tilde{\chi}_1^0}^2\right) \\
E_{\tilde{\chi}_1^0} &= \left(\frac{m_{\tilde{l}}^2 + m_{\tilde{\chi}_1^0}^2}{2m_{\tilde{l}}}\right) \\
E_{\tilde{\chi}_1^0} + p_{\tilde{\chi}_1^0} &= m_{\tilde{l}} \\
p_\gamma &= \frac{m_{\tilde{l}}}{2} \quad \text{which is } > p_1 = \left(\frac{m_{\tilde{l}}^2 - m_{\tilde{\chi}_1^0}^2}{2m_{\tilde{l}}}\right)
\end{aligned}$$

Hence the first case is the maximum.

Inserting the values of the momenta:

$$\begin{aligned}
m_{ll\gamma}^2(\max) &= 4p_\gamma(p_1 + p_2) \\
&= 4 \cdot \frac{m_{\tilde{l}}}{2} \cdot \left(\left(\frac{m_{\tilde{l}}^2 - m_{\tilde{\chi}_1^0}^2}{2m_{\tilde{l}}}\right) + \left(\frac{m_{\tilde{\chi}_2^0}^2 - m_{\tilde{l}}^2}{2m_{\tilde{l}}}\right)\right) \\
&= m_{\tilde{\chi}_2^0}^2 - m_{\tilde{\chi}_1^0}^2
\end{aligned}$$

d) The maximum mass of the photon with the leptons is simply given by

$$
\begin{aligned}
m_{l\gamma}^2(\max) &= 4\,p_l p_\gamma \\
&= m_{\tilde{l}}^2 - m_{\tilde{\chi}_1^0}^2 \quad \text{for lepton 1} \\
&= m_{\tilde{\chi}_2^0}^2 - m_{\tilde{l}}^2 \quad \text{for lepton 2}
\end{aligned}
$$

e) Inserting the numbers we get:

$$
\begin{aligned}
m_{\tilde{\chi}_2^0} &= 224\ \text{GeV} \\
m_{\tilde{l}} &= 119\ \text{GeV} \\
m_{\tilde{\chi}_1^0} &= 164\ \text{GeV}
\end{aligned}
$$

If the LHC discovers a signal which looks like SUSY, methods like this would be used to try to extract the underlying particle masses from the measurements.

5. The equation of motion of the field is given in the lectures as

$$-\frac{d^2\phi}{dx^2} + \lambda^2\phi\left(\phi^2 - \phi_{\min}^2\right) = 0$$

and the solution stated is

$$\phi = \frac{\mu}{\lambda}\tanh\left(\frac{x}{\Delta}\right)$$

Using $\frac{d}{dx}(\tanh(\mathrm{z})) = \frac{1}{\Delta}\mathrm{sech}^2(\mathrm{z})$ $\quad$ $\mathrm{z} = \frac{\mathrm{x}}{\Delta}$ and $\frac{d^2}{dx^2}(\tanh(\mathrm{z})) = -\frac{2}{\Delta^2}\mathrm{sech}^2(\mathrm{z})\tanh(\mathrm{z})$

we obtain

$$\left(\frac{2\mu}{\lambda\Delta^2} - \frac{\mu^3}{\lambda}\right)\mathrm{sech}^2(\mathrm{z})\tanh(\mathrm{z}) = 0$$

and hence $\Delta = \frac{\sqrt{2}}{\mu}$.

We know the potential energy as a function of the field:

$$
\begin{aligned}
V &= \frac{\lambda^2}{4}\left(\phi^2 - \phi_{\min}^2\right)^2 \\
&= \frac{\lambda^2}{4}\frac{\mu^2}{\lambda^2}\left[\tanh^2\left(\frac{x}{\Delta}\right) - 1\right]^2 \\
&= \frac{\mu^2}{4}\operatorname{sech}^4\left(\frac{x}{\Delta}\right)
\end{aligned}
$$

The first approximation Higgs mass is given by $m_H = \sqrt{2}\mu$ and hence $\Delta = 2/m_H$. The Compton wavelength of the Higgs is given by $1/m_H$ so Δ is twice the Compton wavelength (as one might expect from a barrier penetration picture).

6. In a radiation dominated era, the Hubble constant is given by

$$
H = \left(\frac{8\pi G}{3}\right)^{1/2}\rho_r^{1/2} = \frac{1}{2t}
$$

and $\rho_r \propto R^{-4}$. Hence by integration, we obtain

$$
\rho_r = \frac{3}{32\pi G}t^2.
$$

We know that the energy density of the radiation is given by

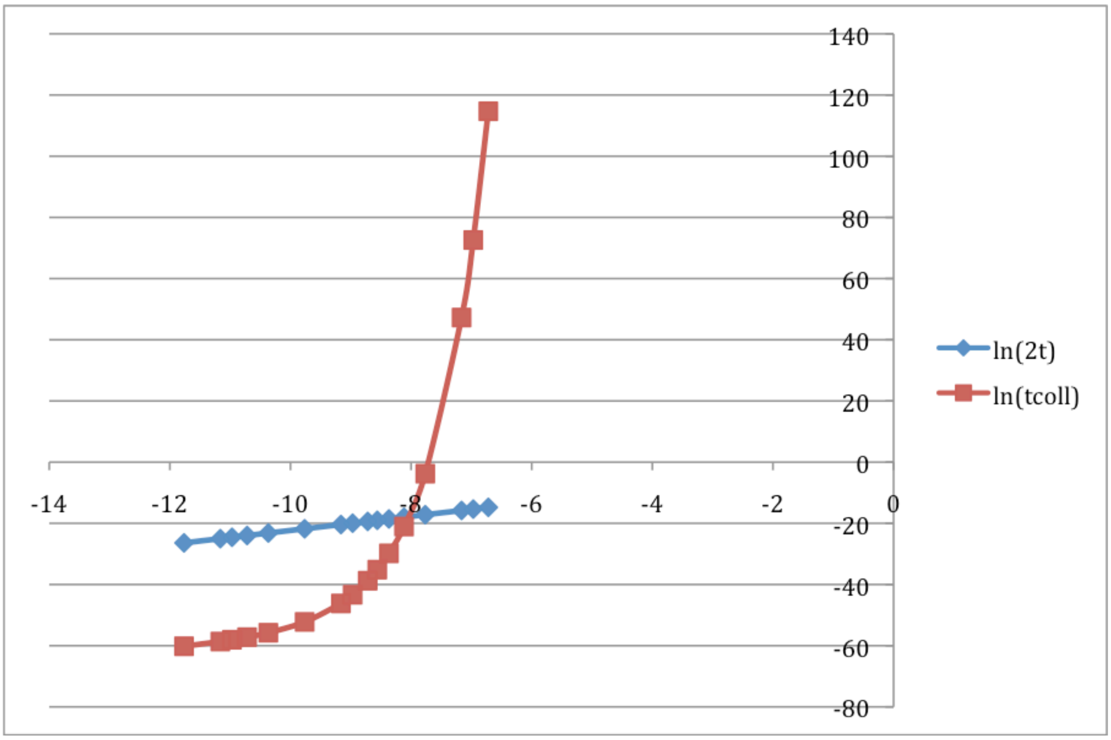
$$
\rho_r c^2 = \frac{4\sigma T^4}{c}
$$

and so

$$
t = \left(\frac{3c^3k^4}{128\pi G\sigma}\right)^{1/2}\frac{1}{(kT)^2}.
$$

The numerical constant is equal to $4.4\text{x}10^{-26}$ J^2s.

Plotting the expansion and collision times versus time (log-log plot) gives



with the two rates equal at around 10^{-8} s, when M/T is around 40.

Using this we have the number density at the freeout temperature T_f and so we can predict the number density today by scaling with a factor $(T_0/T_f)^3$. We can then predict the contribution to the energy density by multiplying by mc^2. We also have the interaction cross section, so we can predict how many such particles would have been produced at the LHC, by multiplying the cross-section by the available integrated luminosity. (Number of events = cross section x integrated luminosity).

7. Simply use the normal method to muliply spinors, not forgetting the γ^0 term in the definition of the conjugate spinors.
Then find the eigenvalues of the neutrino mass matrix given in the lectures. You should get M_R and $-m_D^2/M_R$.

8. Follow method in lectures: integrate over 4-spheres in a 5-dimensional space to get volume of 5-sphere.

$$
\begin{aligned}
V_5(r) &= \int_{-1}^{1} V_4(r\sin\theta)\, d(\cos\theta) \\
&= \int_{0}^{\pi} \frac{1}{2}\pi^2 r^5 \sin^5\theta \, d\theta \\
&= \frac{1}{2}\pi^2 r^5 \frac{16}{15} \\
&= \frac{8}{15}\pi^2 r^5
\end{aligned}
$$

Get surface of 5-sphere by differentiation wrt r: $S_5 = \frac{8\pi^2}{3} r^4$

Hence by Gauss' theorem, the force law in 5-D is: $F = \frac{3G_5 Mm}{2\pi r^4}$

and the potential is $V = \frac{G_5 Mm}{2\pi r^3}$.

If we have a Gaussian surface with a radius r>>R , where R is the radius of the extra dimensions, we can identify this potential with our normal one and so

$$\frac{GMm}{r} = \frac{G_5 Mm}{2\pi R^2 r}$$

giving

$$G = \frac{G_5}{2\pi R^2}$$

Hence we recover the formula in the notes: $\overline{M}_{PL}^2 = M_*^4 R^2 \quad , \quad M_*^4 = \frac{2\pi}{G_5}$

(Note, there are different conventions in the literature for the definition of the bulk Planck mass. We shall use this equation as the definition of M*.)

Setting M* = 1TeV, and using the reduced Planck mass of 2.4×10^{18} GeV gives R=10^{-4} m. (In the lectures I did an order of magnitude estimate, which led to twice this value).